Discrete Optimization

MA2827

Fondements de l'optimisation discrète

https://project.inria.fr/2015ma2827/

Material from P. Van Hentenryck's course

Outline

- Mixed Integer Program
- Examples
  - Warehouse location
  - Knapsack
- Branch and Bound
- Branch and Cut
- TSP

What is an integer program?

$$
\begin{array}{lllllllll}
\min & c_1x_1 & + & \dots & + & c_nx_n & & \\
\text{subject to} & & & & & & & \\
 & a_{11}x_1 & + & \dots & + & a_{1n}x_n & \leq & b_1 \\
 & \dots & & & & & & \\
 & a_{m1}x_1 & + & \dots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & \\
 & x_i \geq 0 & & & & & & \\
 & x_i \text{ integer} & & & & & &
\end{array}
$$

- n variables, m constraints
- Variables are non-negative and integers
- Integrality constraints

What is a mixed integer program?

$$
\begin{array}{llllllll}
\min & c_1 x_1 & + & \dots & + & c_n x_n & & \\
\text{subject to} & & & & & & & \\
 & a_{11} x_1 & + & \dots & + & a_{1n} x_n & \leq & b_1 \\
 & \dots & & & & & & \\
 & a_{m1} x_1 & + & \dots & + & a_{mn} x_n & \leq & b_m \\
 & & & & & & & \\
 & x_i \geq 0 & & & & & & \\
 & x_i \text{ integer} & (i \in I) & & & & &
\end{array}
$$

- n variables, m constraints
- Variables are non-negative and may be integers
- Integrality constraints

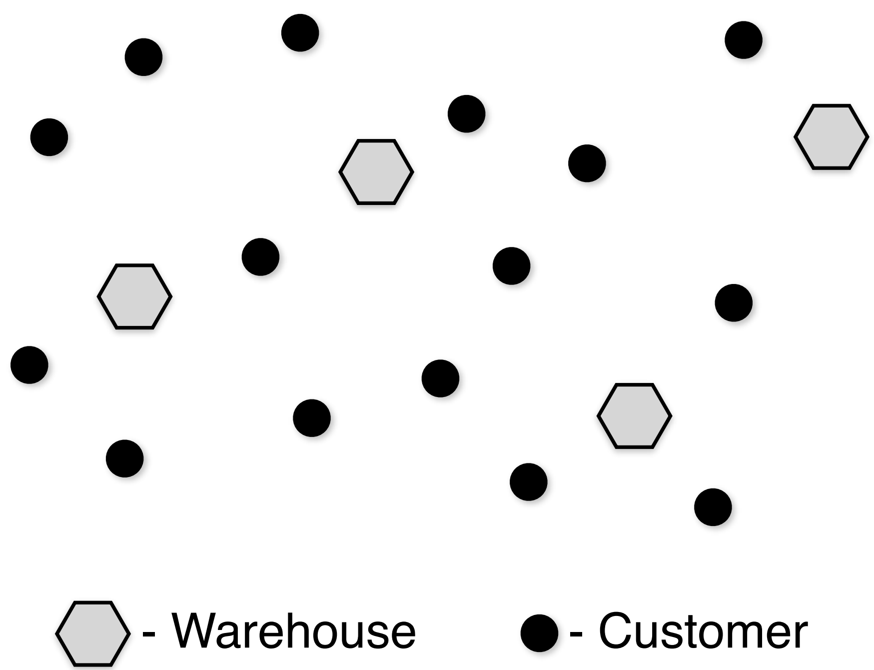
What is a mixed integer program?

- MIP vs LP
  - Integrality constraints
  - Minor? But, P vs NP!

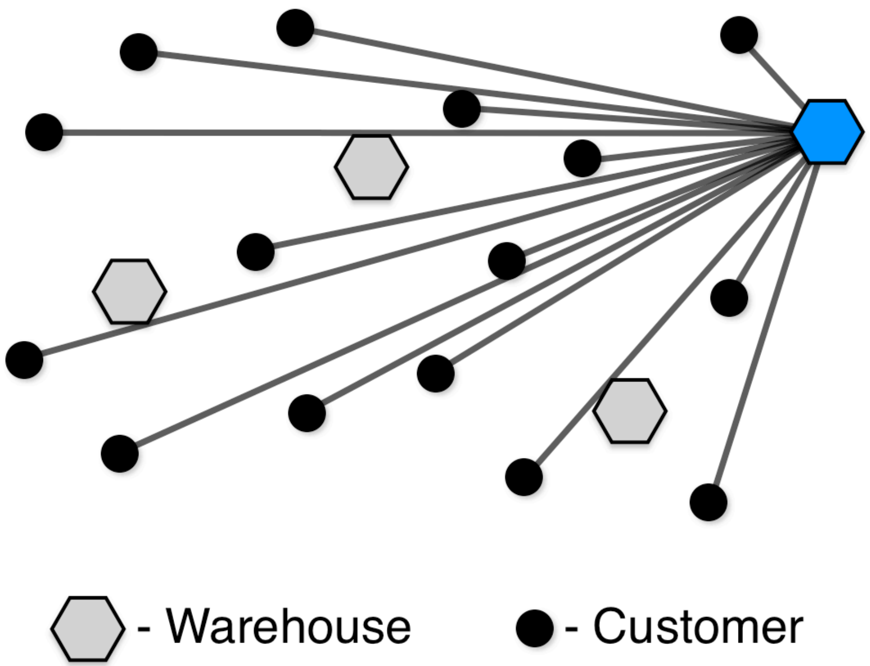
Knapsack problem

$$
\begin{aligned}
&\text{maximize} && \sum_{i \in I} v_i \; x_i \\
&\text{subject to} && \\
& && \sum_{i \in I} w_i x_i \leq K \\
& && x_i \in \{0, 1\} \quad (i \in I)
\end{aligned}
$$

