Discrete Optimization

MA2827

Fondements de l'optimisation discrète

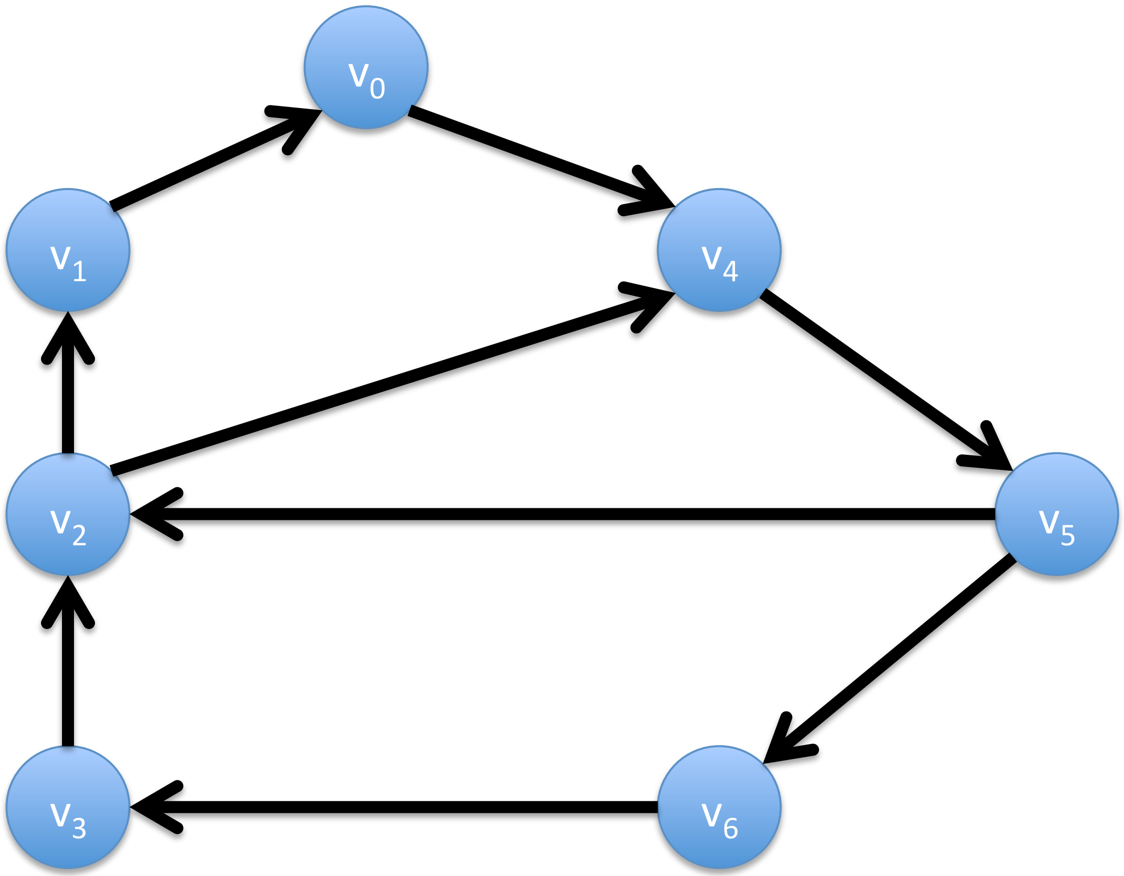
https://project.inria.fr/2015ma2827/

Slides courtesy of M. Pawan Kumar

Outline

- Preliminaries (Recap)
- Menger's Theorem for Disjoint Paths
- Path Packing

Directed Graphs (Digraphs)

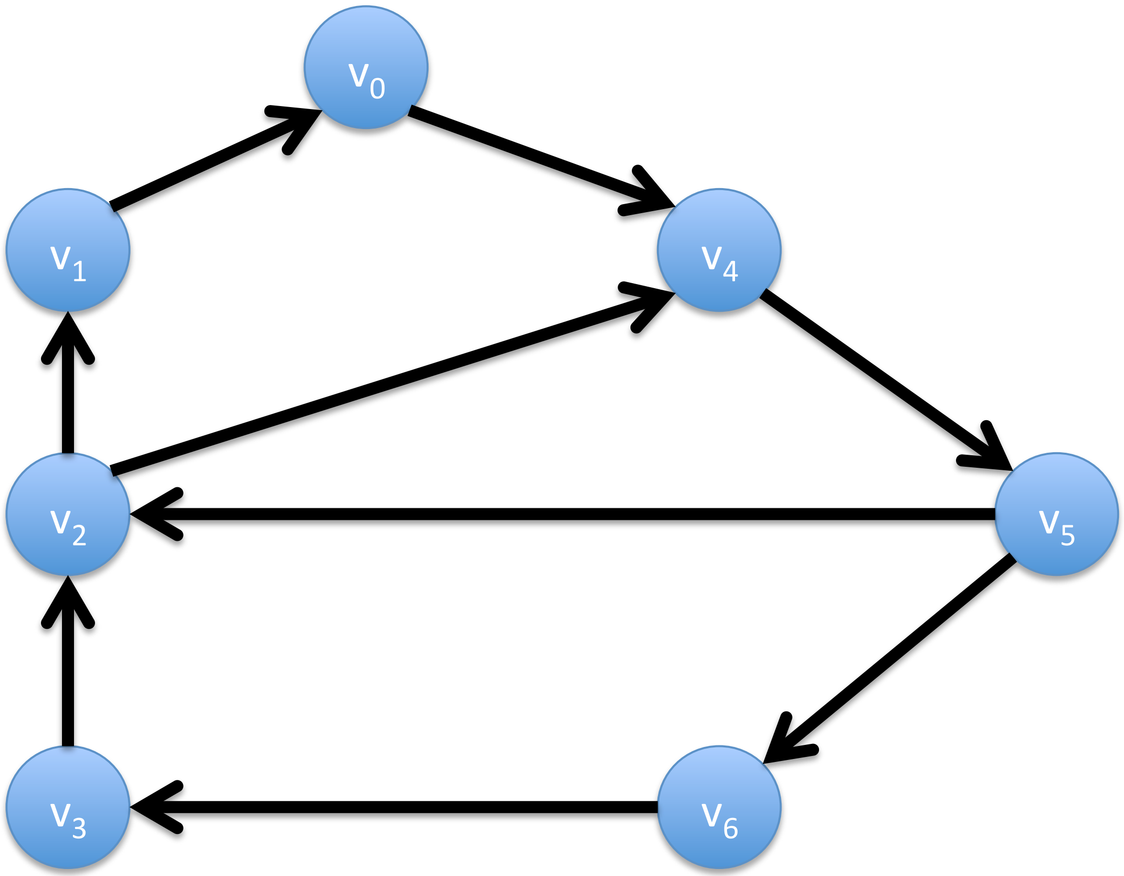


D = (V, A)

