Discrete Optimization

MA2827

Fondements de l'optimisation discrète

https://project.inria.fr/2015ma2827/

Slides courtesy of M. Pawan Kumar

Recap of previous lectures

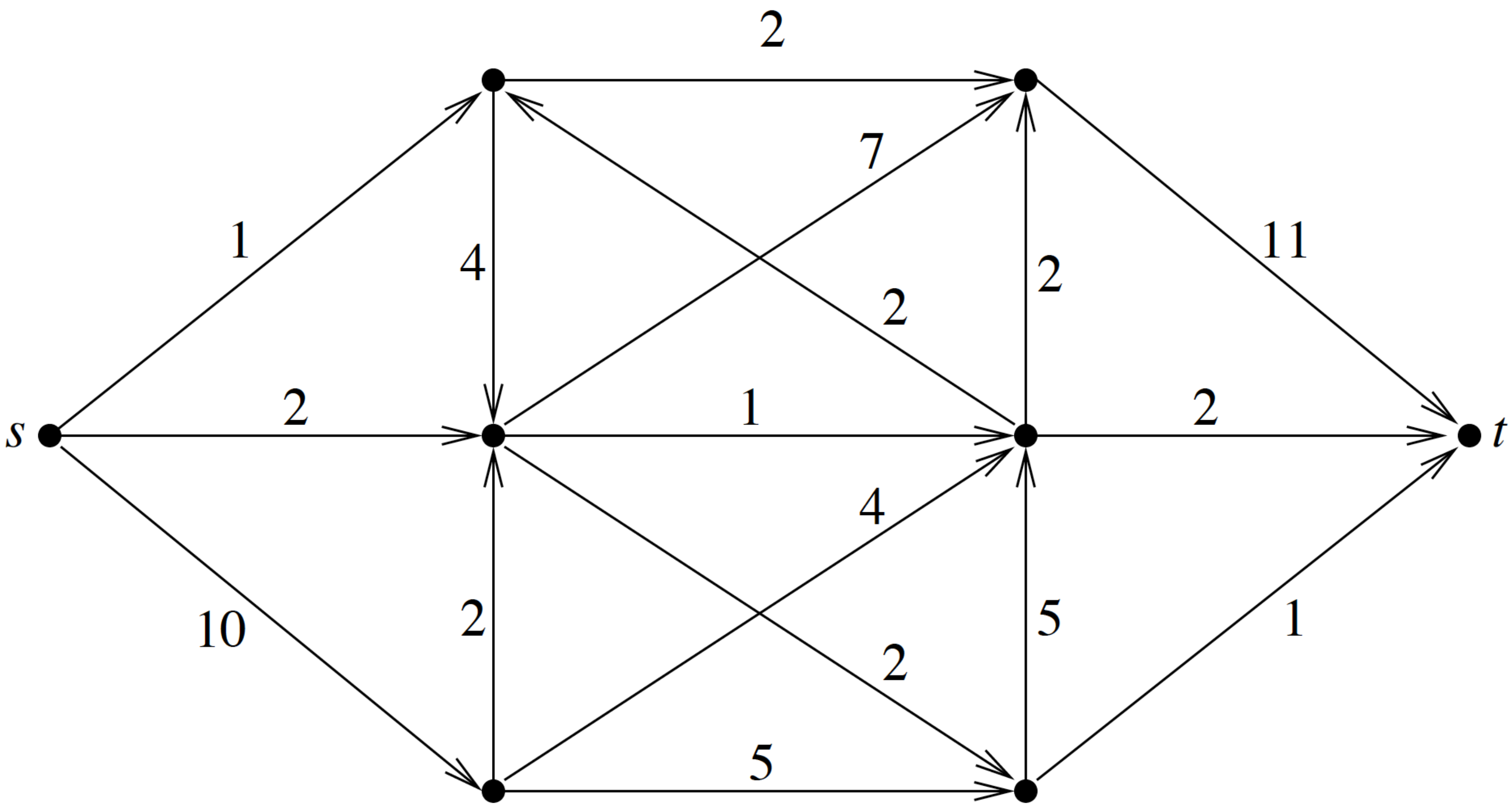
- Lecture 1
  - Graph preliminaries
  - Complexity basics
  - Shortest path algorithms (Dijkstra, Bellman-Ford, Floyd-Warshall)
- Lecture 2
  - Chow-Liu tree
  - Minimum spanning tree (Prim's, Kruskal's)
  - Maximum flow (Ford-Fulkerson, Dinits)

Outline

- Preliminaries
- Menger’s Theorem for Disjoint Paths
- Path Packing

But first…

Find the max flow/min cut.
Show the steps.



Outline

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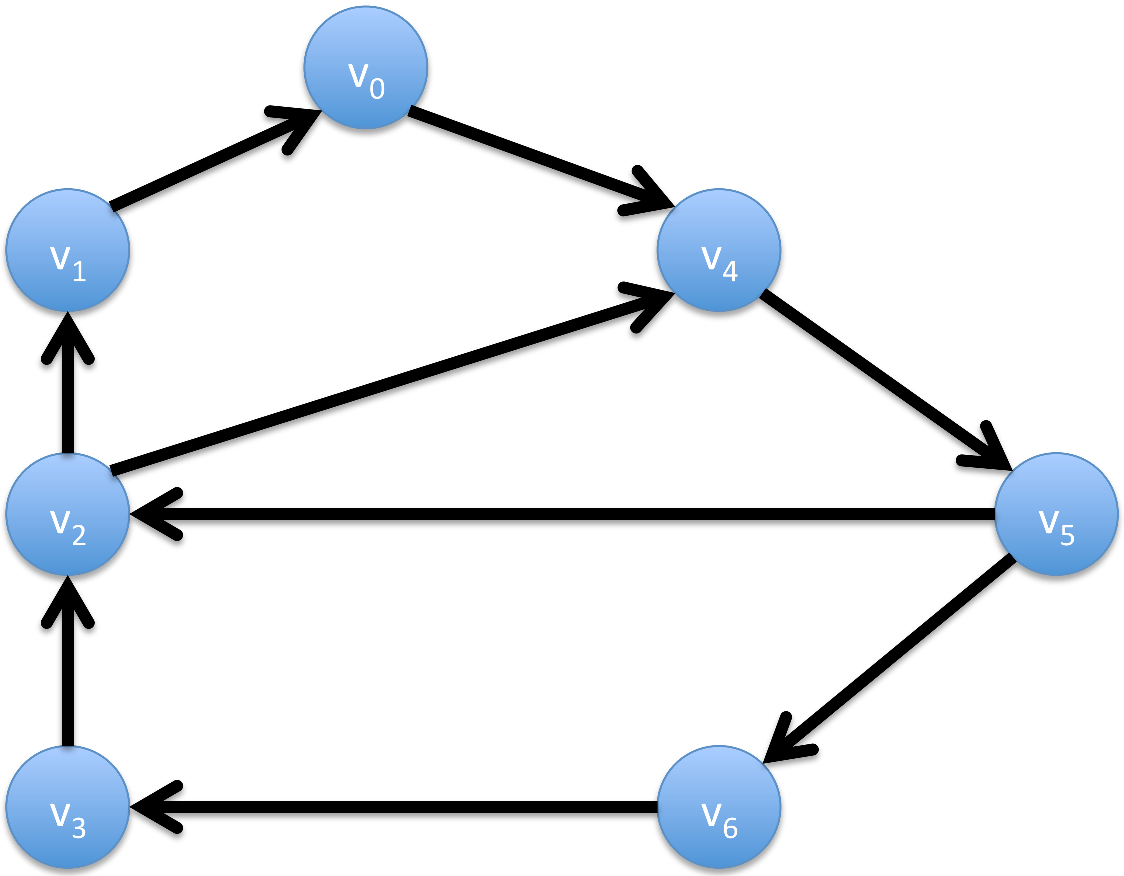
Directed Graphs (Digraphs)

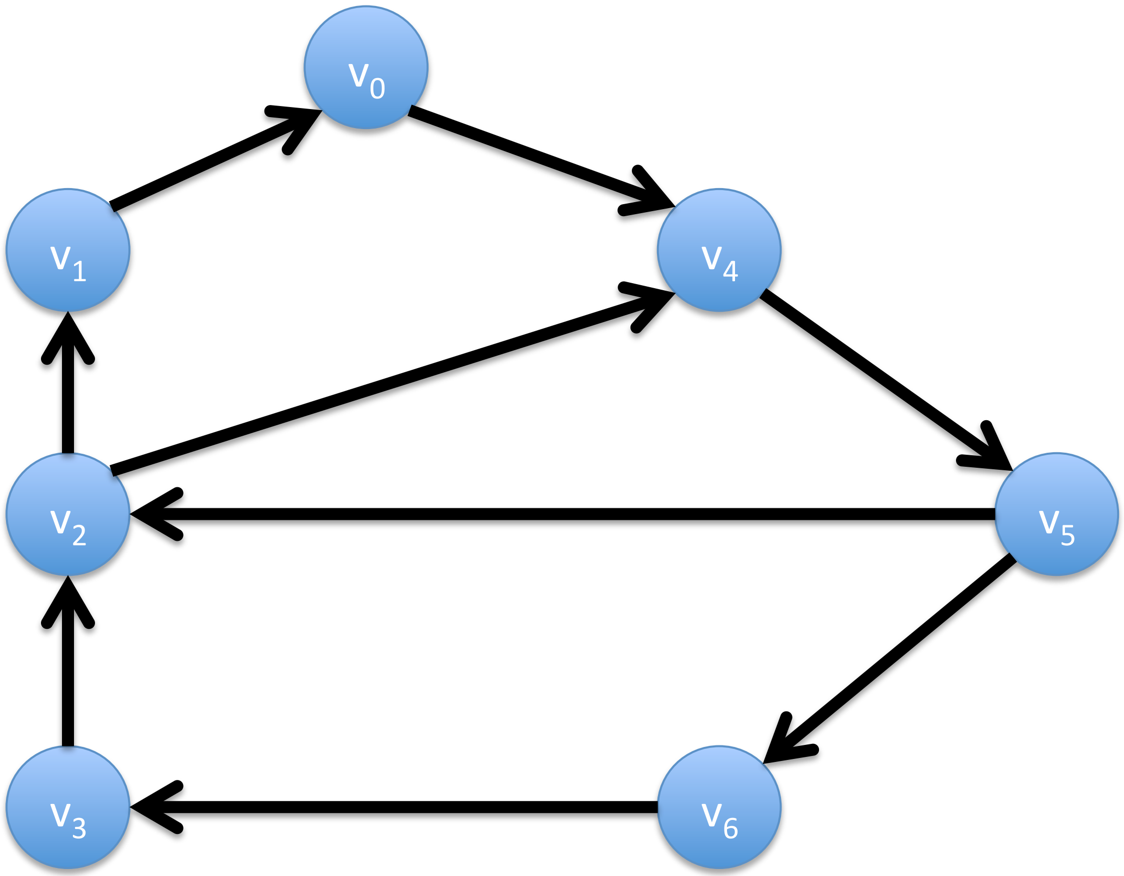


D = (V, A)