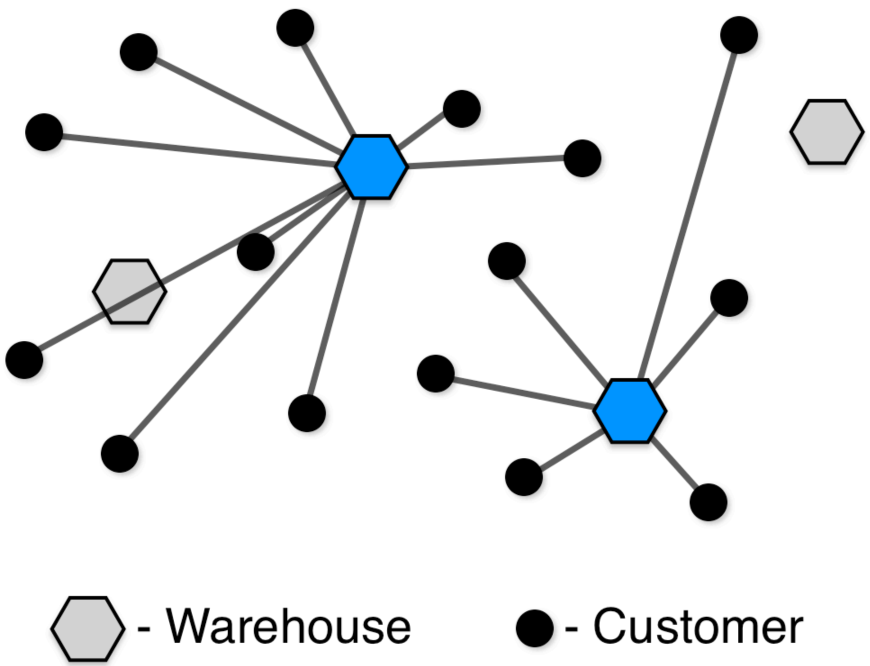
Warehouse location



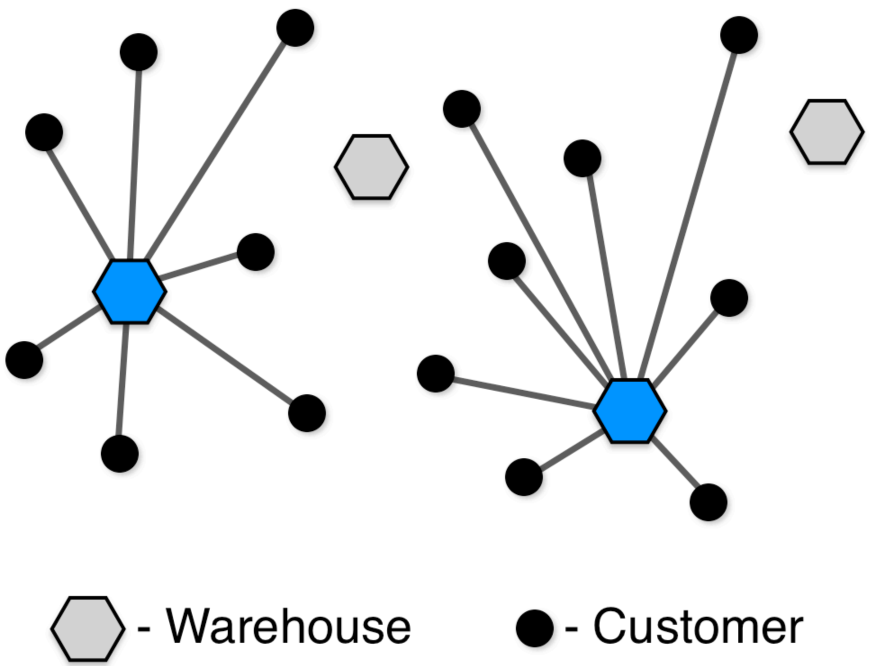
Warehouse location



Warehouse location



Warehouse location



Warehouse location

- Modeling it as a MIP
  - Decision variables?
  - Constraints?
  - Objective function?
- Decision variables
  - Will a warehouse w be opened?
    - i.e., Is $x_w = 1$?
  - Will a warehouse w serve customer c?
    - i.e., Is $y_{wc} = 1$?

Warehouse location

- Decision variables
  - Will a warehouse w be opened?
    - i.e., Is $x_w = 1$?
  - Will a warehouse w serve customer c?
    - i.e., Is $y_{wc} = 1$?
- Constraints
  - A warehouse can serve a customer only if it is open

$$y_{w,c} \leq x_w$$

  - A customer must be served by exactly one warehouse

$$\sum_{w \in W} y_{w,c} = 1$$

Warehouse location

- Decision variables
  - Will a warehouse w be opened?
    - i.e., Is $x_w = 1$?
  - Will a warehouse w serve customer c?
    - i.e., Is $y_{wc} = 1$?
- Objective function

$$\sum_{w \in W} \underbrace{c_w}_{\text{Fixed cost}} \; x_w + \sum_{w \in W, c \in C} \underbrace{t_{w,c}}_{\text{Transportation cost}} \; y_{w,c}$$

Warehouse location

$$
\begin{aligned}
\text{min} \quad & \sum_{w \in W} c_w \, x_w + \sum_{w \in W, c \in C} t_{w,c} \, y_{w,c} \\
\text{subject to} \quad & \\
& y_{w,c} \leq x_w && (w \in W, c \in C) \\
& \sum_{w \in W} y_{w,c} = 1 && (c \in C) \\
& x_w \in \{0,1\} && (w \in W) \\
& y_{w,c} \in \{0,1\} && (w \in W, c \in C)
\end{aligned}
$$

Warehouse location

- Decision variables
  - Will a warehouse w be opened?
    - i.e., Is $x_w = 1$?
  - Will a warehouse w serve customer c?
    - i.e., Is $y_{wc} = 1$?
- Why not
  - y_c denotes the warehouse serving customer c?

Mixed integer programming

- Typically 0/1 variables in a MIP
- Easy to transform them to linear constraints
- Other possible models to consider though
  - Decision variables
  - Constraints
  - Objective function

Mixed integer programming

- How to solve?
  - Active area of research
- Branch and bound
  - Bounding: finding an optimistic relaxation
  - Branching: splitting the problem into subproblems
- MIP gives rise to a natural relaxation
  - Linear relaxation
  - i.e., remove integrality constraints

Branch and bound

- Solve the linear relaxation
- If the linear relaxation is
  - worse than the best solution found so far, prune this node (because the associated problem is suboptimal)
  - integral, we have found a feasible solution (update the best feasible solution if necessary)
- Otherwise,
  - find an integer variable x with fractional value f, create two subproblems $\mathrm{x} \leq \lfloor f \rfloor, \ \mathrm{x} \geq \lceil f \rceil$ and repeat

Branch and bound

- Focus on the objective
  - Relaxation gives an optimistic bound
- Pruning based on sub-optimality
  - Prune provably suboptimal nodes
- Relax feasibility
  - Relax the integrality constraints
- Global view of relaxation
  - Consider all problem constraints

Knapsack problem

- Revisit this problem

$$
\begin{aligned}
&\text{maximize} \quad && \sum_{i \in I} v_i \; x_i \\
&\text{subject to} && \\
& && \sum_{i \in I} w_i x_i \leq K \\
& && x_i \in \{0, 1\} \quad (i \in I)
\end{aligned}
$$

Knapsack problem

- Linear relaxation

$$
\begin{aligned}
&\text{maximize} && \sum_{i \in I} v_i \; x_i \\
&\text{subject to} && \\
& && \sum_{i \in I} w_i x_i \leq K \\
& && 0 \leq x_i \leq 1 \quad (i \in I)
\end{aligned}
$$