'n' vertices or nodes V

'm' arcs A: ordered pairs from V

Indegree of a Vertex

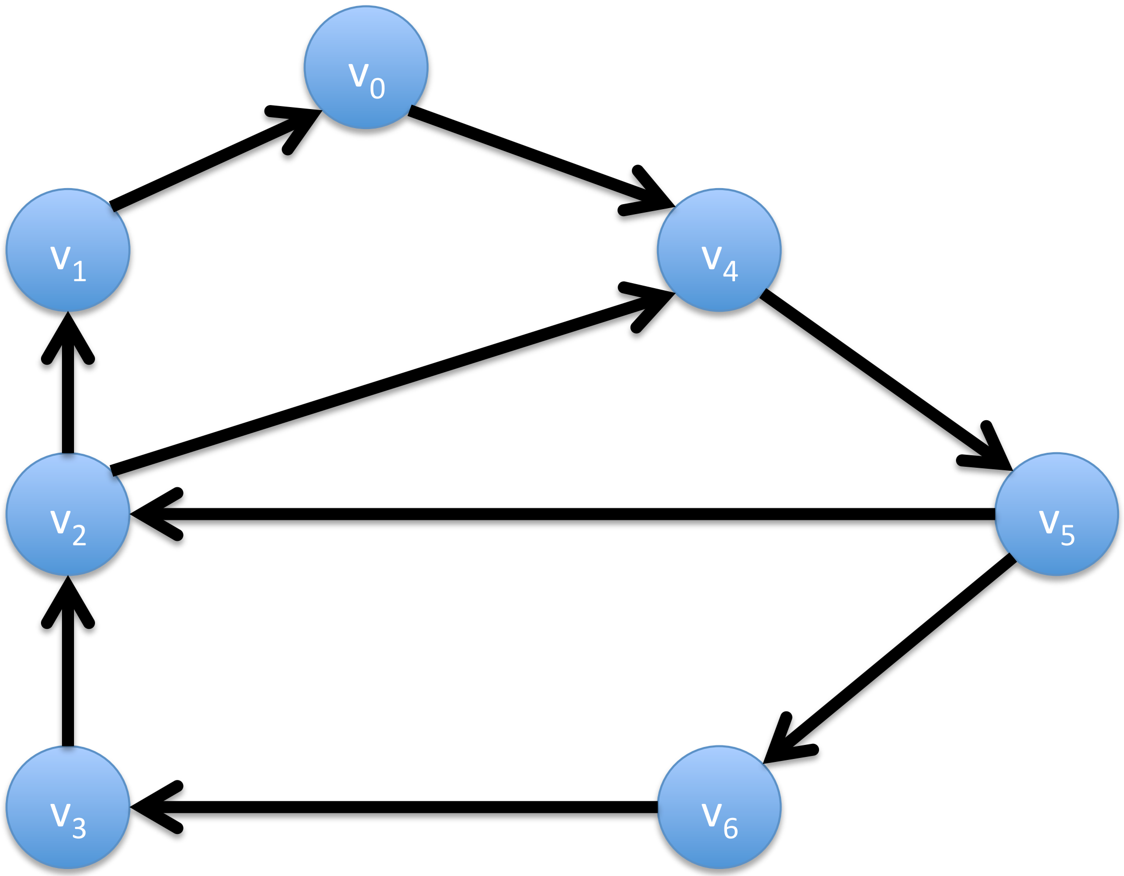


D = (V, A)

Number of arcs entering the vertex.

$indeg(v_0) = 1$, $indeg(v_1) = 1$, $indeg(v_4) = 2$, ...

Indegree of a Subset of Vertices

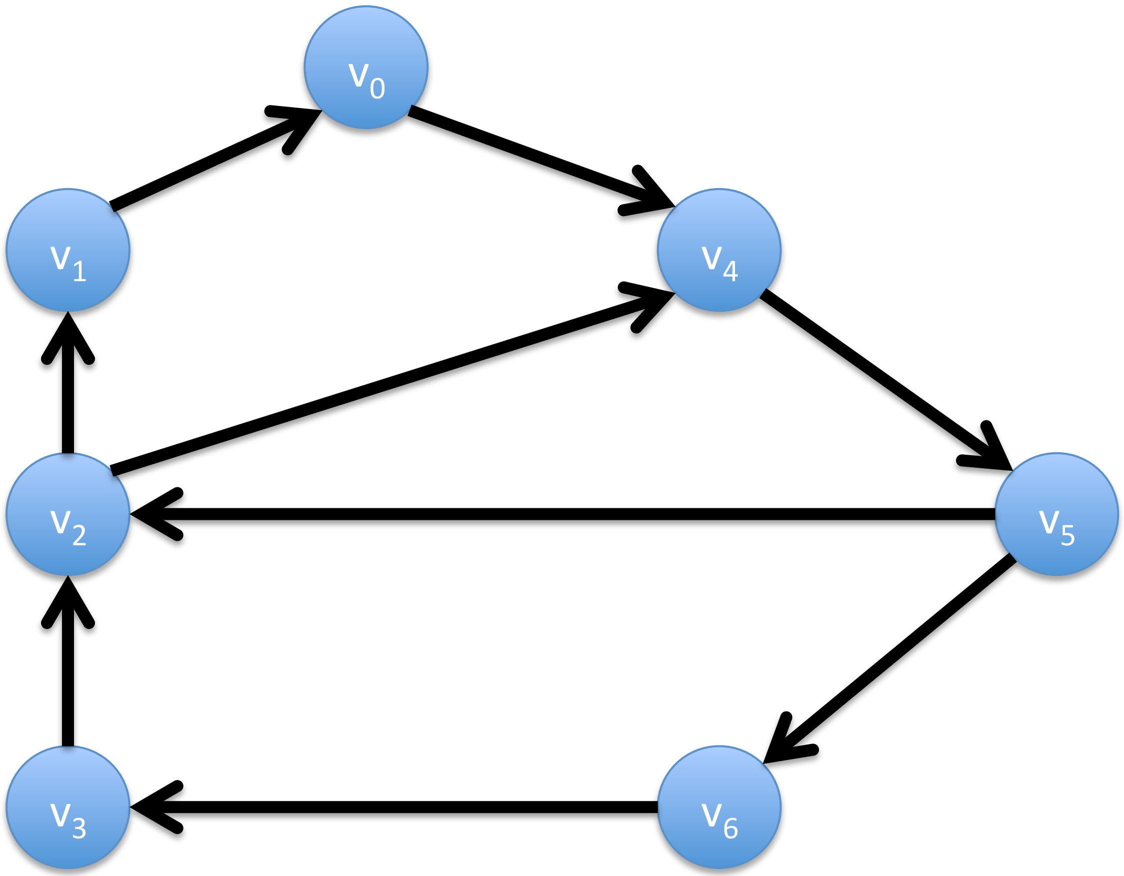


D = (V, A)

Number of arcs entering the subset.