'n' vertices or nodes V

'm' arcs A: ordered pairs from V

Indegree of a Vertex

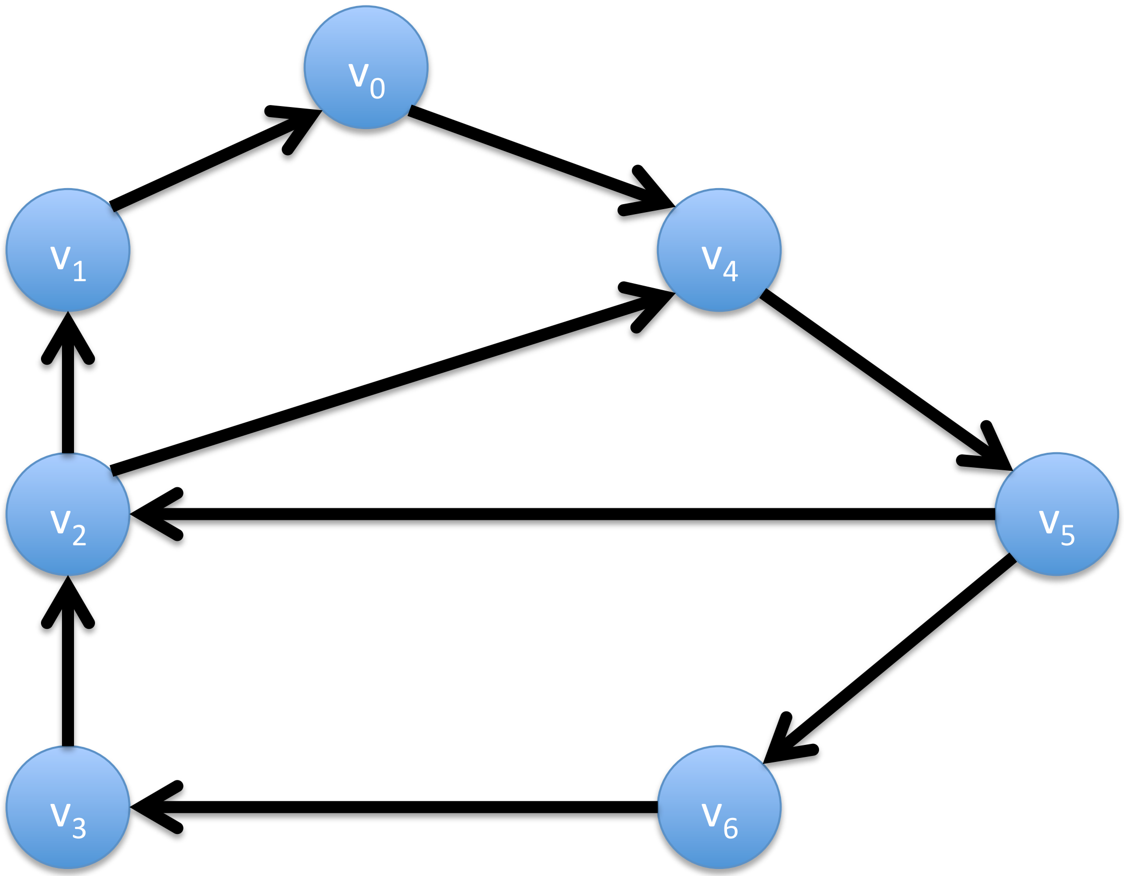


D = (V, A)

Number of arcs entering the vertex.

$indeg(v_0) = 1$, $indeg(v_1) = 1$, $indeg(v_4) = 2$, …

Indegree of a Subset of Vertices

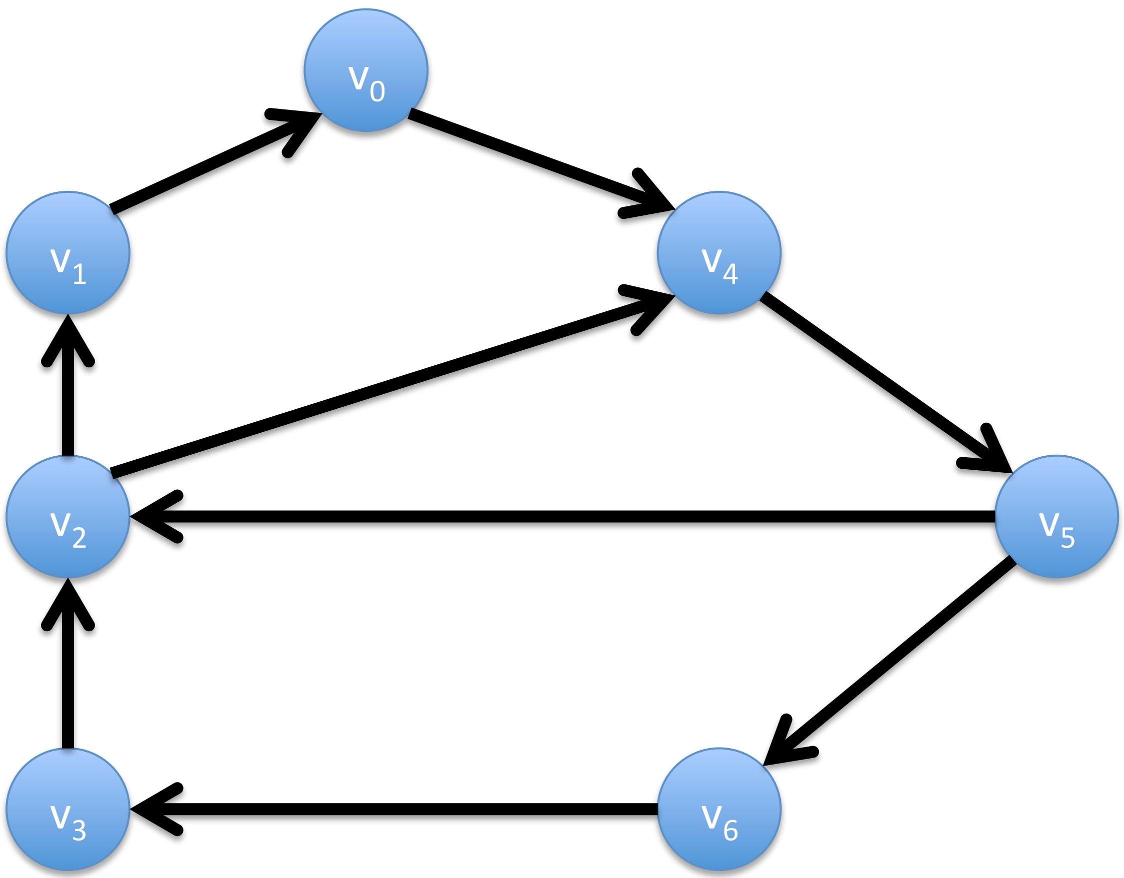


D = (V, A)

Number of arcs entering the subset.

indeg($\{v_0,v_1\}$) = 1, indeg($\{v_1,v_4\}$) = 3, ...

Outdegree of a Vertex

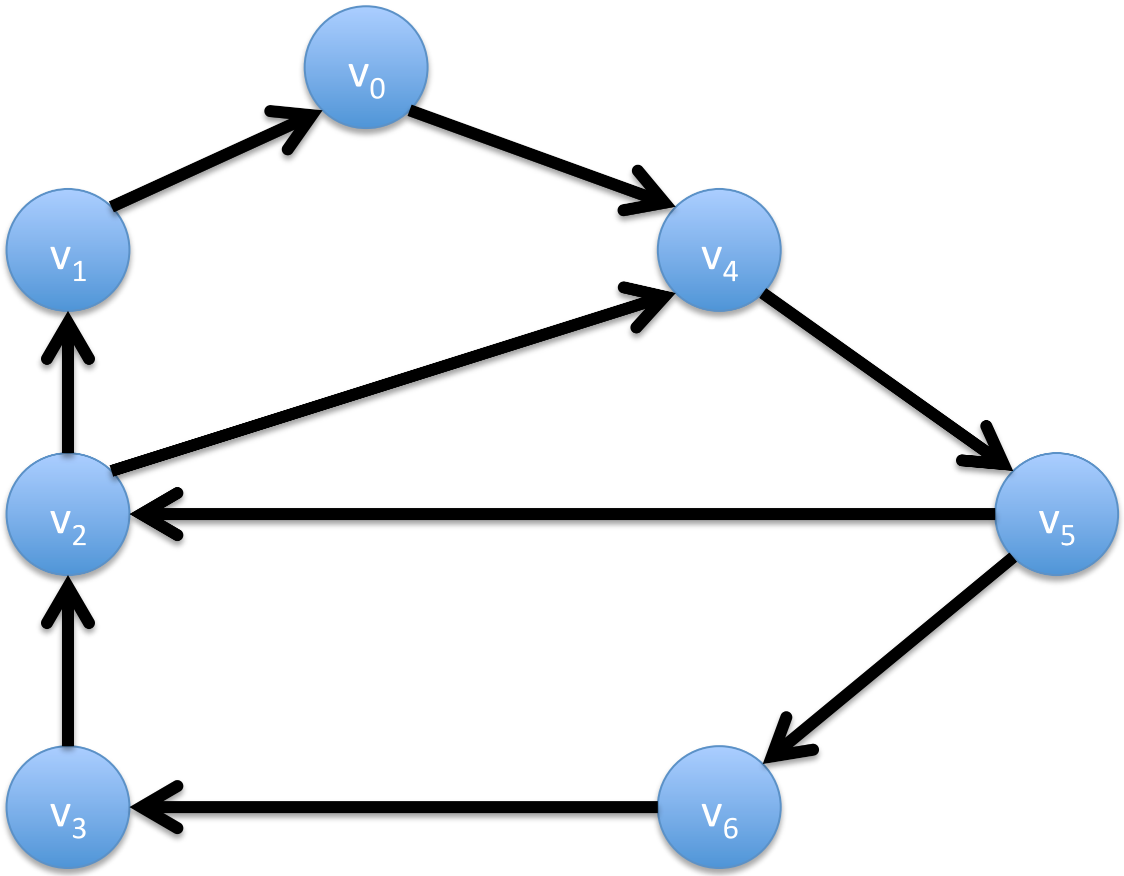


D = (V, A)

Number of arcs leaving the vertex.

$outdeg(v_0) = 1$, $outdeg(v_1) = 1$, $outdeg(v_2) = 2$, ...

Outdegree of a Subset of Vertices



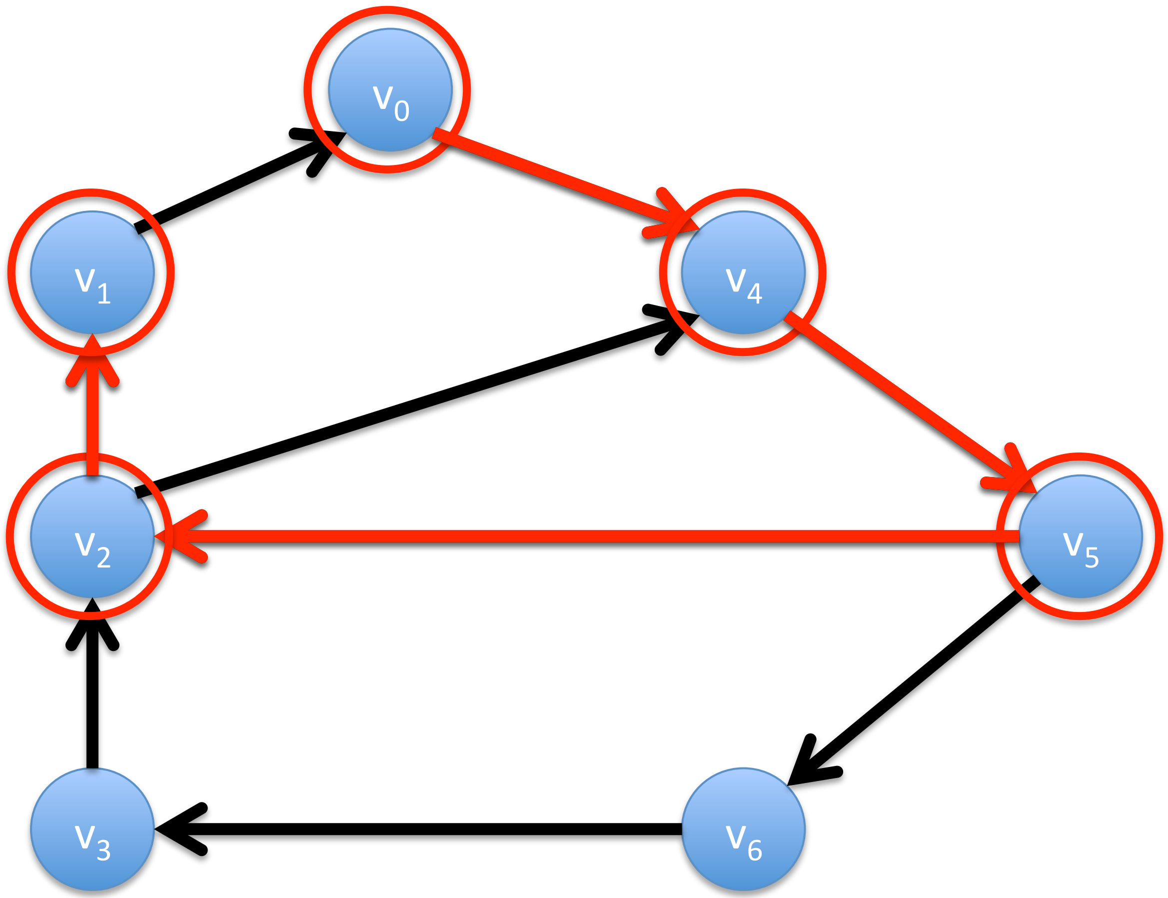
D = (V, A)