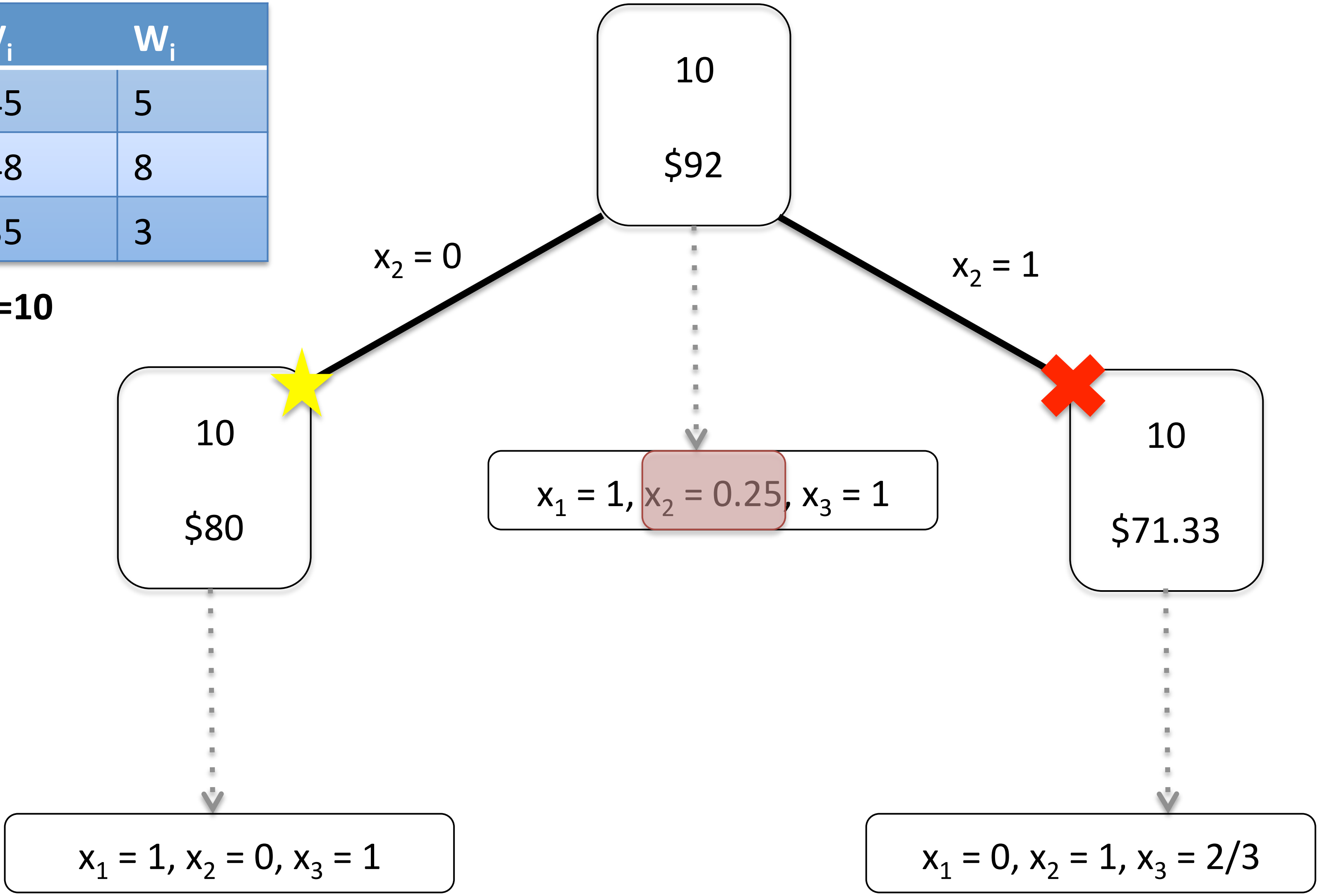
Knapsack problem

- Branch and bound for this problem
- Linear relaxation
  - Related to the greedy solution
- Branching
  - Variable with a fractional value
  - i.e., most valuable item (x_i) that cannot fit entirely
- Bounding (subproblems)
  - Do not take the item ($x_i = 0$)
  - Take the item ($x_i = 0$)

Knapsack problem

i	V_i	W_i
1	45	5
2	48	8
3	35	3

K=10



Branch and bound

- When is it effective?
  - Necessary condition: the linear relaxation is strong
  - Is it sufficient? (homework)

- What is a good MIP model?
  - One with a good linear relaxation

- Which variable to branch on?
  - One with the most fractional value

Outline

- Mixed Integer Program
- Examples
  - Warehouse location
  - Knapsack
- Branch and Bound
- Branch and Cut
- TSP

Cover cuts

- Constraints: $\sum_{j=1}^{n} a_j x_j \leq b$
- Find facets of these constraints?
- Cover: a set $C \subseteq N = \{1, \ldots, n\}$ is a cover if

$$\sum_{j \in C} a_j > b$$

  – A cover is minimal if $C \setminus \{j\}$ is not a cover for any j

Cover cuts

- Constraints: $\sum_{j=1}^{n} a_j x_j \leq b$

- Find facets of these constraints?
- If $C \subseteq N = \{1, \ldots, n\}$ is a cover, then

$$\sum_{j \in C} x_j \leq |C| - 1$$

is a valid inequality

Cover cuts

- Example

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

- Minimal cover inequalities

$$x_1 + x_2 + x_3 \leq 2$$

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

Stronger cover cuts

- If $C \subseteq N = \{1, \ldots, n\}$ is a cover, then

$$\sum_{j \in E(C)} x_j \leq |C| - 1$$

is a valid inequality, where

$$E(C) = C \cup \{j \mid \forall i \in C : a_j \geq a_i\}$$

Stronger cover Cuts

- Example

$$11x_1 + 6x_2 + 6x_3 + 5x_4 + 5x_5 + 4x_6 + x_7 \leq 19$$

- Cover inequality

$$x_3 + x_4 + x_5 + x_6 \leq 3$$

- Stronger cover inequality

$$x_1 + \ldots + x_6 \leq 3$$

Branch and cut

- Overall idea
  1. Formulate MIP for the application (e.g., TSP)
  2. Solve the linear relaxation; if the solution is integral, terminate
  3. Find a polyhedral cut which prunes the linear relaxation and is a facet, if possible
     - If found, repeat from step 2
     - Otherwise, do a branch

Separation problem

- Consider a solution x* to the linear relaxation (may already be enhanced by cuts)

- Goal: To know whether there exists a cover cut for x*

Separation for cover cuts

- The cover inequality $\sum_{j \in C} x_j \leq |C| - 1$

  can be rewritten as $\sum_{j \in C} (1 - x_j) \geq 1$

- Does there exist a $C \subseteq N$ that satisfies

$$\sum_{j \in C} (1 - x_j^*) < 1$$

$$\sum_{j \in C} a_j > b$$

Separation for cover cuts

- This is equivalent to

$$\begin{aligned}
\min \quad & \sum_{j \in N} (1 - x_j^*) z_j \\
s.t. \quad & \\
& \sum_{j \in N} a_j z_j > b \\
& z_j \in \{0, 1\}
\end{aligned}$$

- If the minimum value is less than 1, a cut exists. All the variables assigned to 1 are a cover.