indeg({v_0,v_1}) = 1, indeg({v_1,v_4}) = 3, …

Outdegree of a Vertex

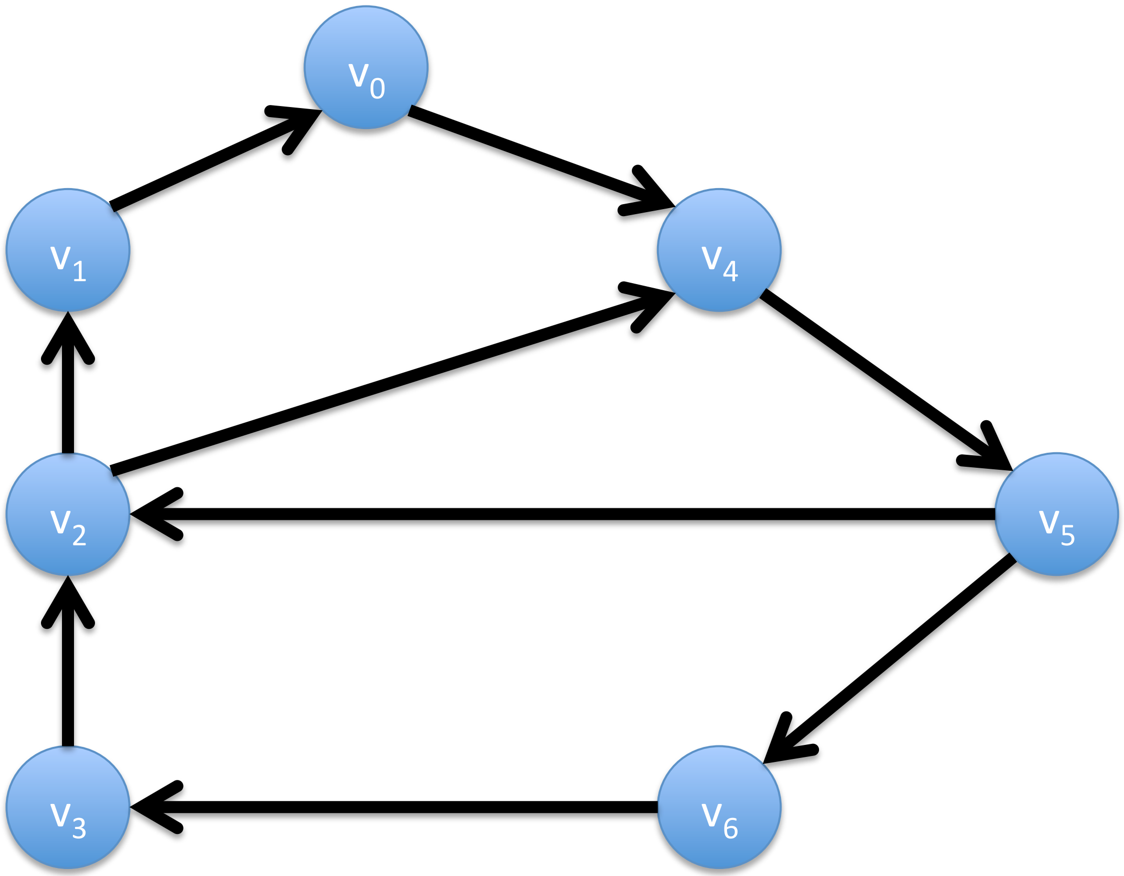


D = (V, A)

Number of arcs leaving the vertex.

$\text{outdeg}(v_0) = 1$, $\text{outdeg}(v_1) = 1$, $\text{outdeg}(v_2) = 2$, ...

Outdegree of a Subset of Vertices

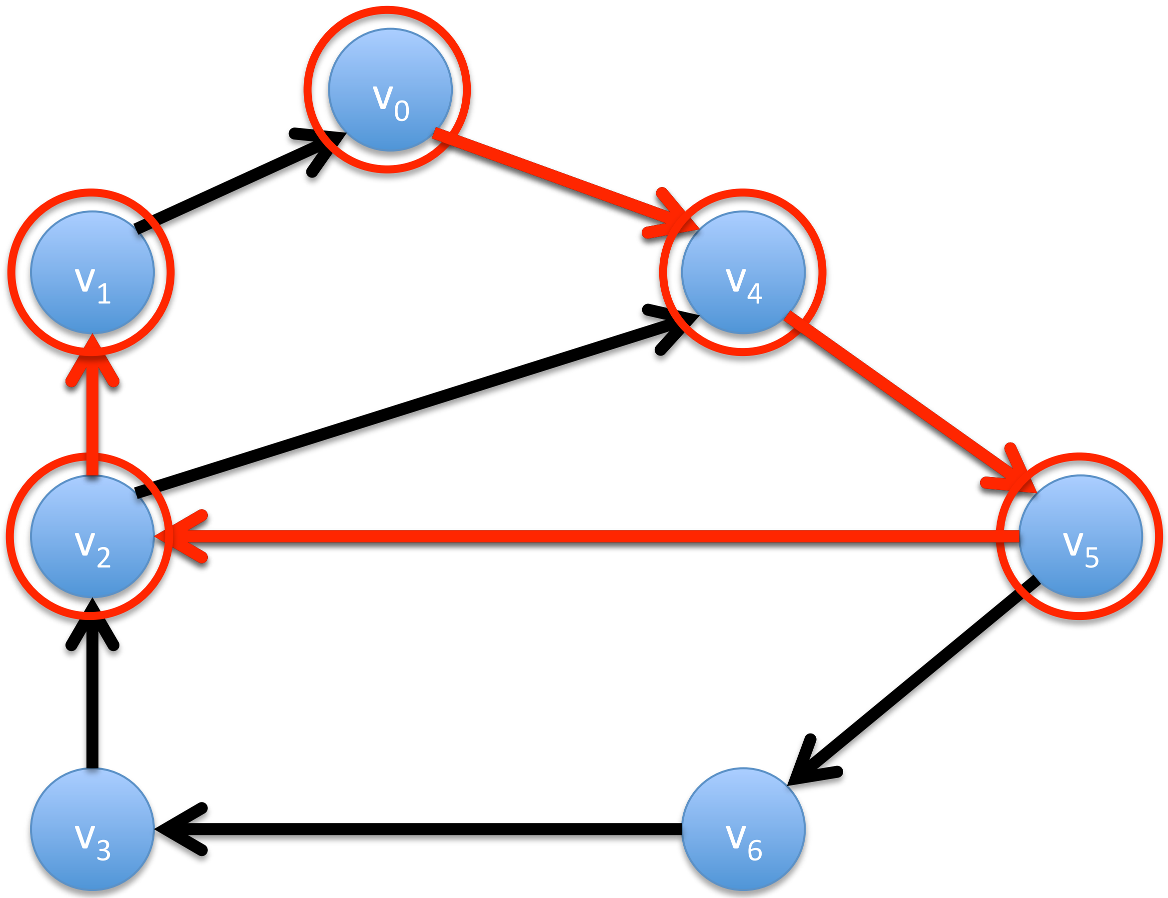


D = (V, A)

Number of arcs leaving the subset.

outdeg($\{v_0,v_1\}$) = 1, outdeg($\{v_1,v_4\}$) = 2, ...