Number of arcs leaving the subset.

outdeg($\{v_0, v_1\}$) = 1, outdeg($\{v_1, v_4\}$) = 2, ...

s-t Path

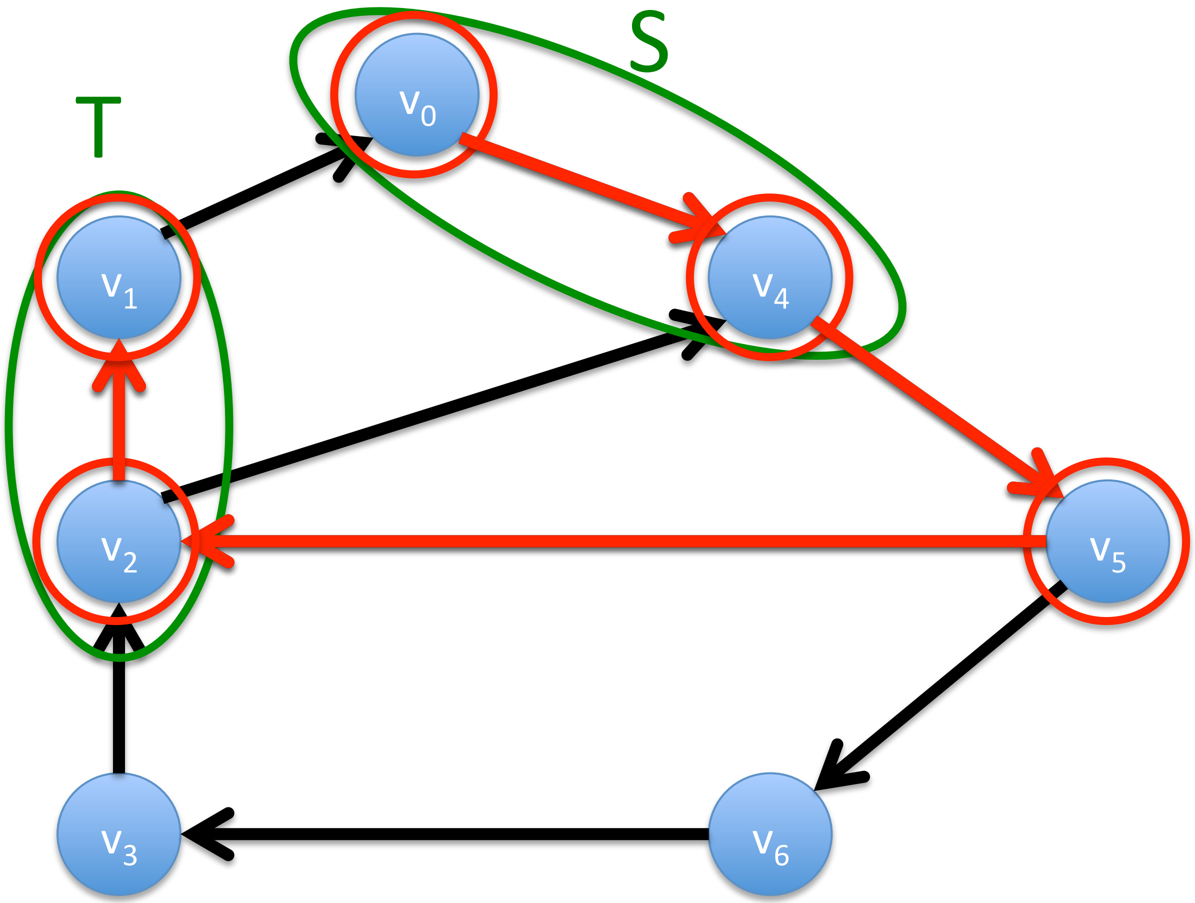
D = (V, A)



Sequence $P = (s=v_0,a_1,v_1,\ldots,a_k,t=v_k)$, $a_i = (v_{i-1},v_i)$

Vertices $s=v_0,v_1,\ldots,t=v_k$ are distinct

S-T Path

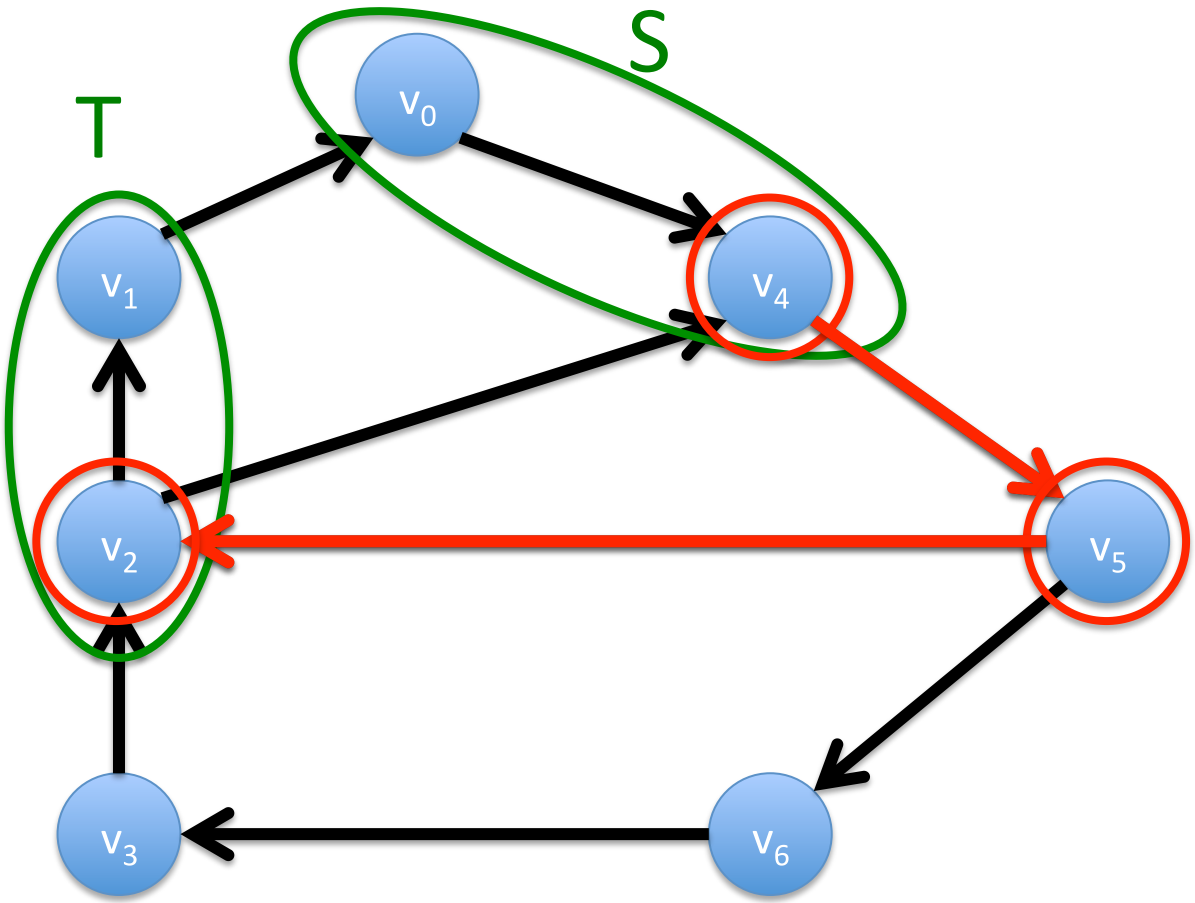


D = (V, A)

S and T are subsets of V

Any st-path where s ∈ S and t ∈ T

S-T Path



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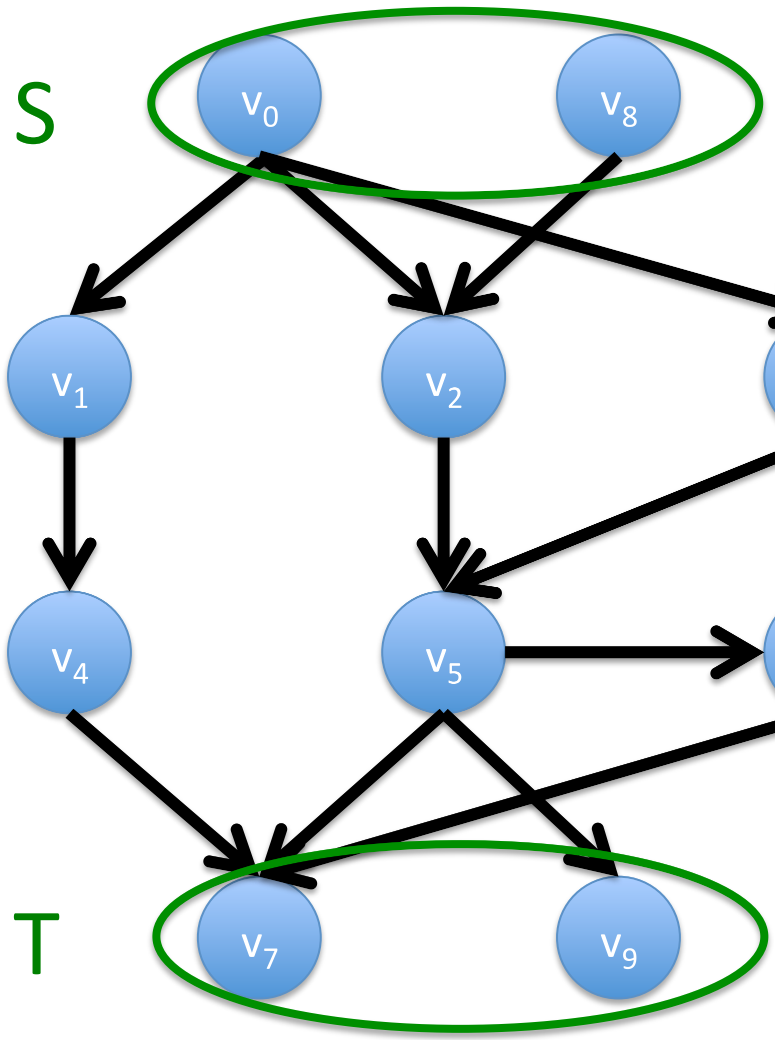
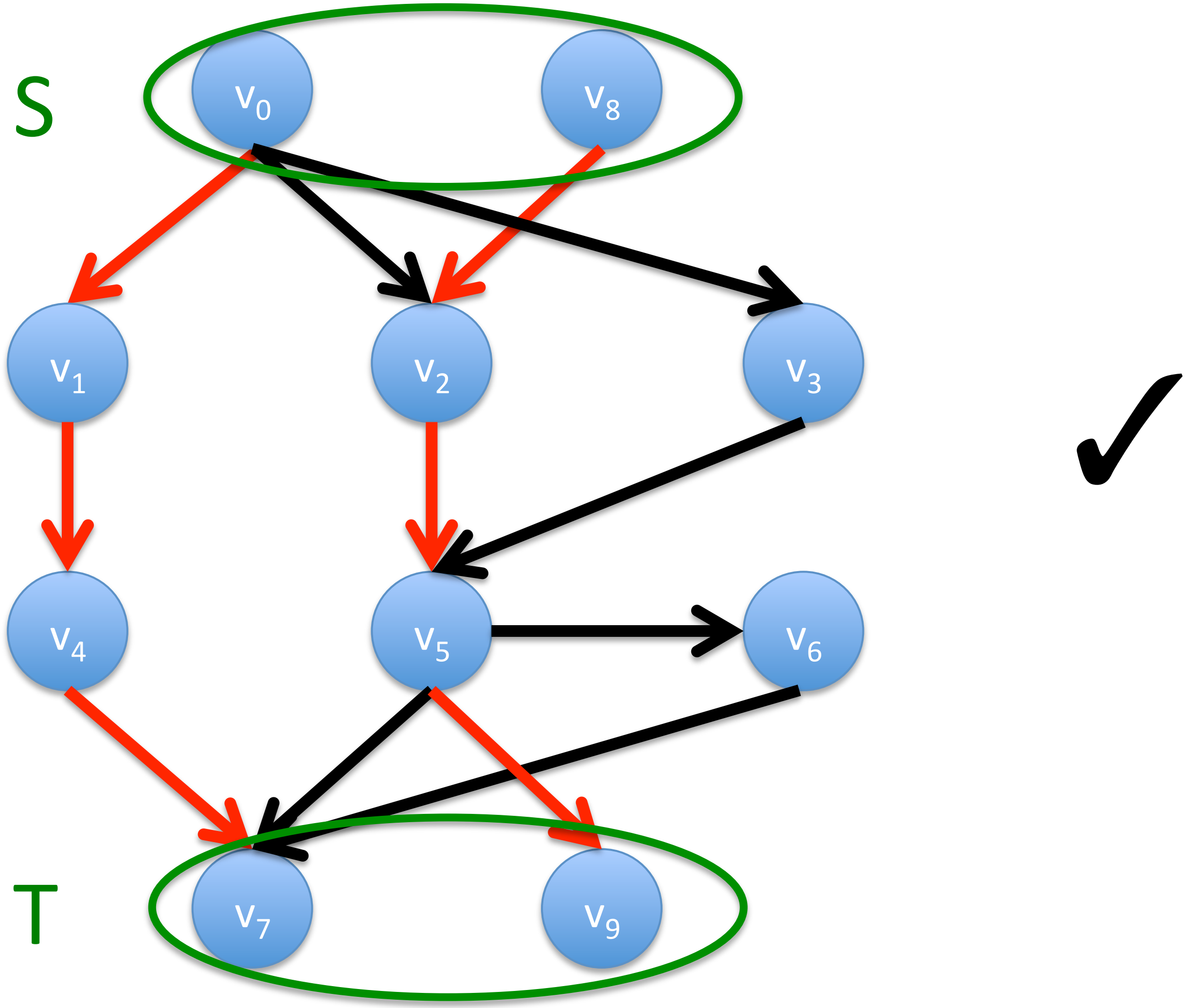
- **Menger's Theorem for Disjoint Paths**
- Path Packing