Separation for cover cuts

- Example

$$45x_1 + 46x_2 + 79x_3 + 54x_4 + 53x_5 + 125x_6 \leq 178$$

Fractional solution: $x^* = (0, 0, \frac{3}{4}, \frac{1}{2}, 1, 0)$

- Separation problem is given by

$$\begin{array}{lllllllll} min & z_1 & +z_2 & +\frac{1}{4}z_3 & +\frac{1}{2}z_4 & & +z_6 & \\ s.t & & & & & & & \\ & 45z_1 & +46z_2 & +79z_3 & +54z_4 & +53z_5 & +125z_6 & > 178 \end{array}$$

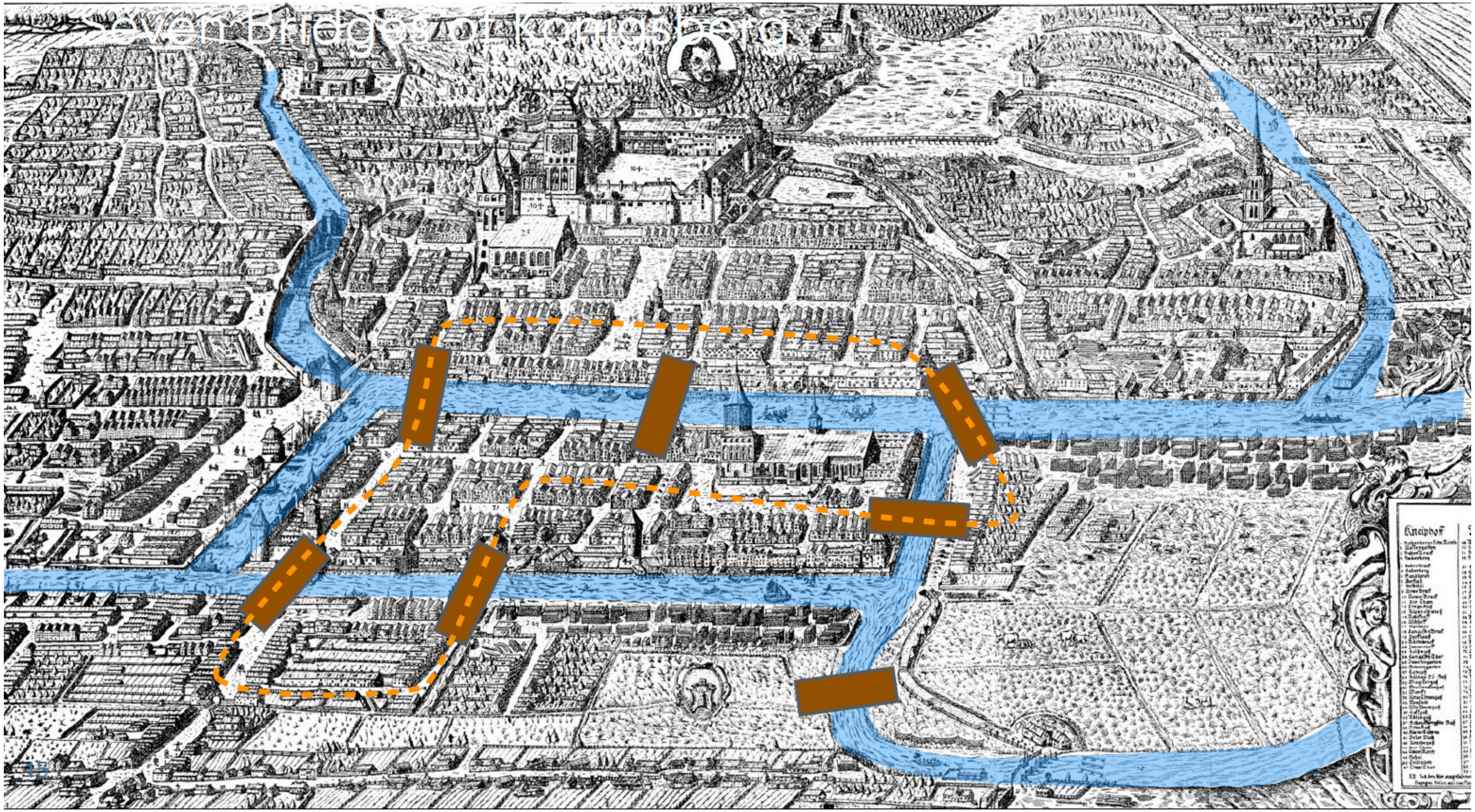
Separation for cover cuts

- Problem

$$\begin{aligned}
\min \quad & \sum_{j \in N} (1 - x_j^*) z_j \\
s.t. \quad & \\
& \sum_{j \in N} a_j z_j > b \\
& z_j \in \{0, 1\}
\end{aligned}$$