s-t Path

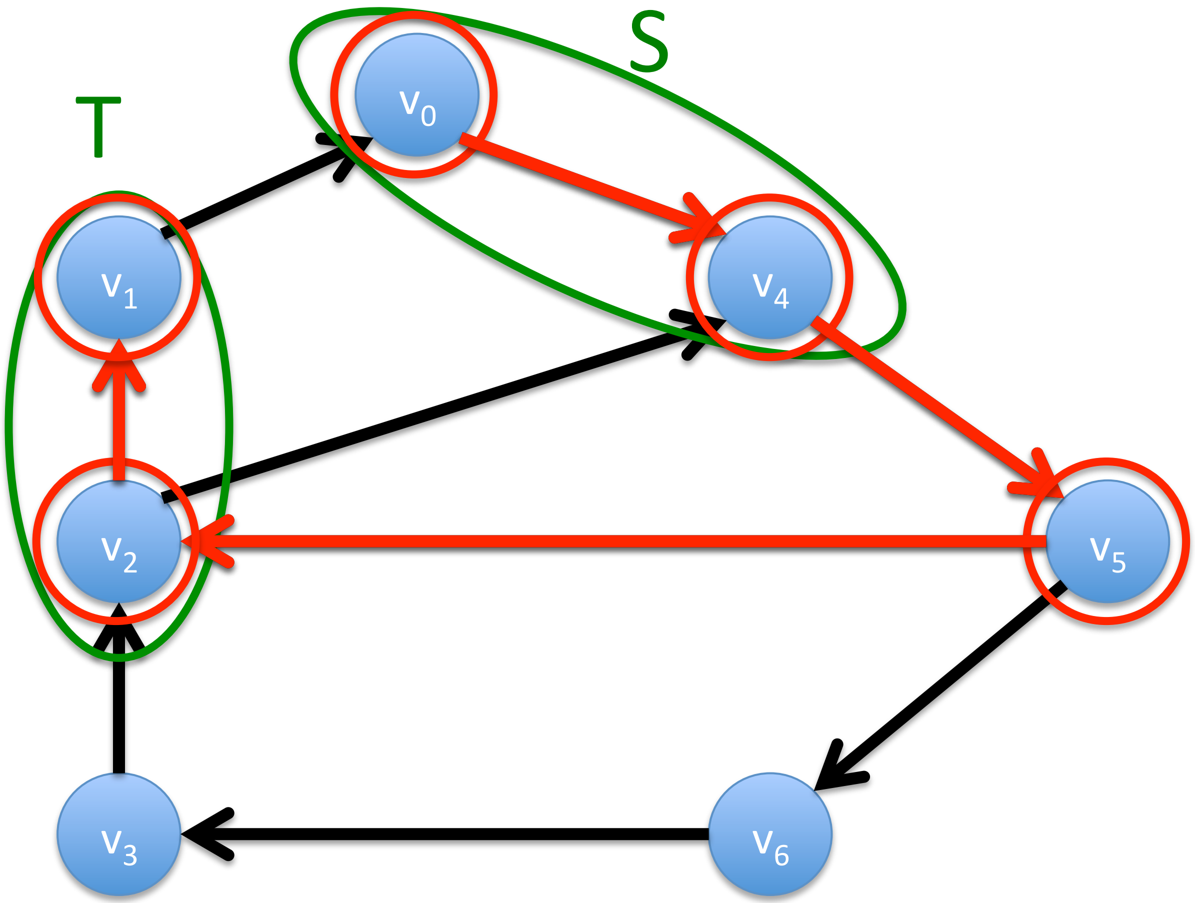


D = (V, A)

Sequence $P = (s=v_0,a_1,v_1,\ldots,a_k,t=v_k)$, $a_i = (v_{i-1},v_i)$

Vertices $s=v_0,v_1,\ldots,t=v_k$ are distinct

S-T Path

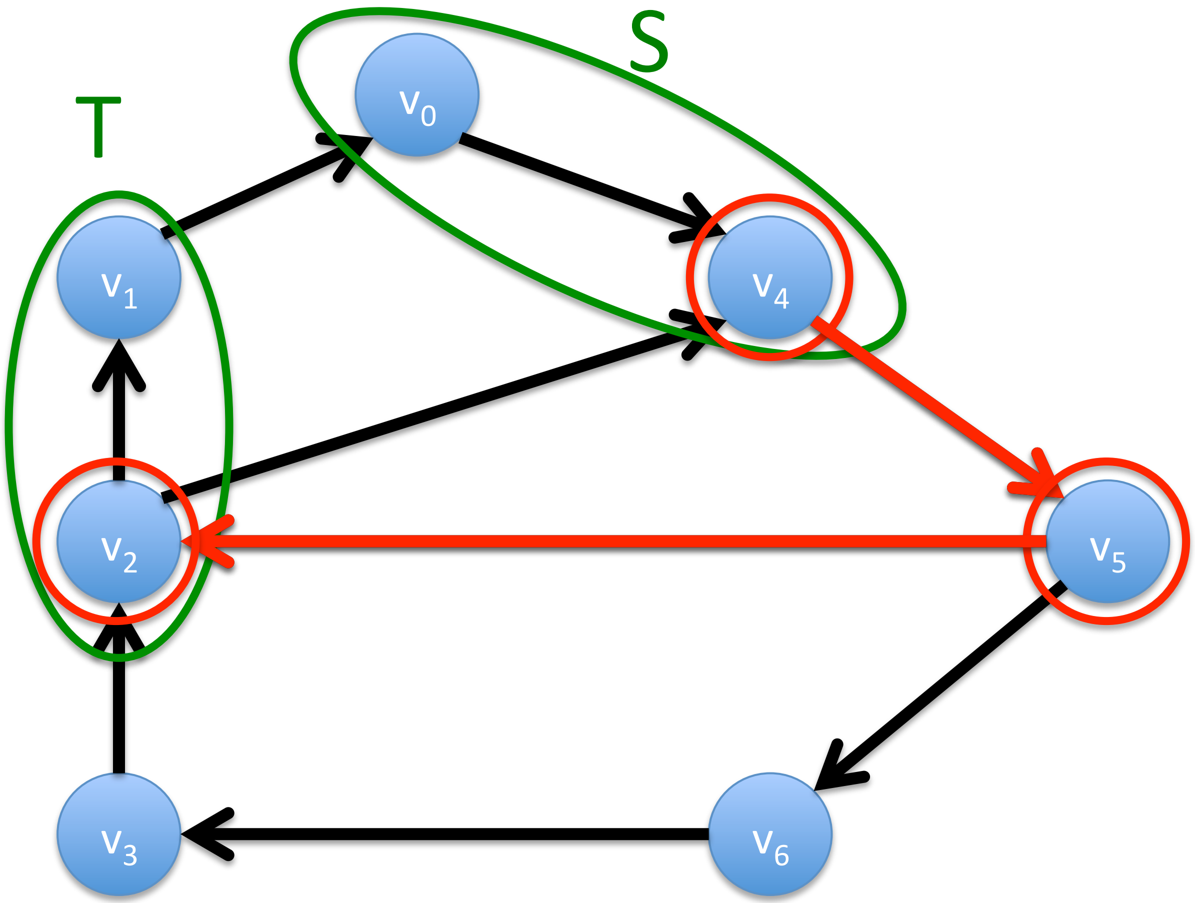


D = (V, A)

S and T are subsets of V

Any st-path where $s \in S$ and $t \in T$

S-T Path



D = (V, A)