Vertex Disjoint S-T Paths



Set of S-T Paths with no common vertex

Vertex Disjoint S-T Paths



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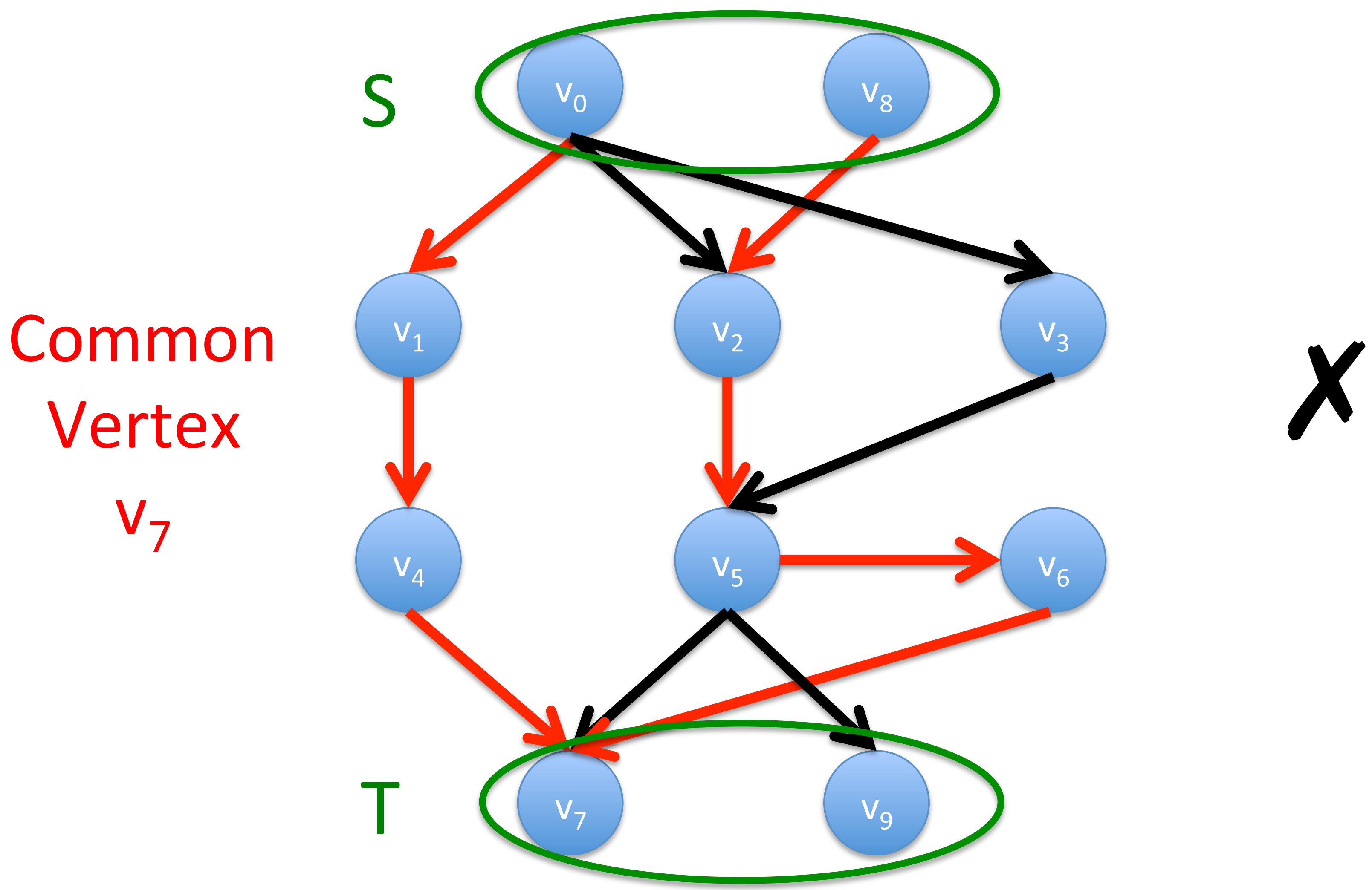
Vertex Disjoint S-T Paths



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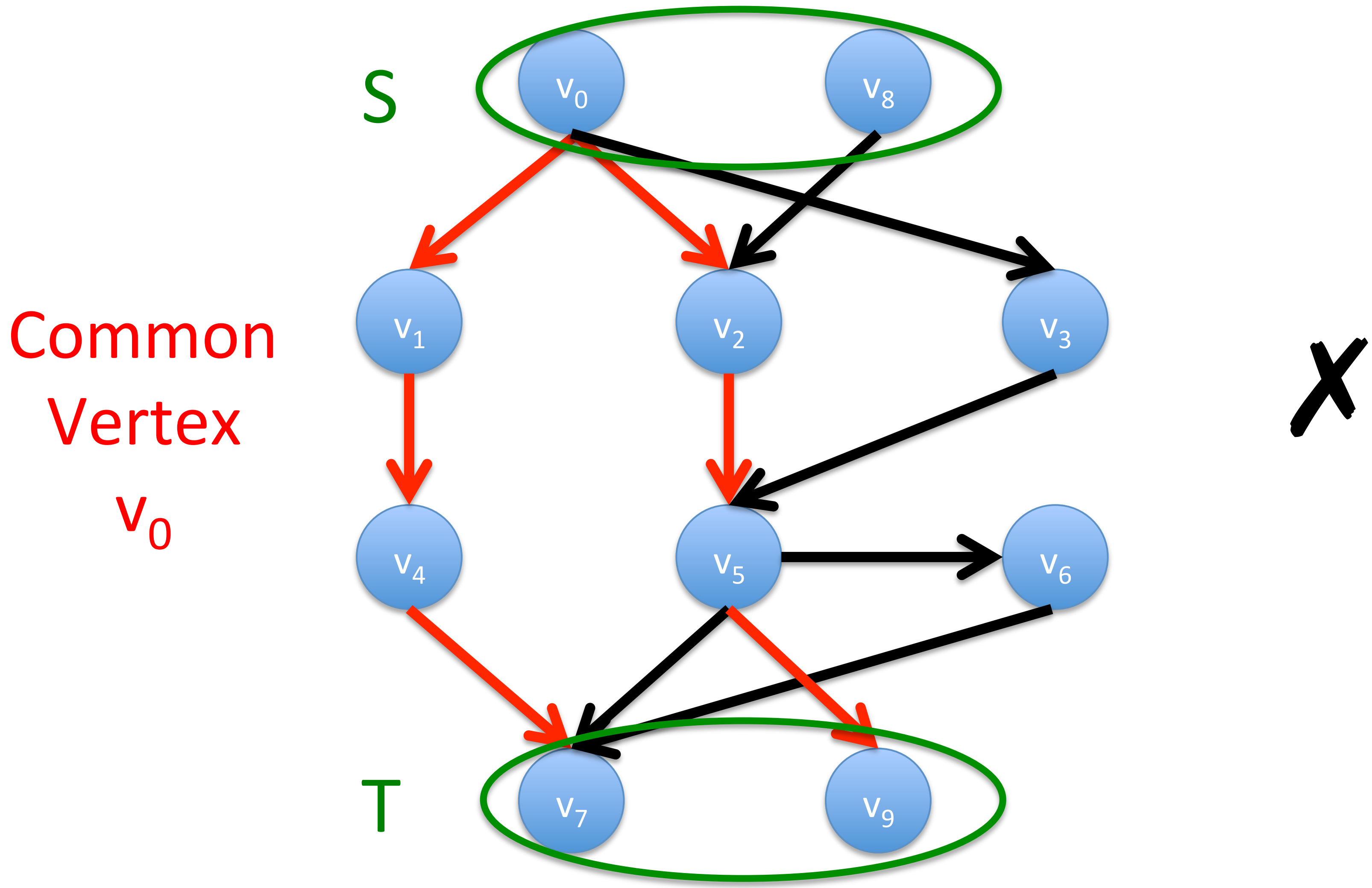
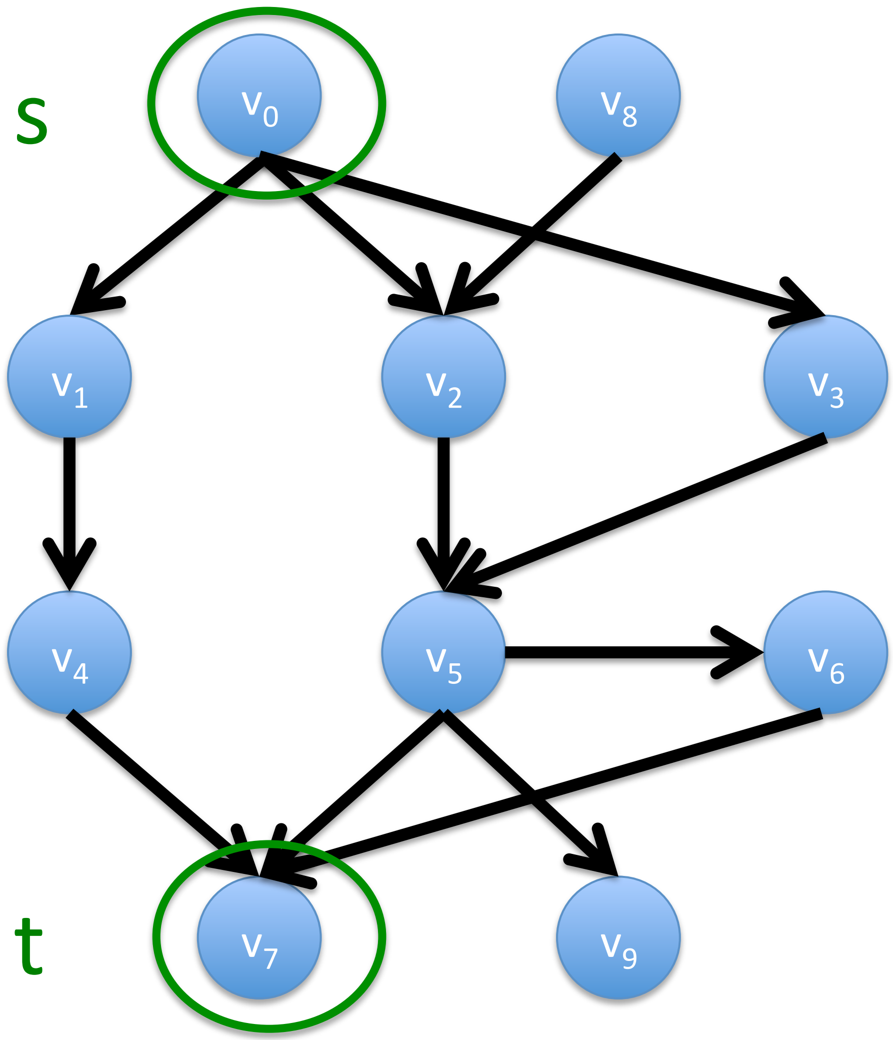
Vertex Disjoint S-T Paths