- Related to anything you know already?
  - Replace z_j by $(1 - y_j)$

TSP, but first seven bridges of Könisberg

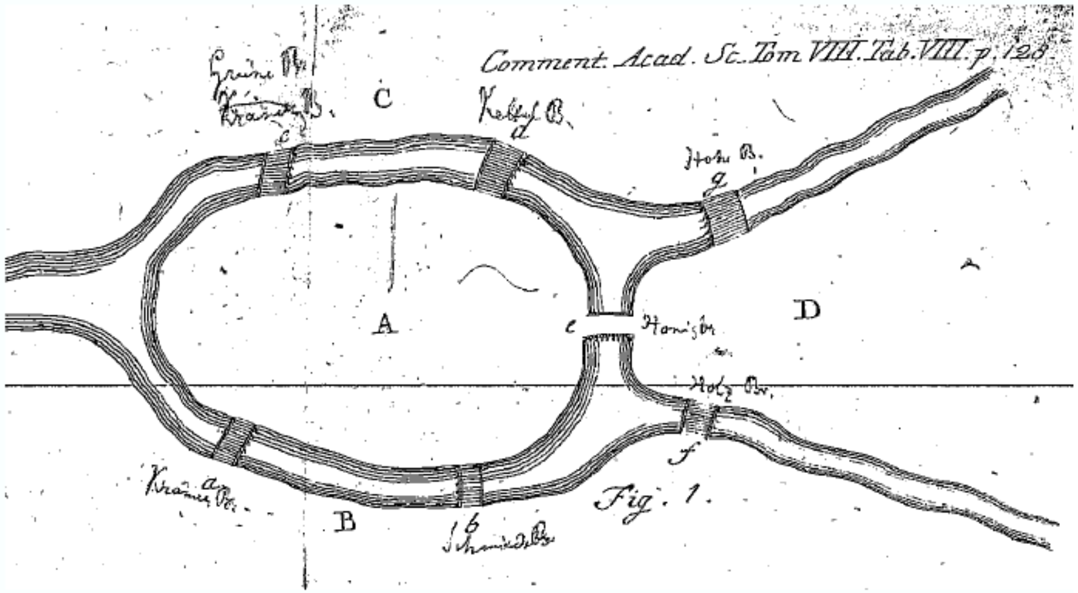


https://commons.wikimedia.org/wiki/File:Koenigsberg,_Map_by_Bering_1613.jpg#globalusage

Seven bridges of Könisberg

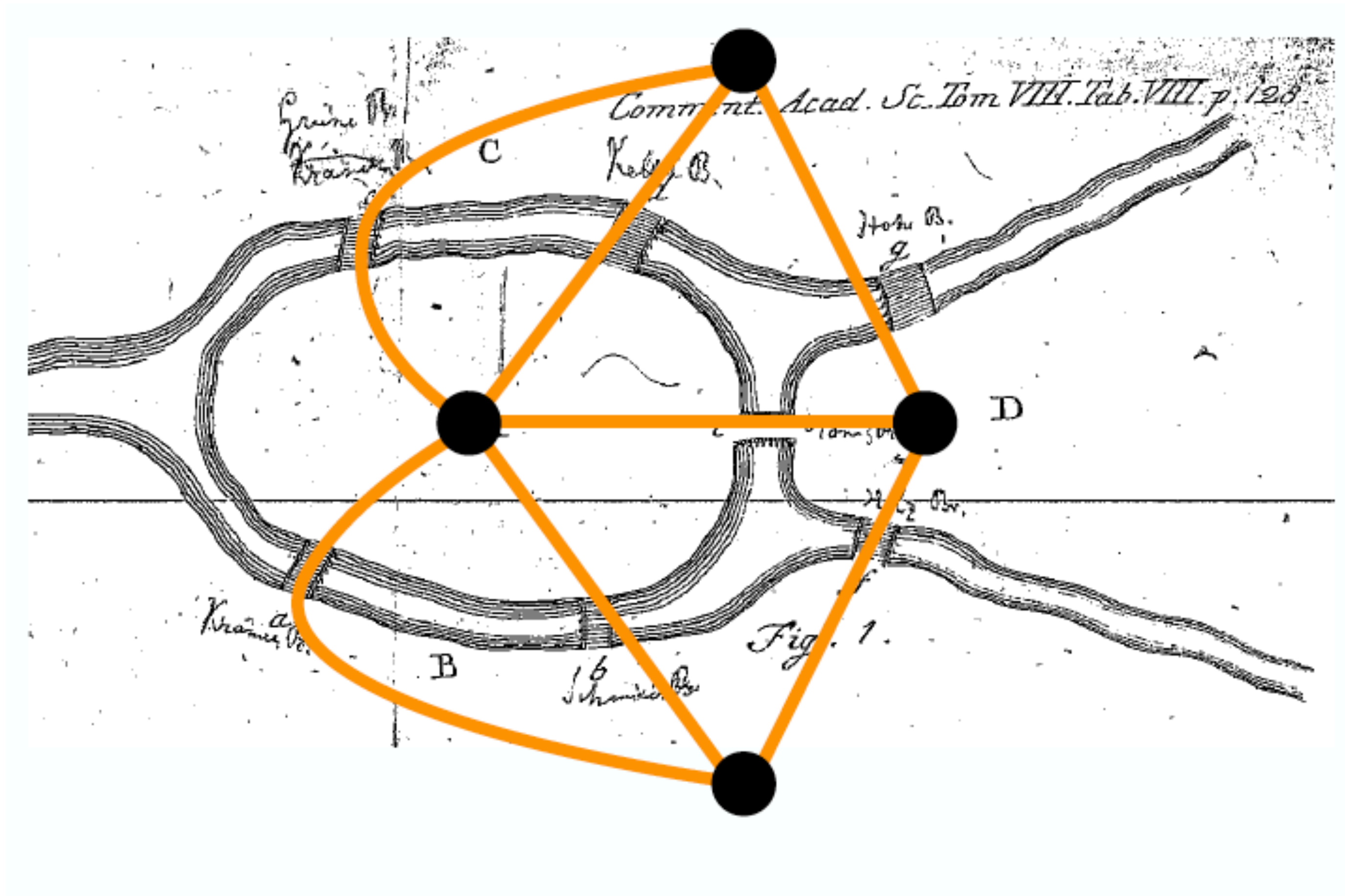
https://commons.wikimedia.org/wiki/File:Leonhard_Euler_2.jpg