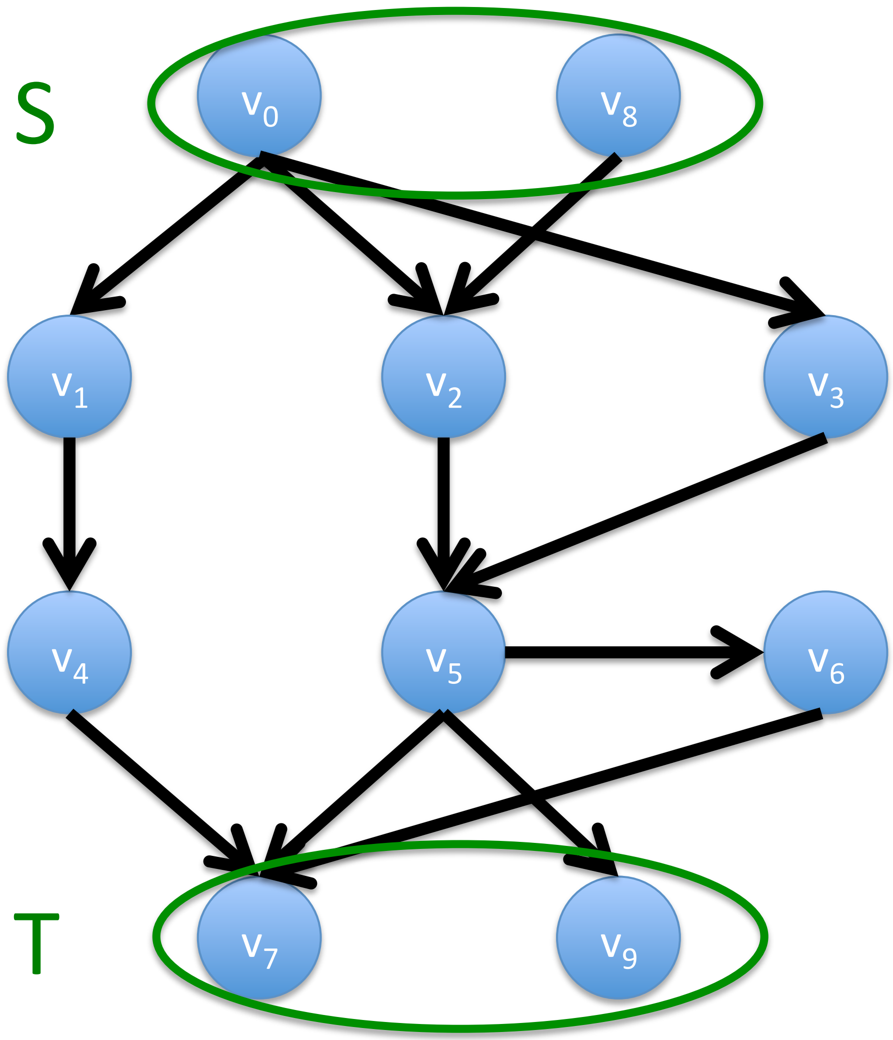
S and T are subsets of V

Any st-path where $s \in S$ and $t \in T$

Outline

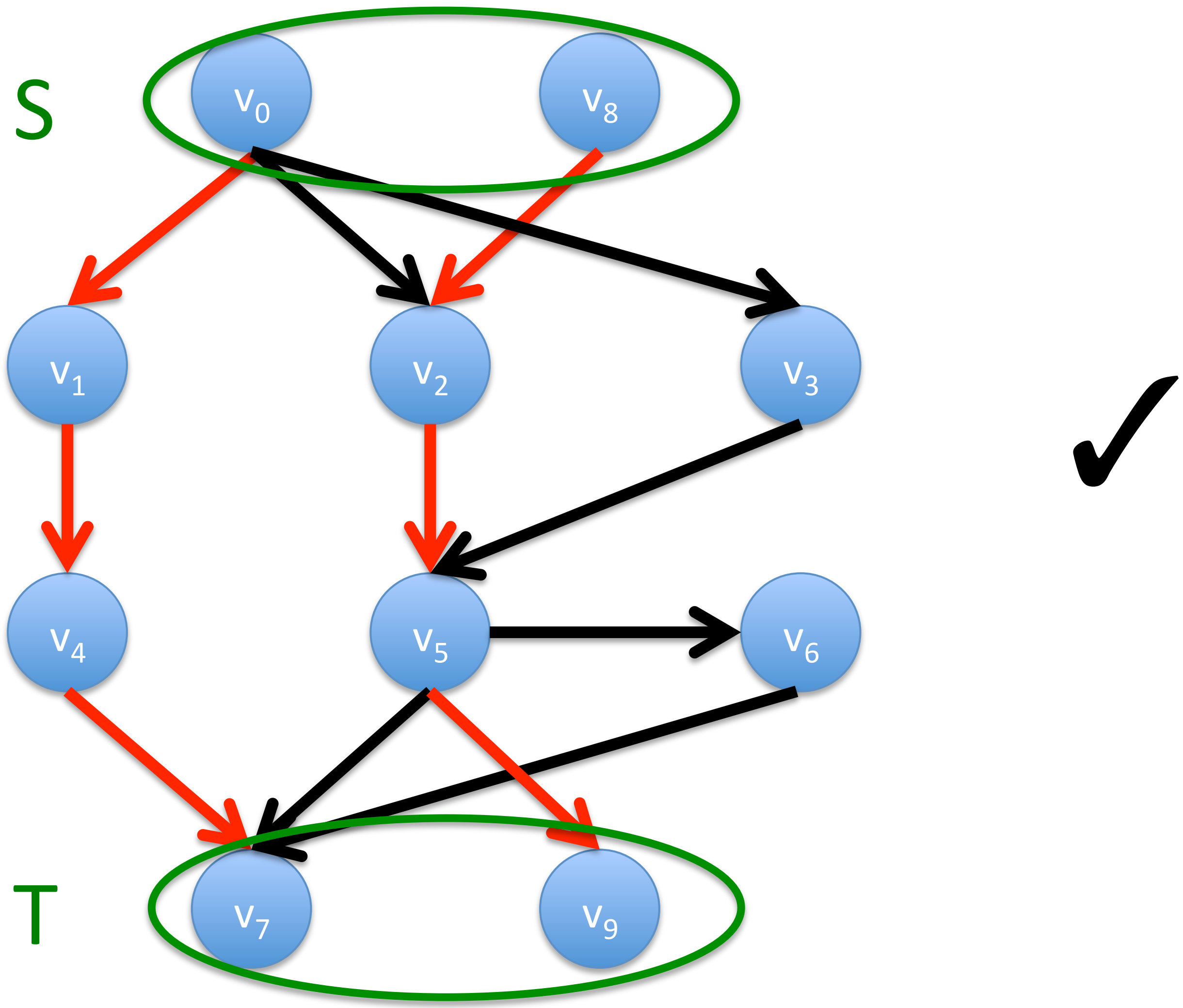
- Preliminaries
- **Menger's Theorem for Disjoint Paths**
- Path Packing