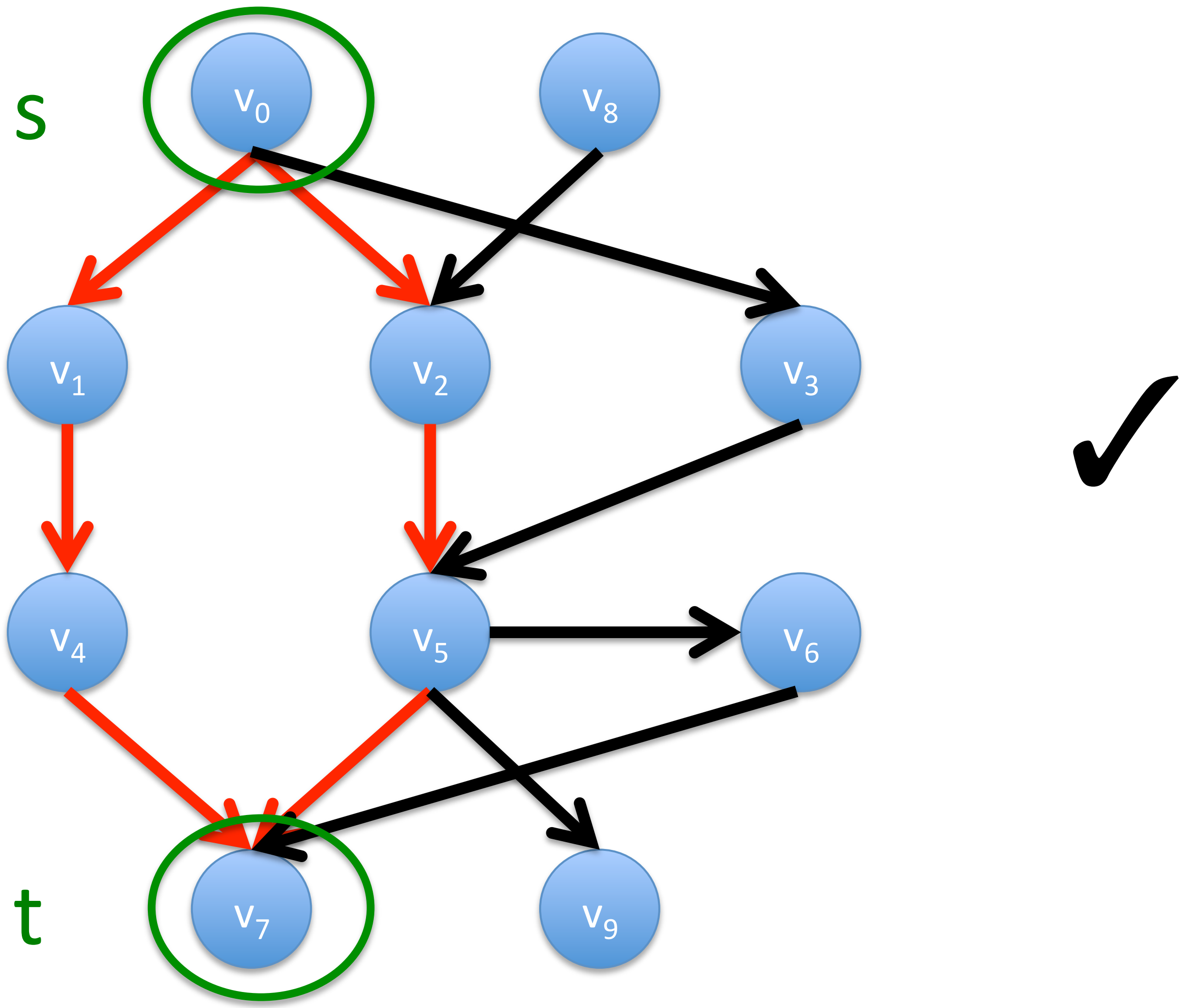
Set of S-T Paths with no common vertex

Internally Vertex Disjoint s-t Paths



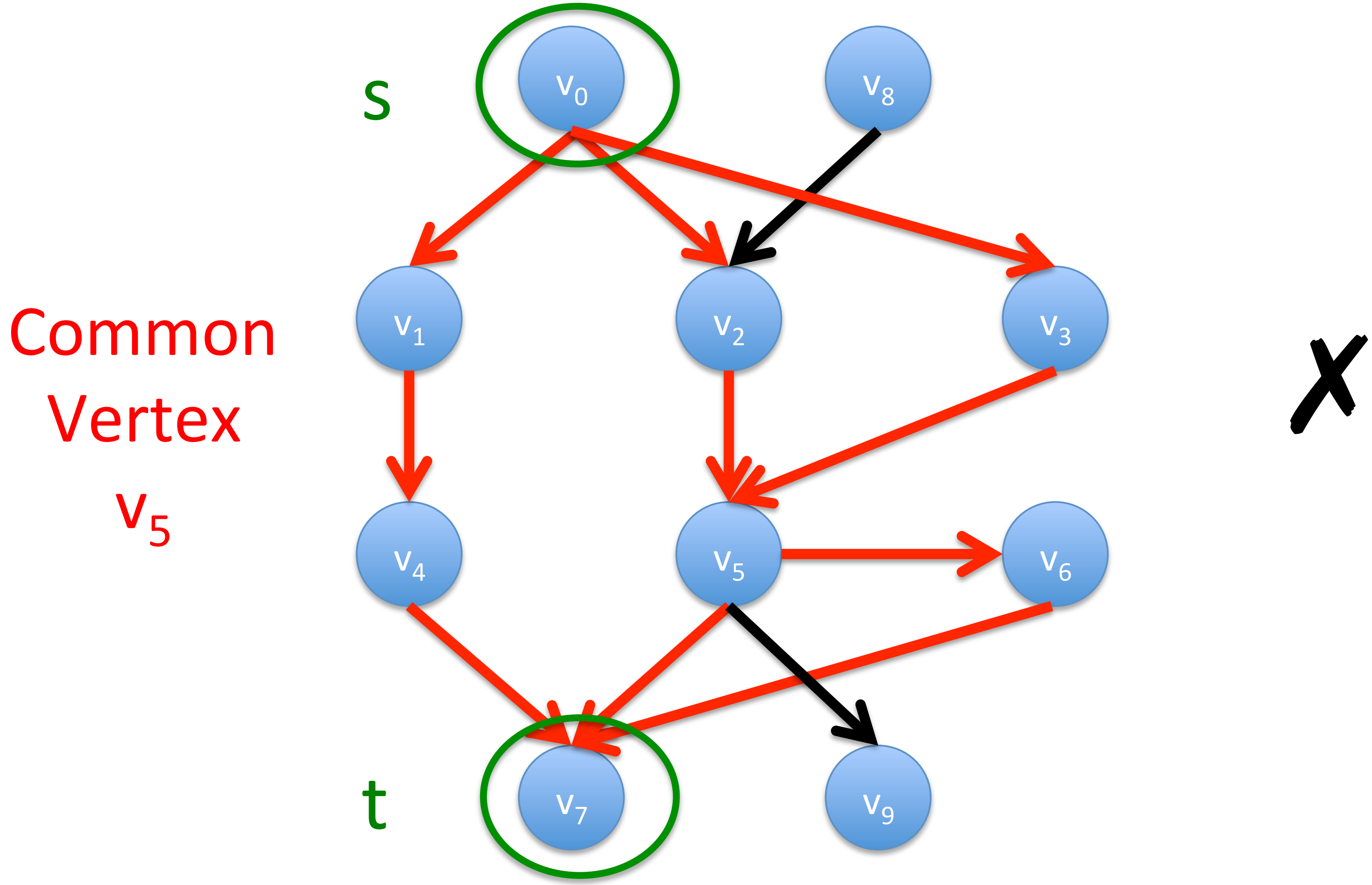
Set of s-t Paths with no common internal vertex

Internally Vertex Disjoint s-t Paths



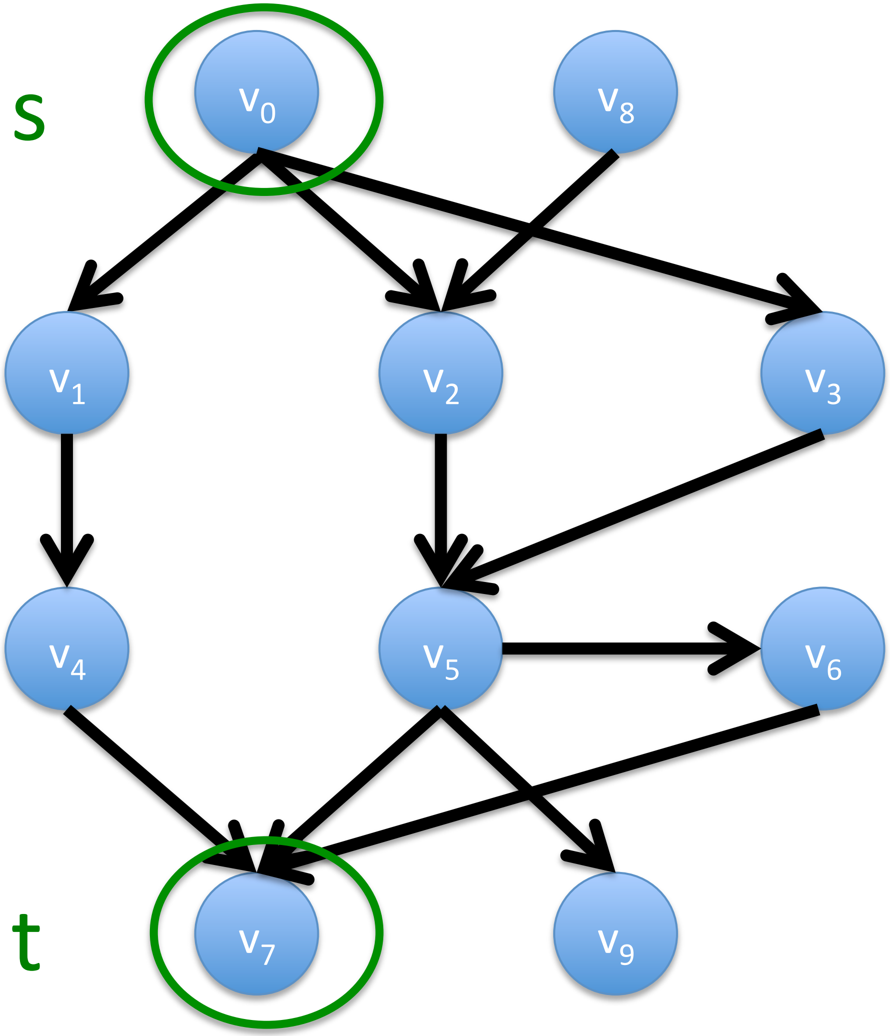
Set of s-t Paths with no common internal vertex