Leonhard Euler
By Jakob Emanuel Handmann

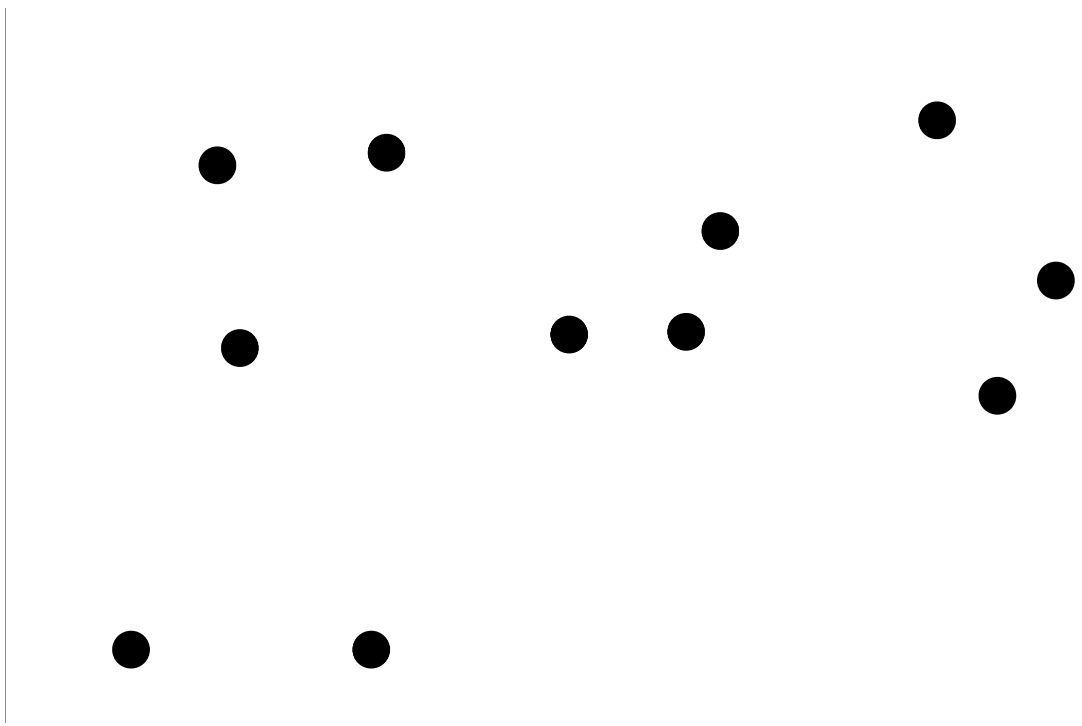


Bridges of Königsberg, 1741

Seven bridges of Könisberg



Traveling salesman problem



Traveling salesman problem

- MIP for TSP
  - Decision variables, constraints, objectives?
- Decision variables
  - Is an edge part of the tour or not?
- Constraints
  - Degree constraints: Each node has exactly two edges selected

MIP for TSP

- Decision variables
  - x_e is 1 if edge e is selected
- Notation
  - V: set of vertices
  - E: set of edges
  - δ(v): edges adjacent to vertex v
  - δ(S): edges with exactly one vertex in S (subset of V)
  - γ(S): edges with both vertices in S
  - $x_{\{e_1,\ldots,e_n\}} = x_{e_1} + \ldots + x_{e_n}$

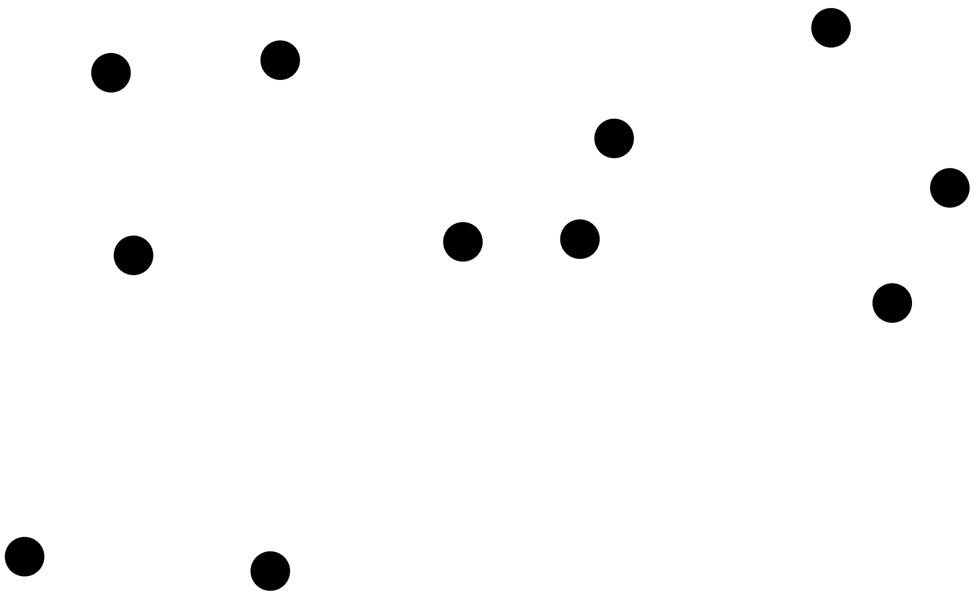
MIP for TSP

flow conservation

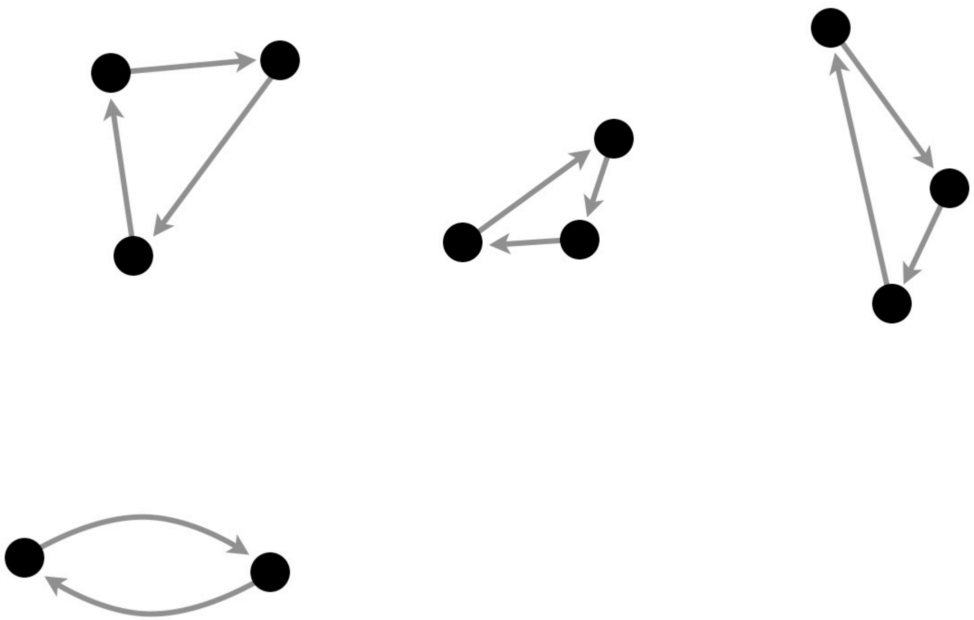
$$
\begin{array}{lll}
\min & \displaystyle\sum_{e \in E} c_e x_e & \\
\text{subject to} & & \\
& x_{\delta(v)} = 2 & v \in V \\
& x_e \in \{0,1\} & e \in E
\end{array}
$$