Vertex Disjoint S-T Paths



Set of S-T Paths with no common vertex

Vertex Disjoint S-T Paths



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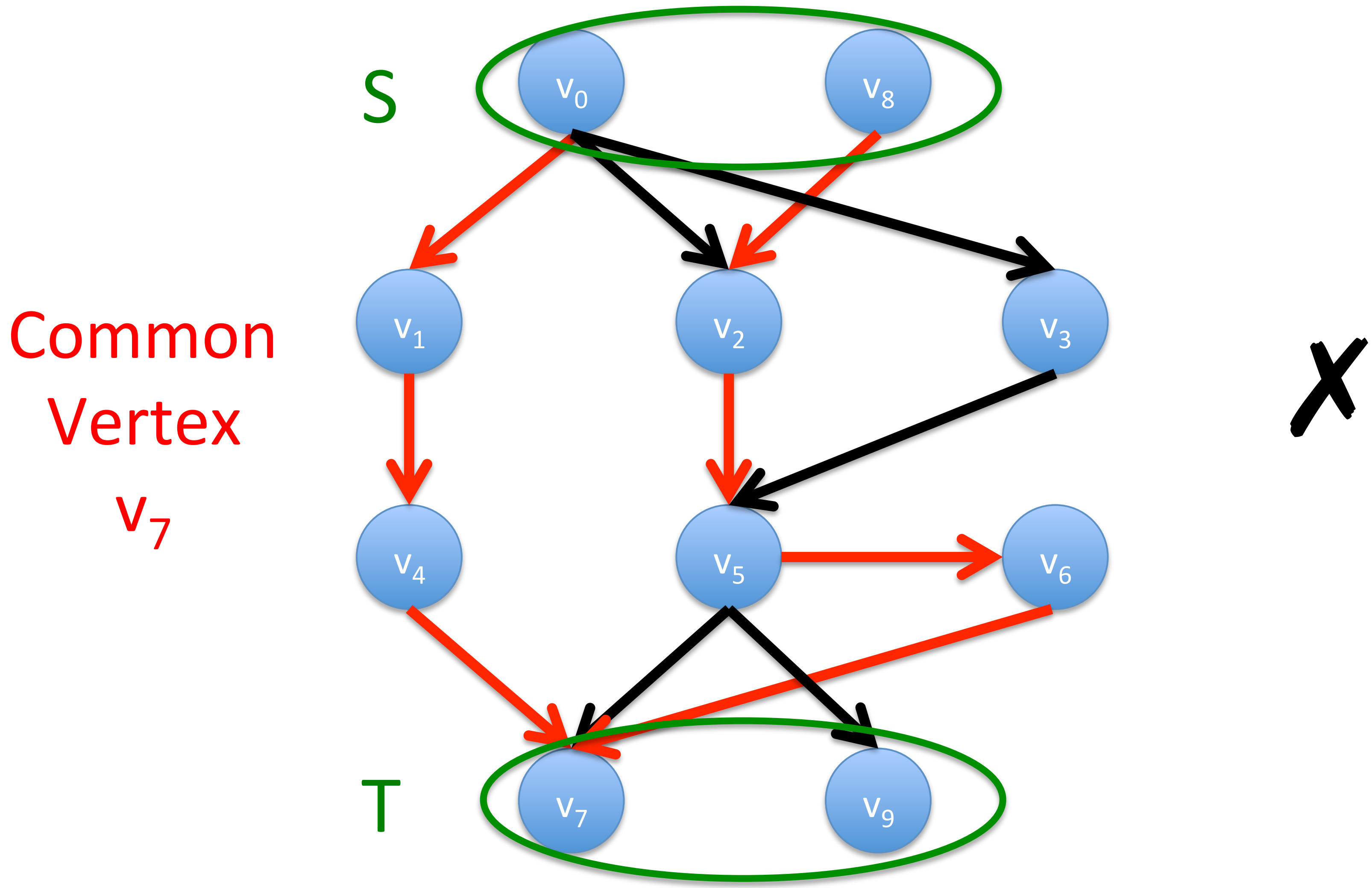
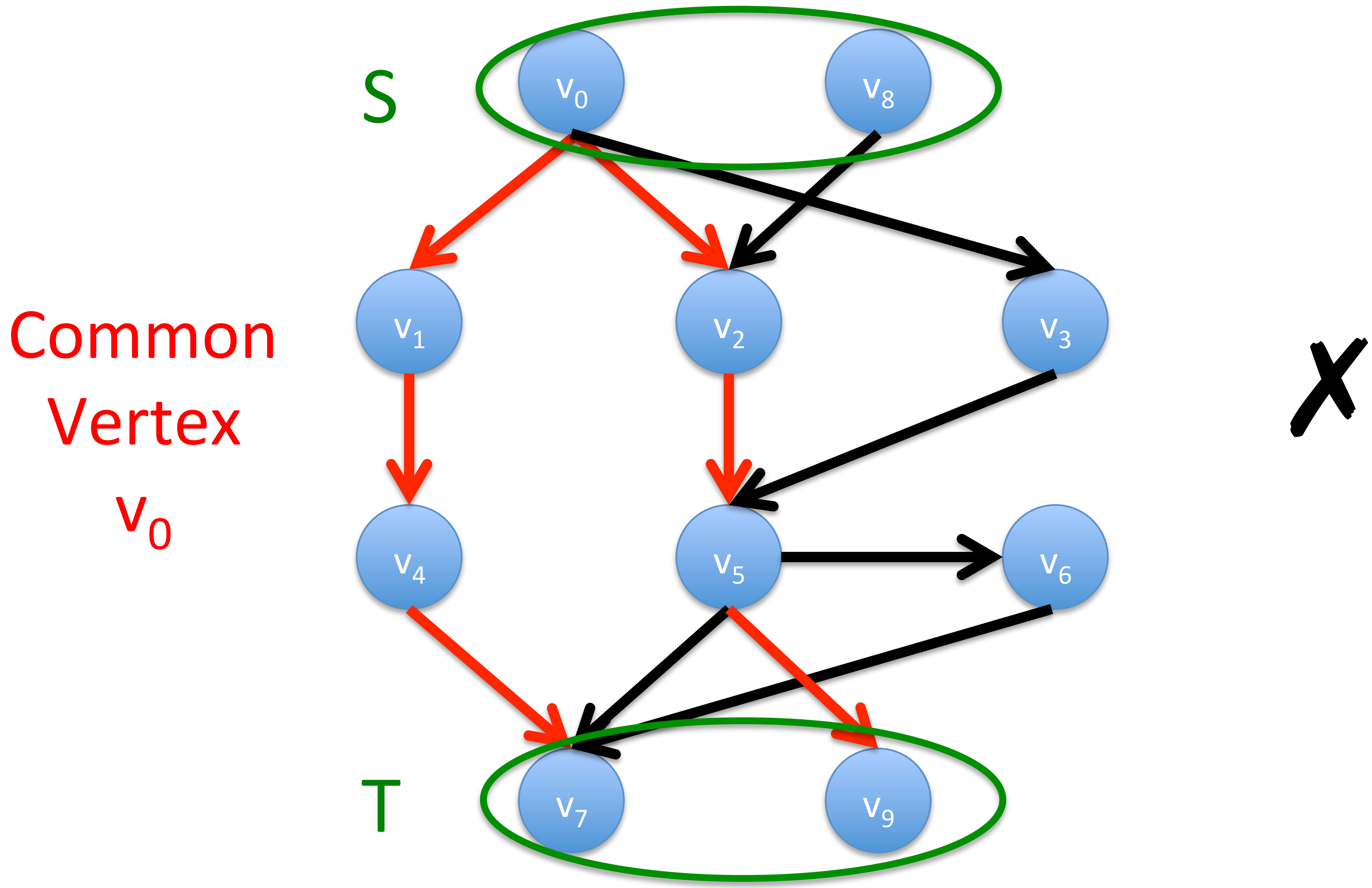
Vertex Disjoint S-T Paths