Internally Vertex Disjoint s-t Paths



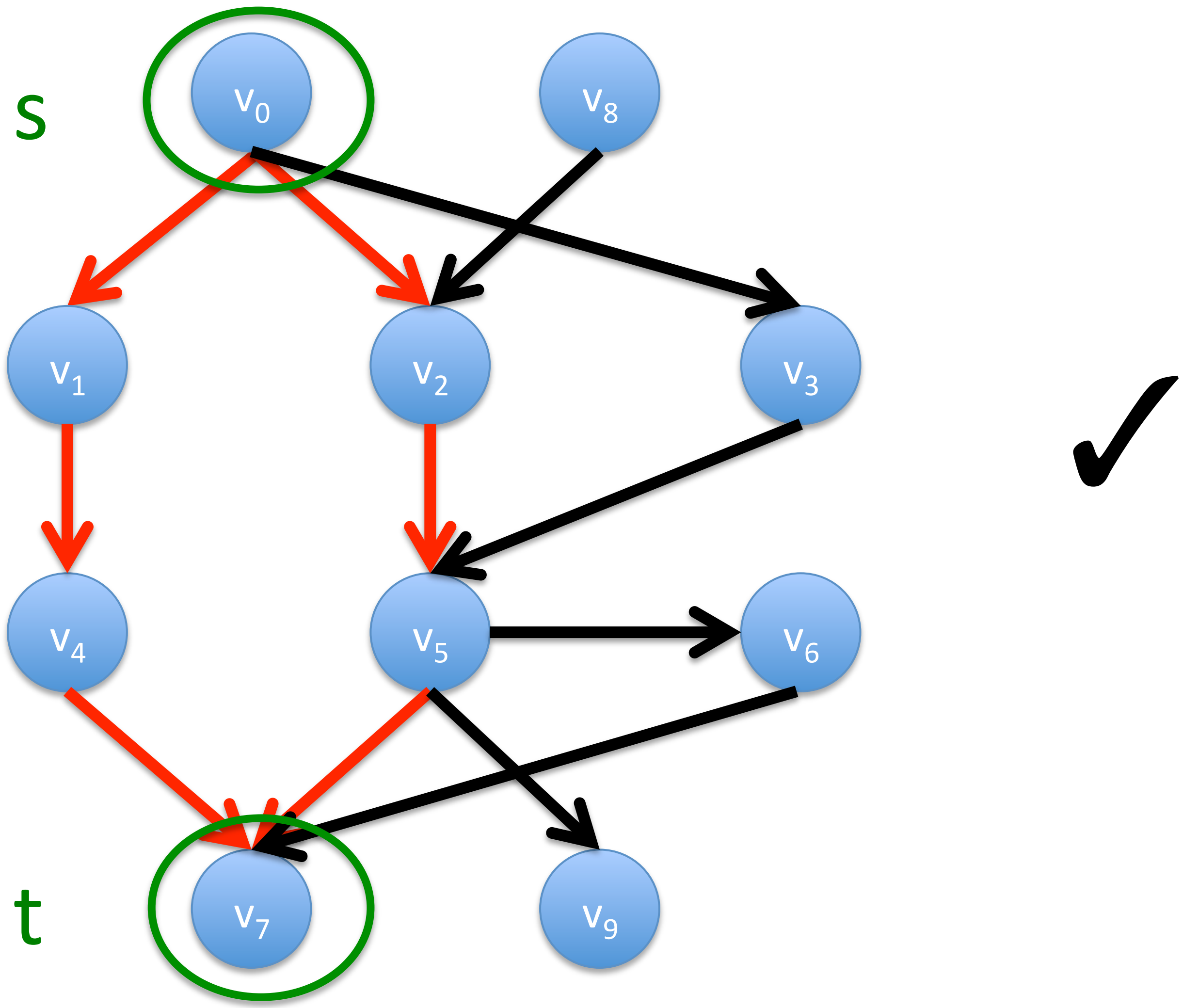
Set of s-t Paths with no common internal vertex

Arc Disjoint s-t Paths



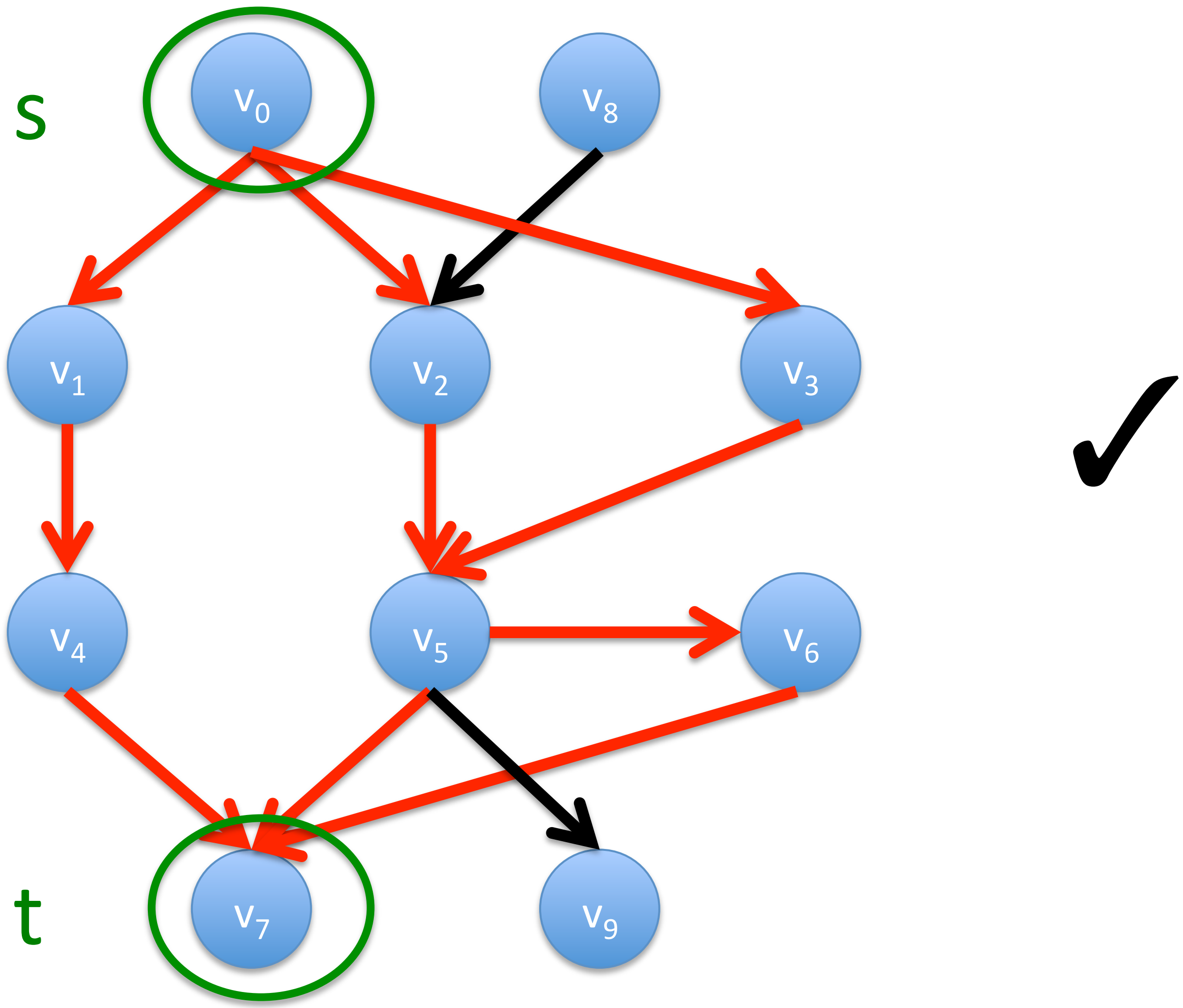
Set of s-t Paths with no common arcs

Arc Disjoint s-t Paths



Set of s-t Paths with no common arcs

Arc Disjoint s-t Paths



Set of s-t Paths with no common arcs

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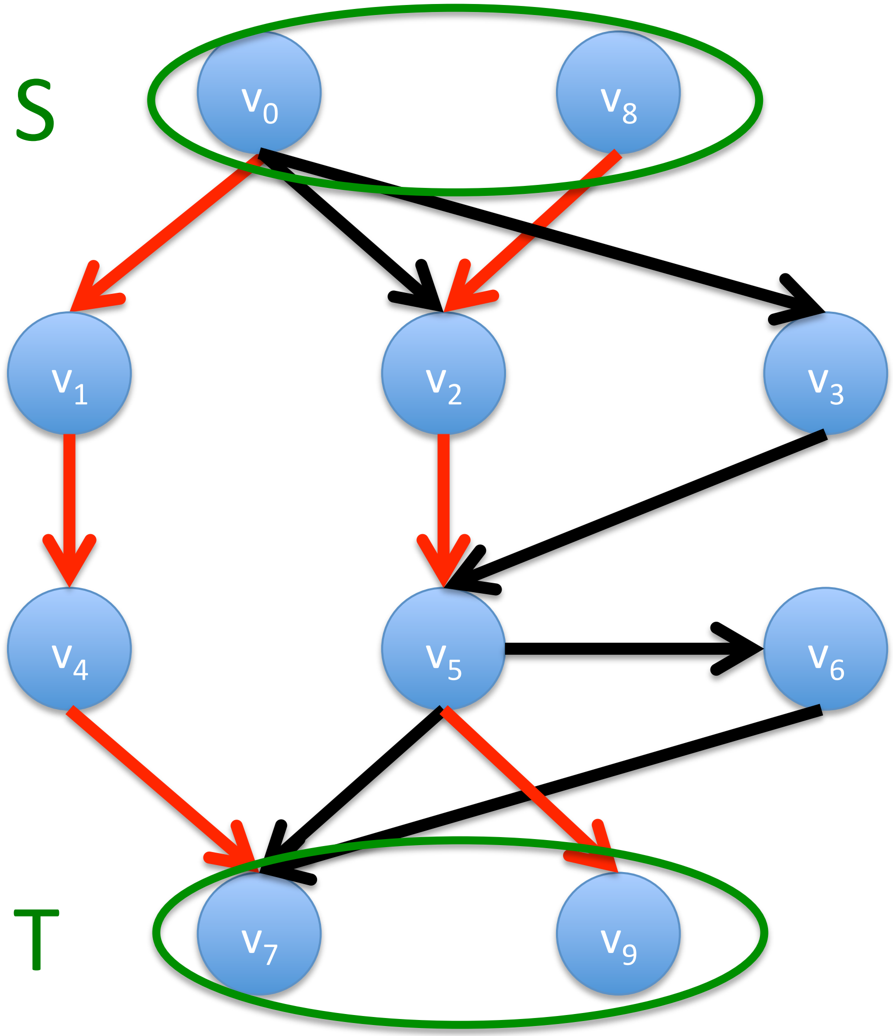
- Preliminaries
- Menger's Theorem for Disjoint Paths
  - **Vertex Disjoint S-T Paths**
  - Internally Vertex Disjoint s-t Paths
  - Arc Disjoint s-t Paths
- Path Packing