minimize cost

MIP for TSP

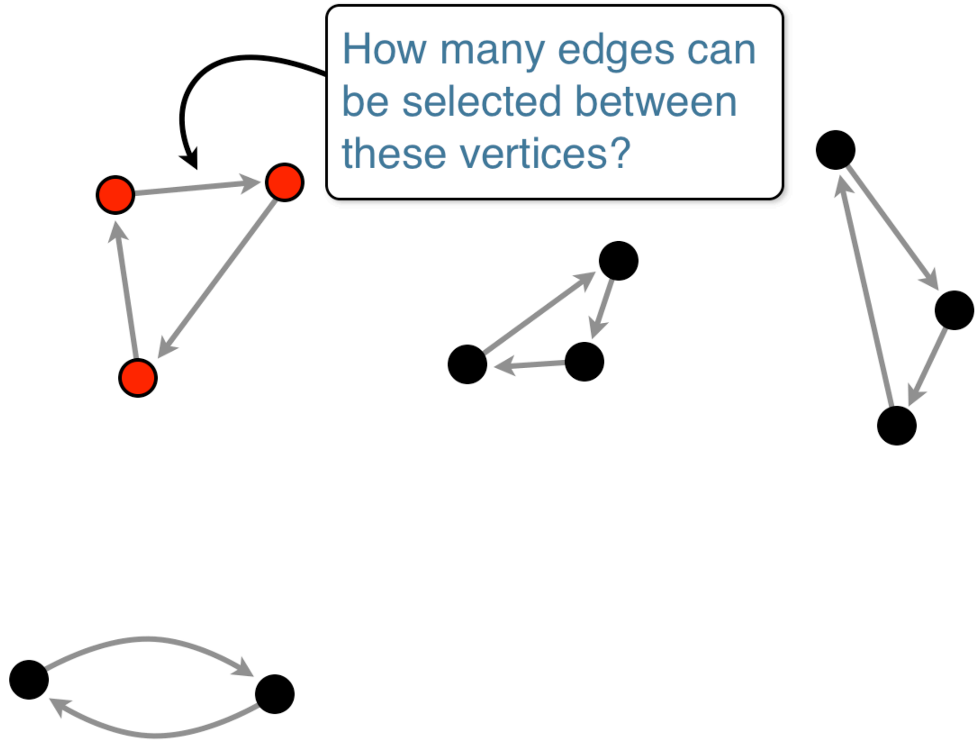


MIP for TSP



MIP for TSP

- Eliminate these subtours



Subtour elimination

$$
\begin{array}{lll}
\min & \sum_{e \in E} c_e x_e & \\
\text{subject to} & & \\
& x_{\delta(v)} = 2 & v \in V \\
& \boxed{x_{\gamma(S)} \leq |S| - 1 \quad S \subset V} & \\
& x_e \in \{0, 1\} & e \in E
\end{array}
$$

- Great, but too many (exponential no.) of them
- Branch and cut
  - Generate them on demand

Subtour elimination

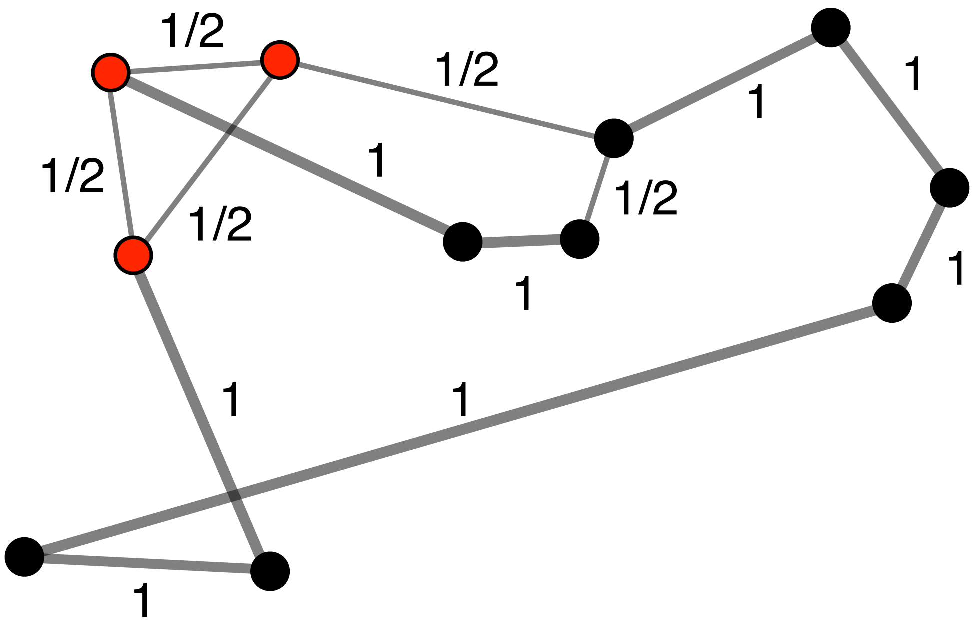
$$
\begin{array}{lll}
\min & \sum_{e \in E} c_e x_e & \\
\text{subject to} & & \\
& x_{\delta(v)} = 2 & v \in V \\
& \boxed{x_{\delta(S)} \geq 2 \qquad S \subset V} & \\
& x_e \in \{0, 1\} & e \in E
\end{array}
$$

- How to separate subtour constraints?

Separation of subtour constraints

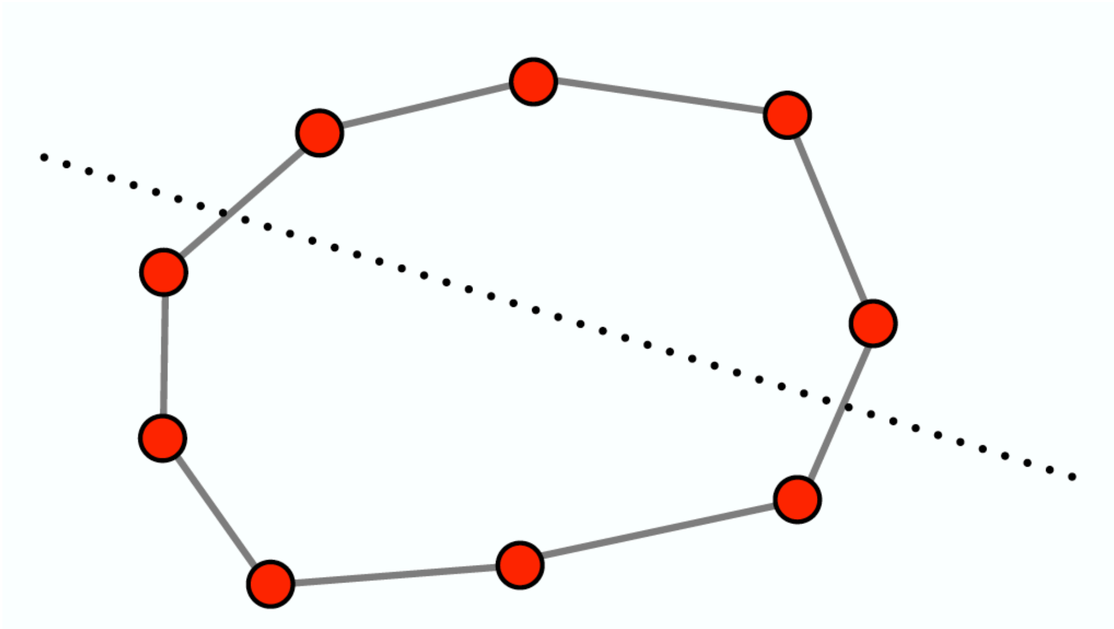
- Build a graph G* = (V, E) where
  - weight of an edge e, w(e) = x_e*
- Finding a separation is equivalent to finding
  - A minimum cut in G*
  - If the cost of the cut is less than 2, then we have isolated a subtour constraint violated by the linear relaxation
  - Recall: Finding the cut takes polynomial time