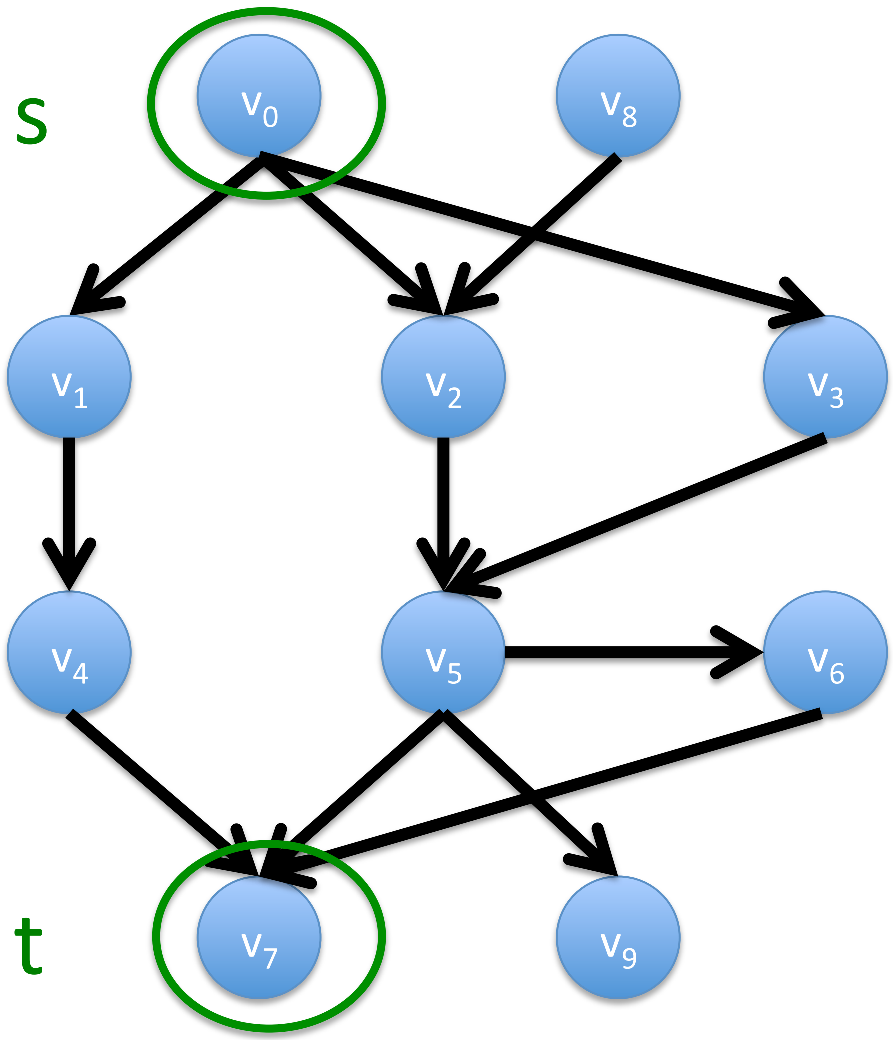
Set of S-T Paths with no common vertex

Vertex Disjoint S-T Paths



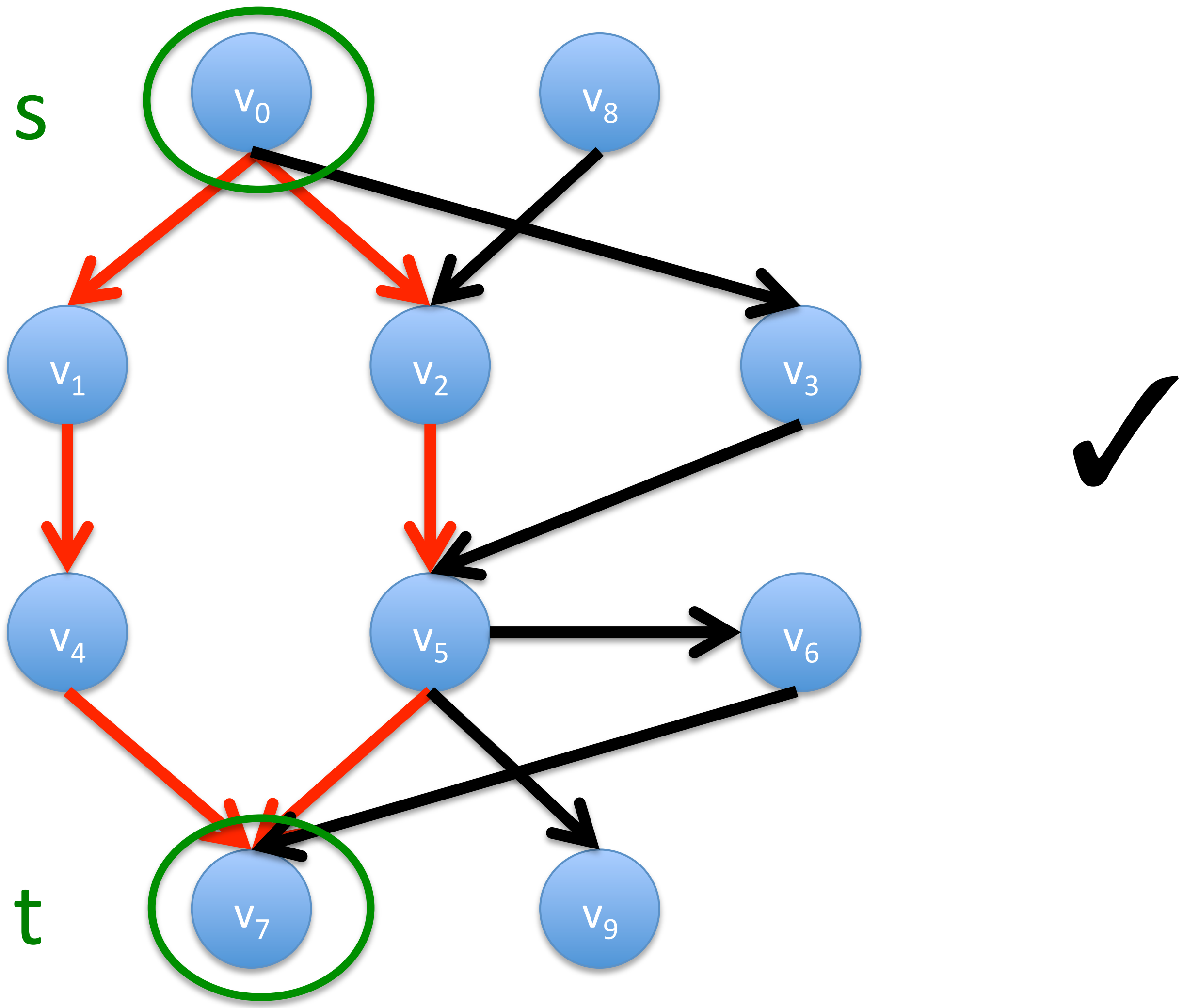
Set of S-T Paths with no common vertex

Internally Vertex Disjoint s-t Paths



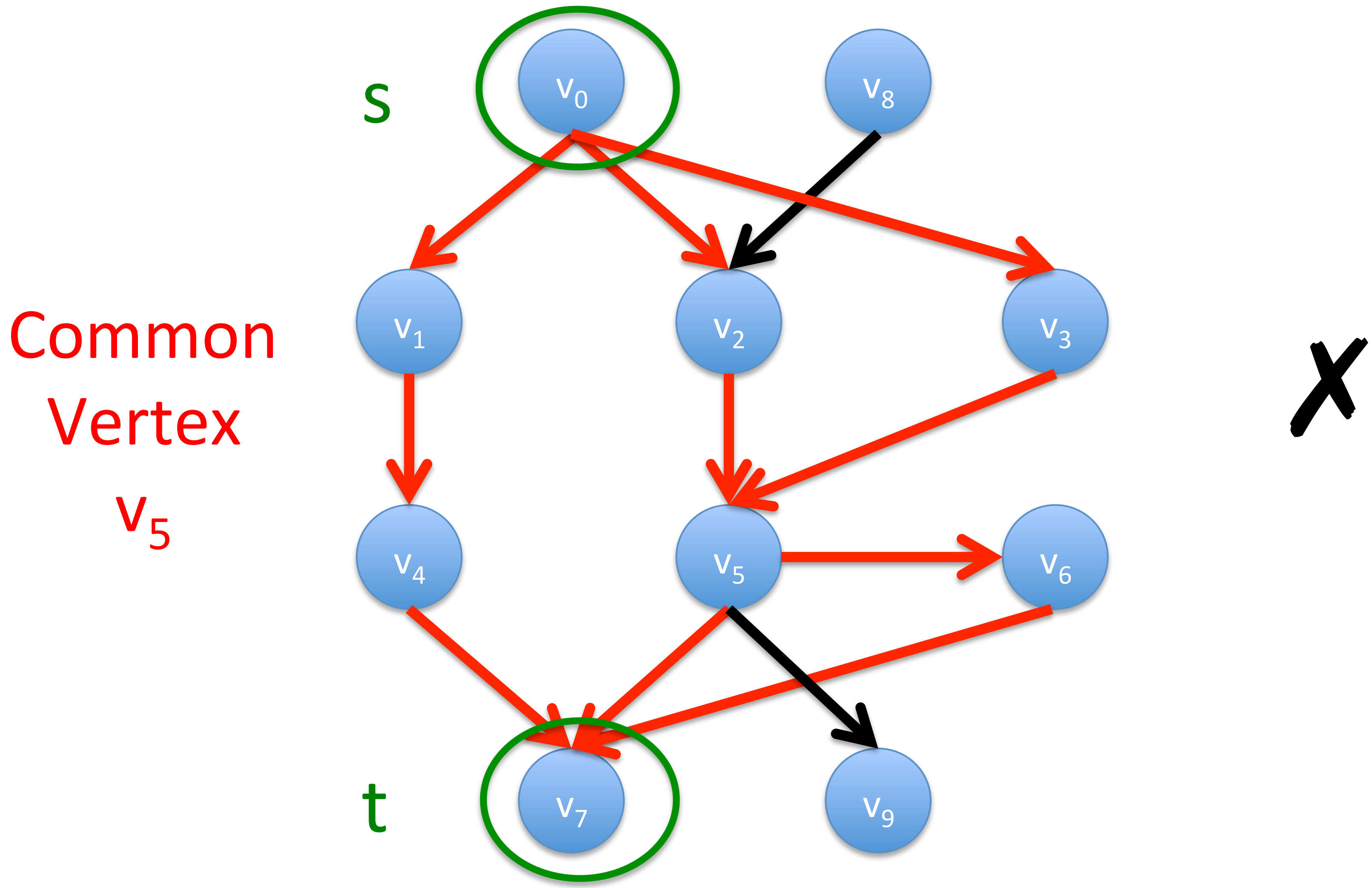
Set of s-t Paths with no common internal vertex