Vertex Disjoint S-T Paths

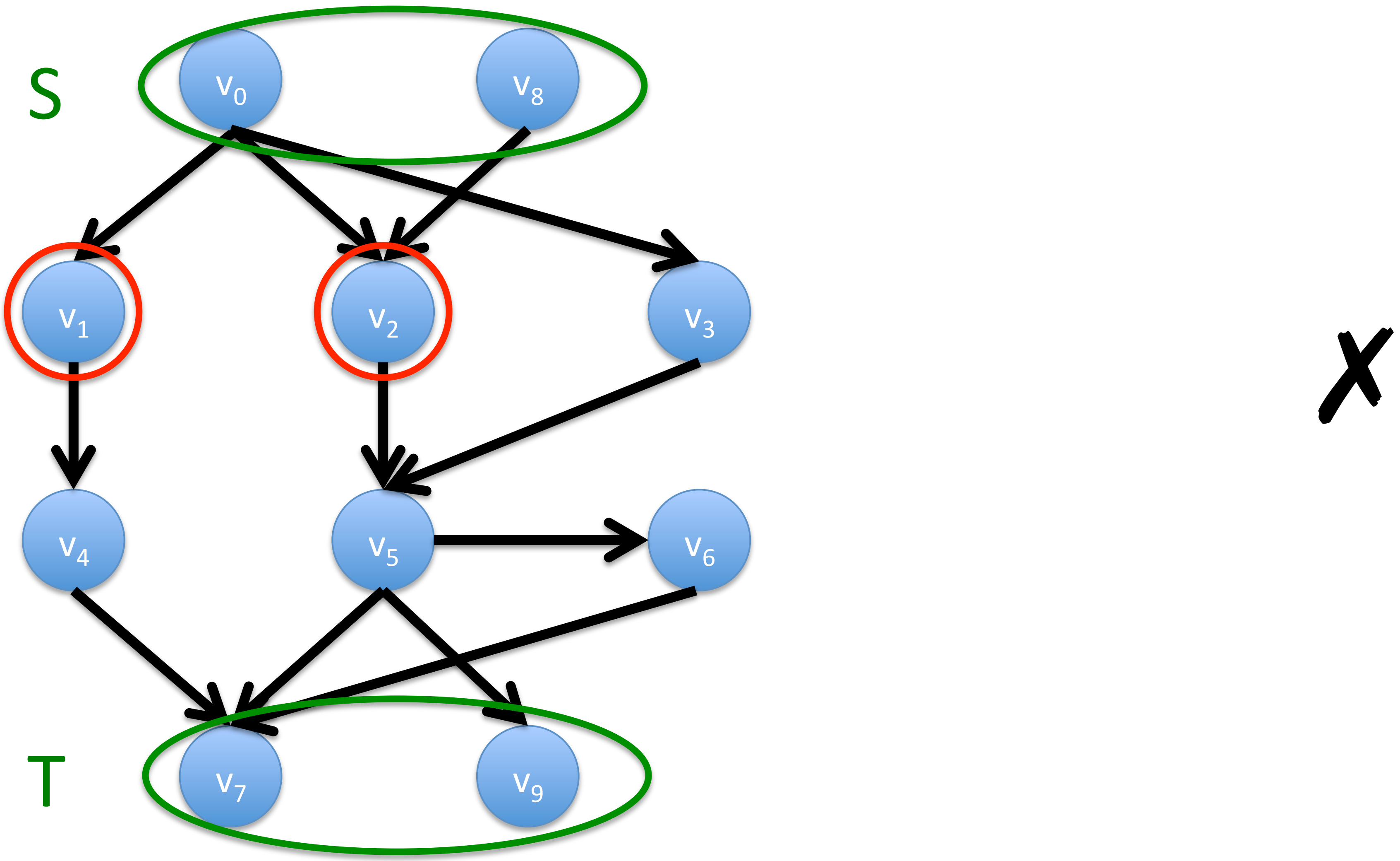


Maximum number of disjoint paths?

Minimum size of S-T disconnecting vertex set !!

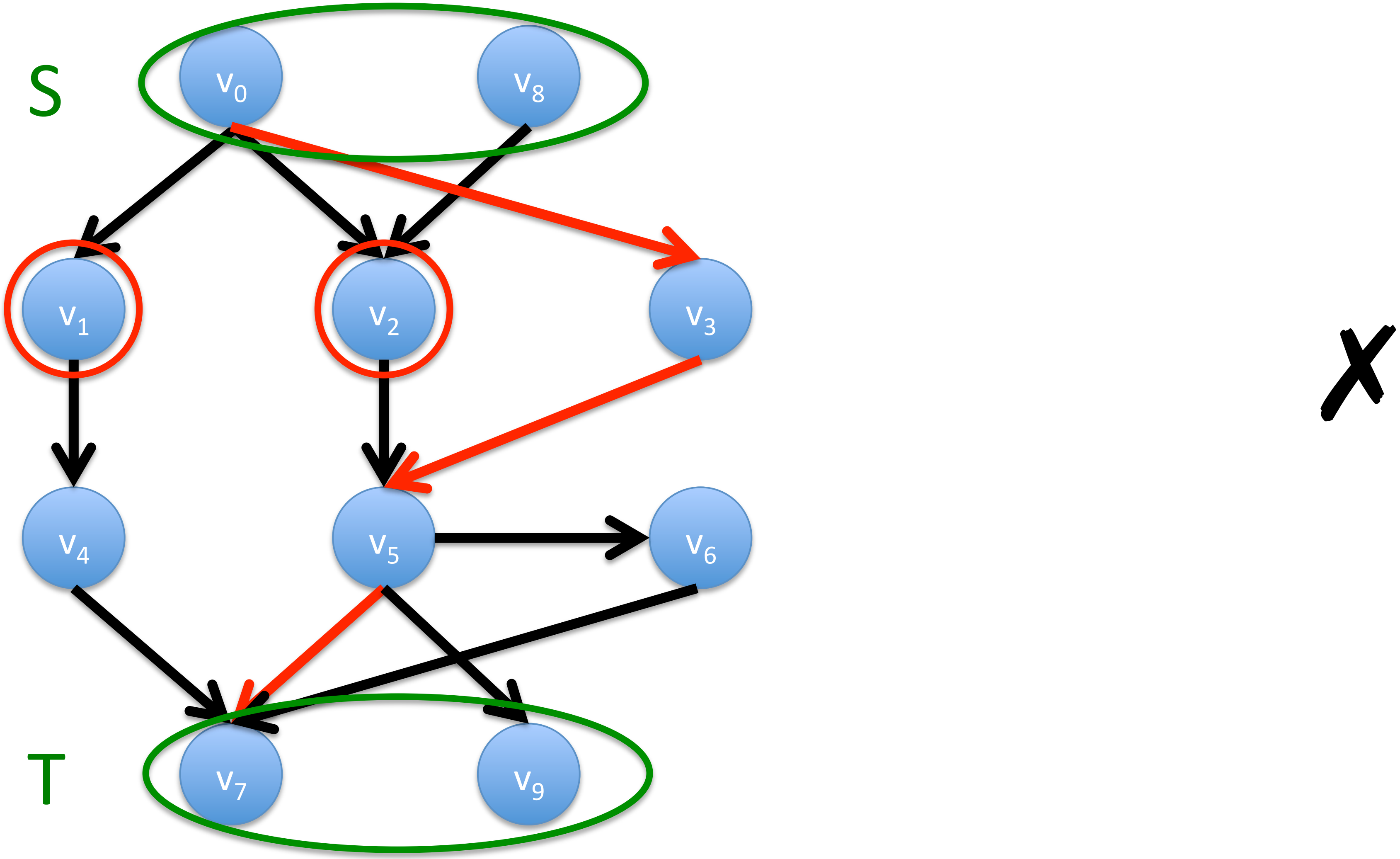
Set of S-T Paths with no common vertex

S-T Disconnecting Vertex Set



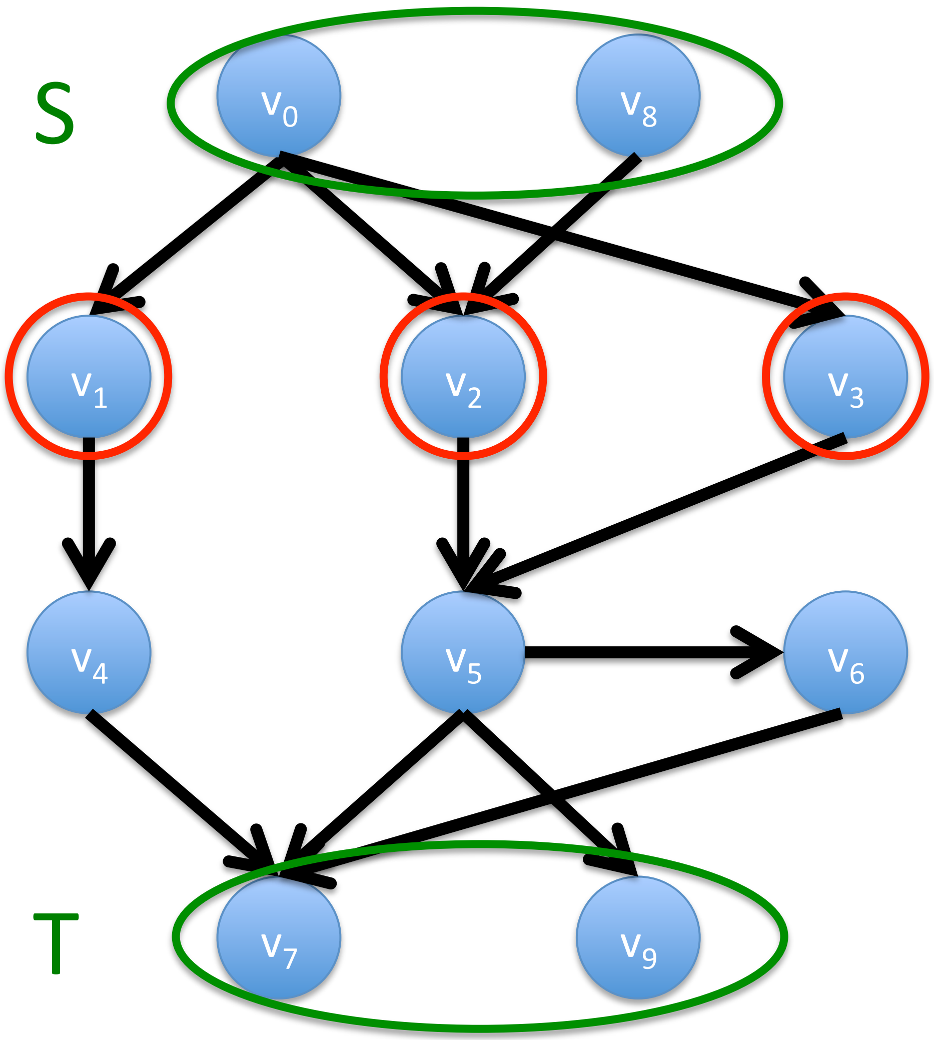
Subset U of V which intersects with all S-T Paths