MIP for TSP



MIP for TSP

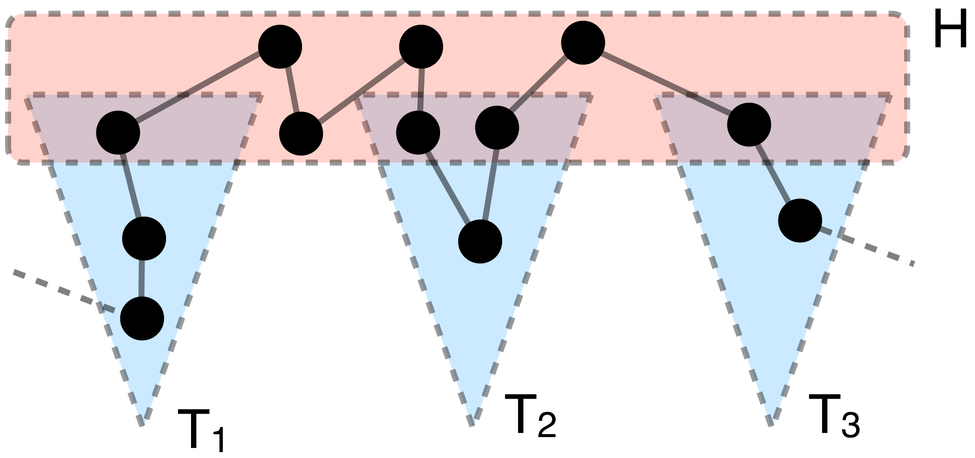
- Comb constraints



- Number of edges crossed?

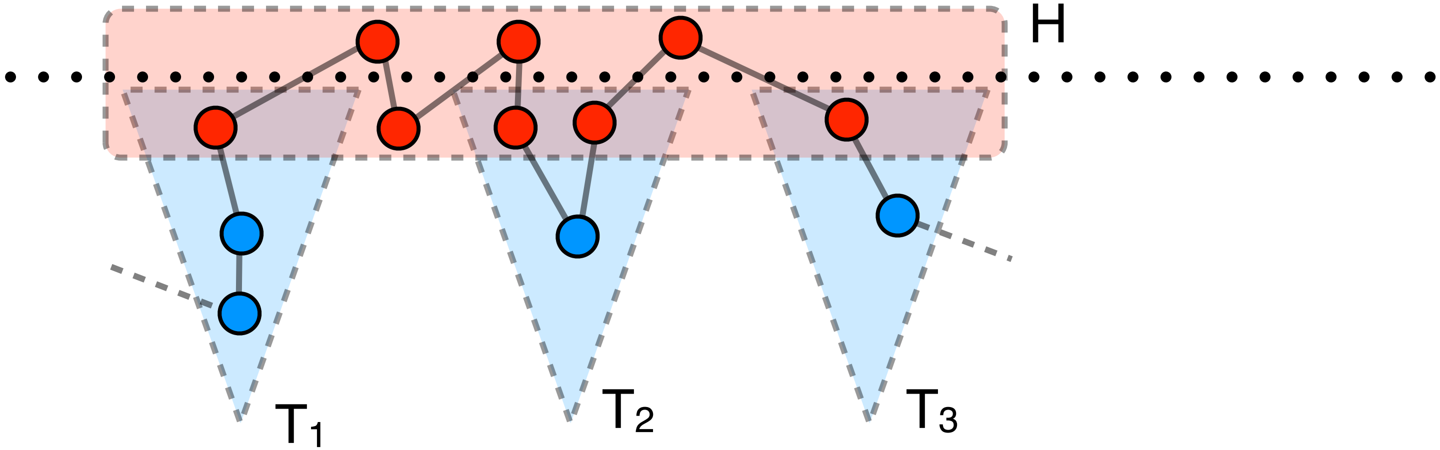
MIP for TSP

- Comb constraints



MIP for TSP

- Comb constraints



▸ comb inequalities

$$x_{\gamma(H)} + \sum_{i=1}^{t} x_{\gamma(T_i)} \leq |H| + \sum_{i=1}^{k} |T_i| - \lceil \frac{3k}{2} \rceil$$

Branch and cut for TSP

- On the TSPLIB benchmark
  - Subtour elimination: 2% of optimality gap
  - Subtour + comb cuts: 0.5% of optimality gap

- Other constraints are needed for large instances

Outline

- Mixed Integer Program
- Examples
  - Warehouse location
  - Knapsack
- Branch and Bound
- Branch and Cut
- TSP
- Bonus! (Duality)

Duality

https://en.wikipedia.org/wiki/File:German_postcard_from_1888.png

Duality

$$\begin{array}{ll} \text{min} & c\ x \\ \text{subject to} & \\ & Ax \geq b \\ & x_j \geq 0 \end{array}$$

primal

$$\begin{array}{ll} \text{max} & y\ b \\ \text{subject to} & \\ & yA \leq c \\ & y_i \geq 0 \end{array}$$

dual

Duality

$$
\begin{array}{lrcrcrl}
\min & 3x_1 & + & 2x_2 & + & 4x_3 & \\
\text{subject to} & & & & & & \\
 & 2x_1 & + & x_2 & + & & \geq 2 \\
 & 2x_1 & - & x_2 & + & x_3 & \geq 5
\end{array}
$$

$$
\begin{array}{lrcrl}
\max & 2y_1 & + & 5y_2 & \\
\text{subject to} & & & & \\
 & 2y_1 & + & 2y_2 & \leq 3 \\
 & y_1 & - & y_2 & \leq 2 \\
 & & & y_2 & \leq 4
\end{array}
$$

Duality

$$
\begin{array}{llllll}
\text{min} & 3x_1 + 2x_2 + 4x_3 & & & & \\
\text{subject to} & & & & & \\
& 2x_1 + x_2 + & \geq & 2 & & y_1 \\
& 2x_1 - x_2 + x_3 & \geq & 5 & & y_2
\end{array}
$$

$$
\begin{array}{llll}
\text{max} & 2y_1 + 5y_2 & & \\
\text{subject to} & & & \\
& 2y_1 + 2y_2 & \leq & 3 \\
& y_1 - y_2 & \leq & 2 \\
& y_2 & \leq & 4
\end{array}
$$

Duality

$$\min \quad \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\max \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

subject to

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$