S-T Disconnecting Vertex Set



Subset U of V which intersects with all S-T Paths

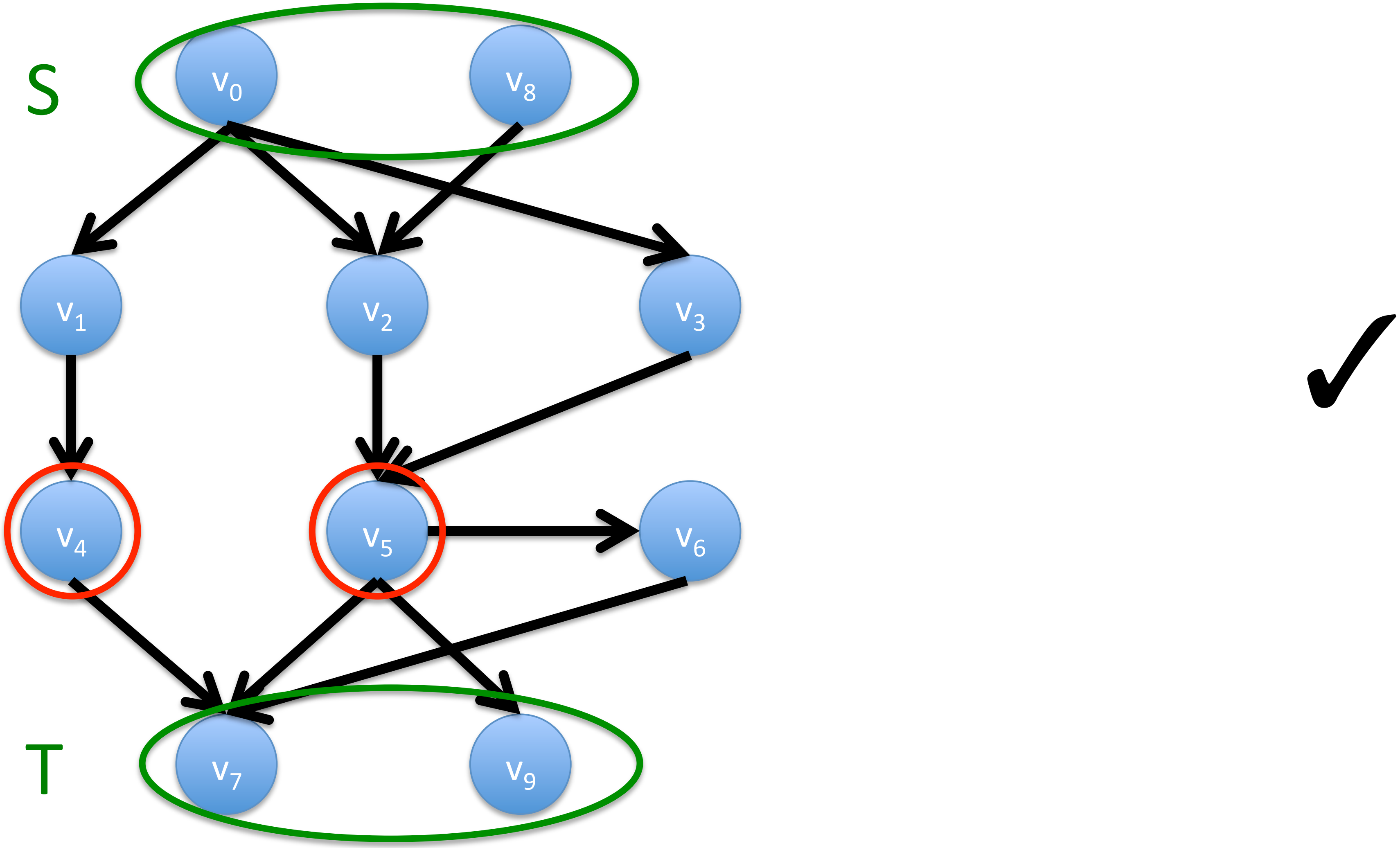
S-T Disconnecting Vertex Set



✓

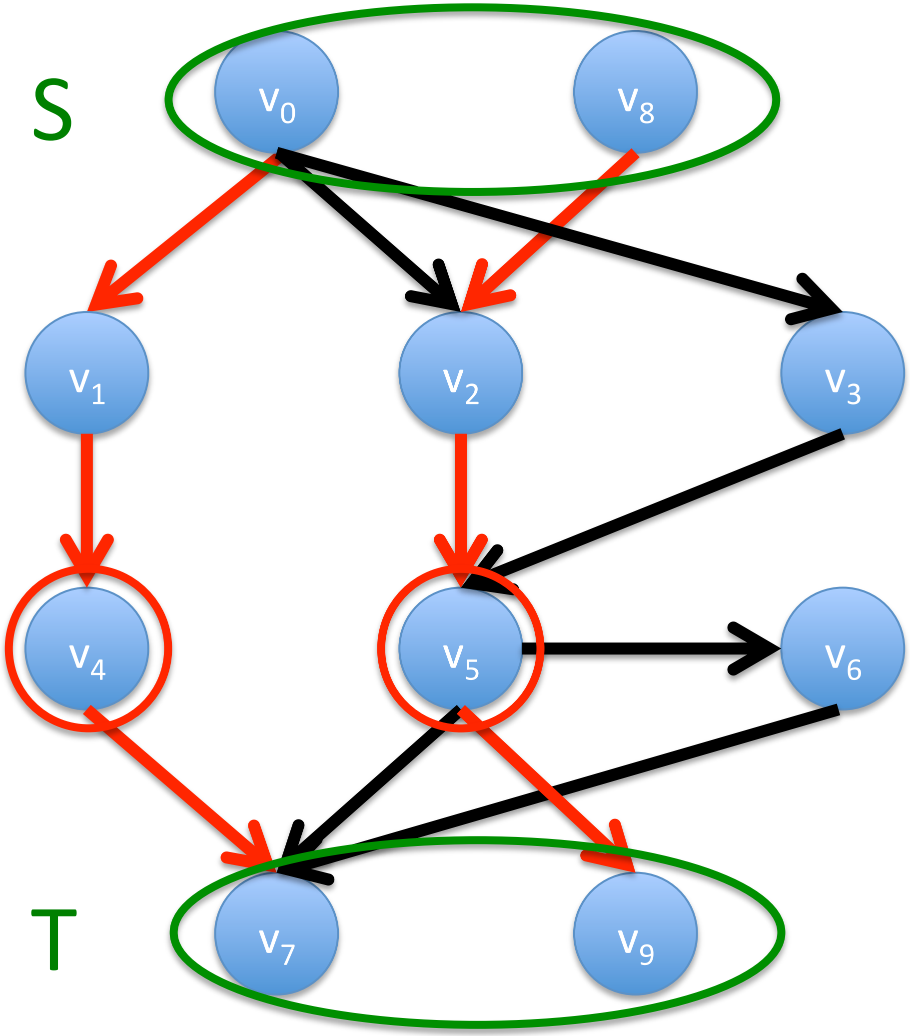
Subset U of V which intersects with all S-T Paths

S-T Disconnecting Vertex Set



Subset U of V which intersects with all S-T Paths

Connection

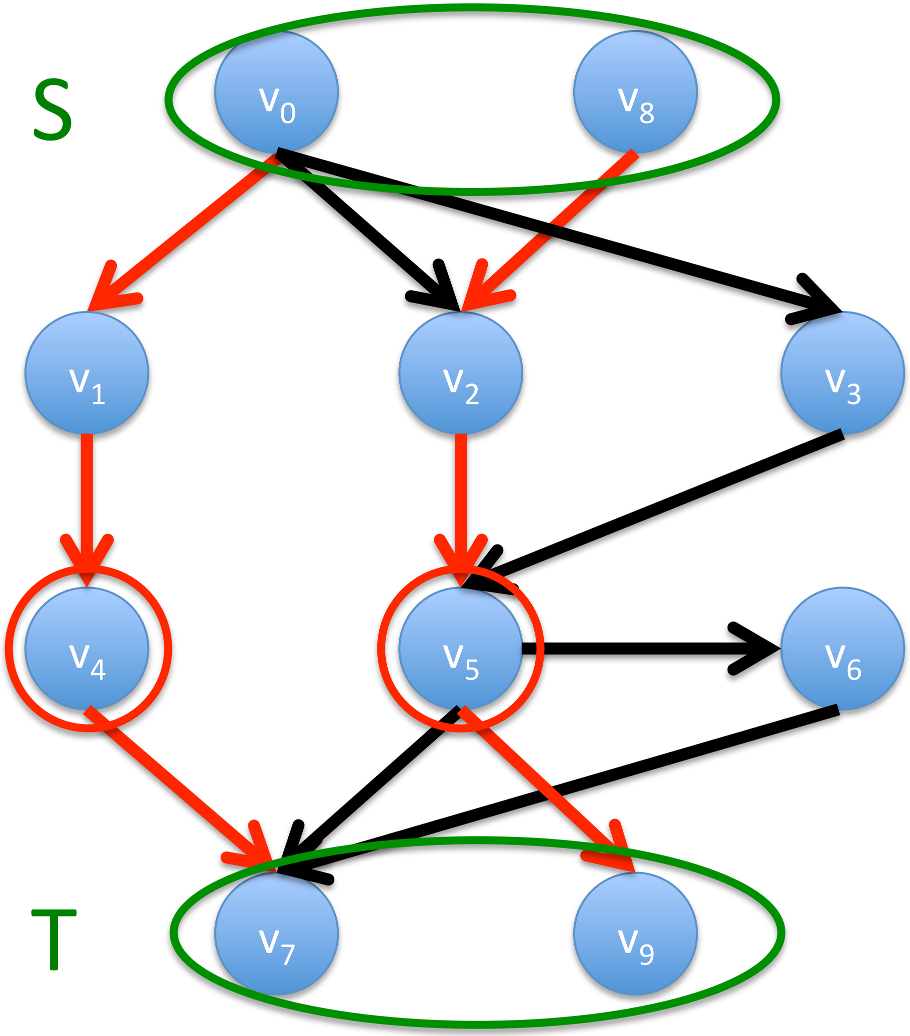


Maximum number of disjoint paths

≤

Minimum size of S-T disconnecting vertex set !!

Menger's Theorem



Maximum number of disjoint paths

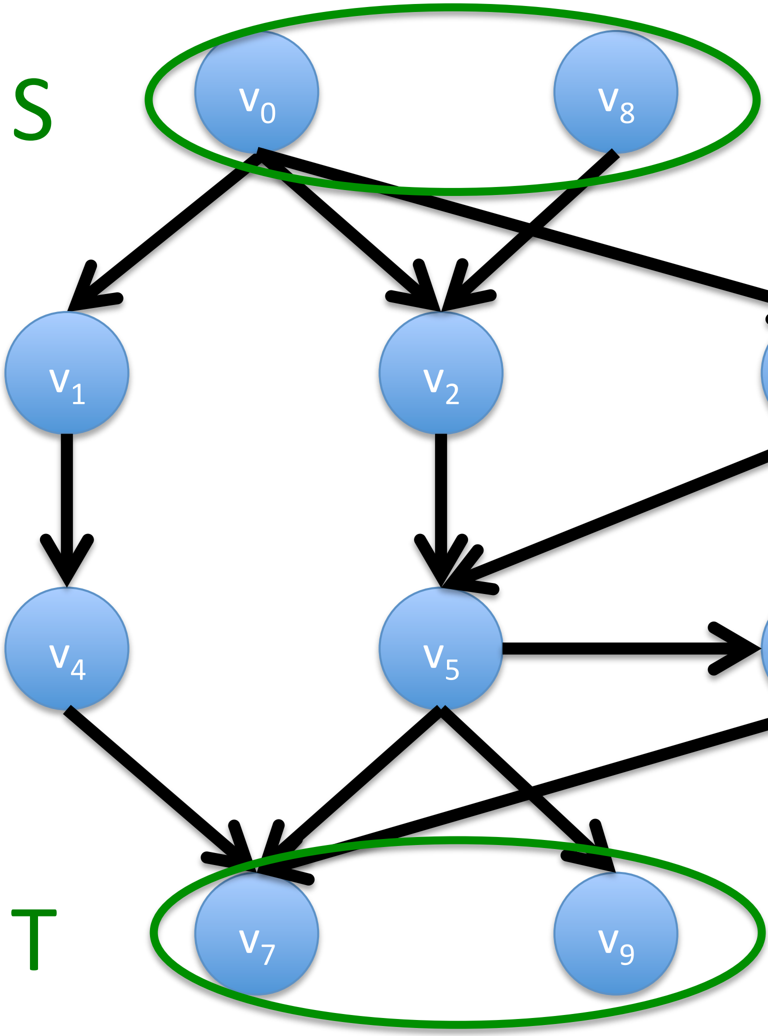
=

Minimum size of S-T disconnecting vertex set !!

Proof ?

Mathematical Induction on |A|

Menger's Theorem



True for |A| = 0

Assume it is
true for |A| < m

To be continued…
(Try working out the rest)

Mathematical Induction on |A|