Internally Vertex Disjoint s-t Paths



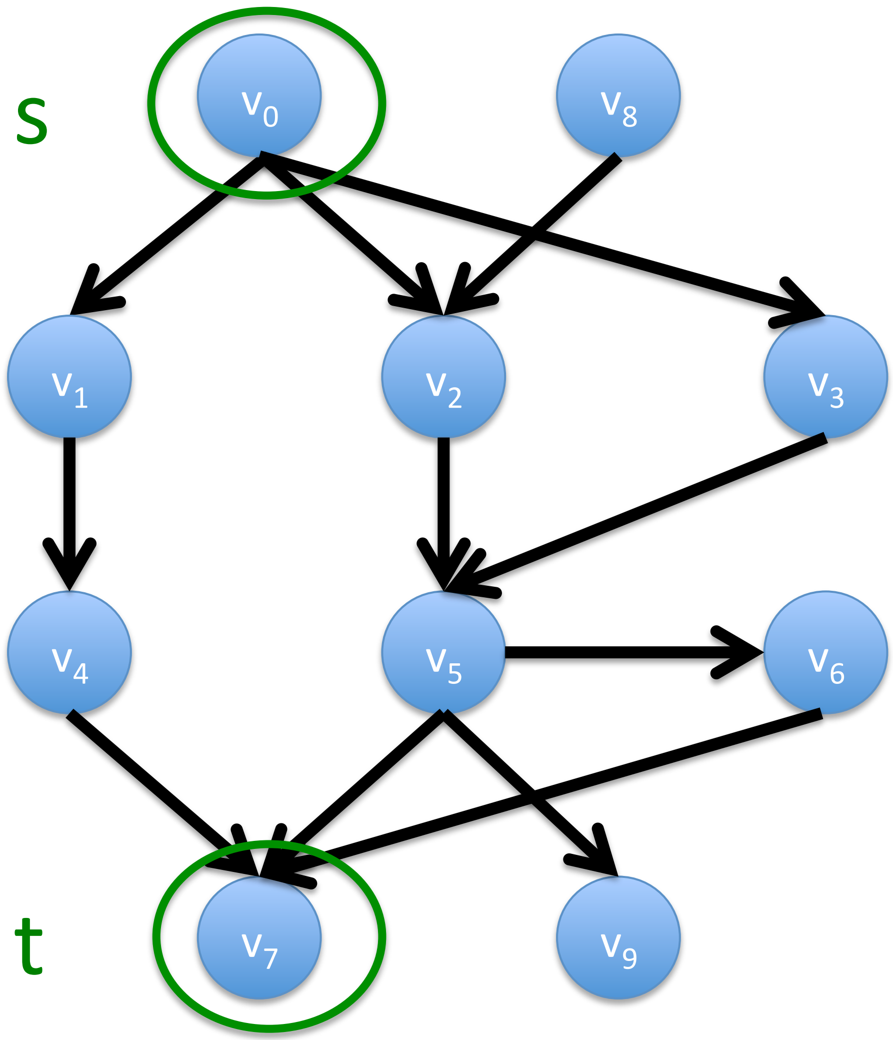
Set of s-t Paths with no common internal vertex

Internally Vertex Disjoint s-t Paths



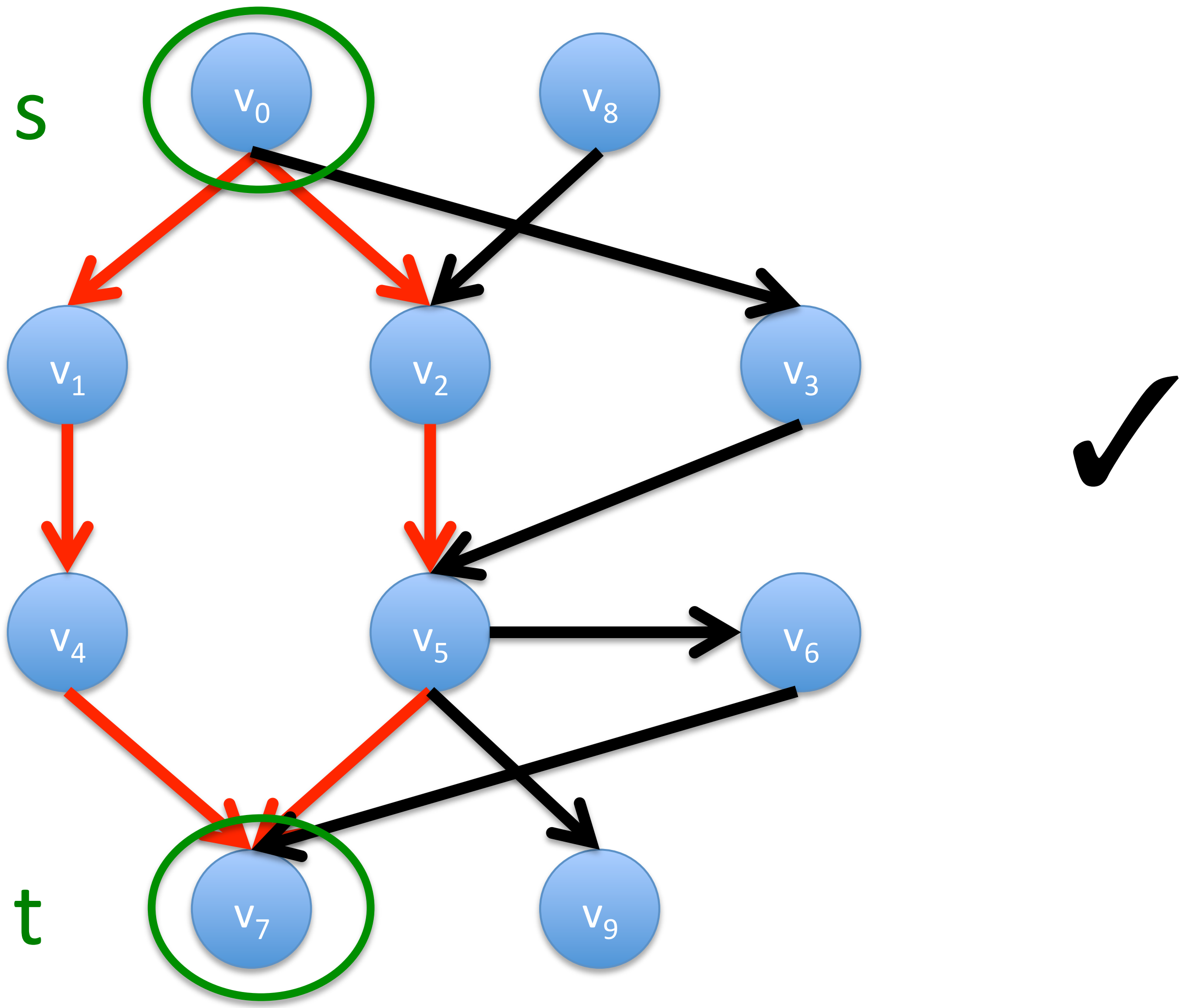
Set of s-t Paths with no common internal vertex

Arc Disjoint s-t Paths



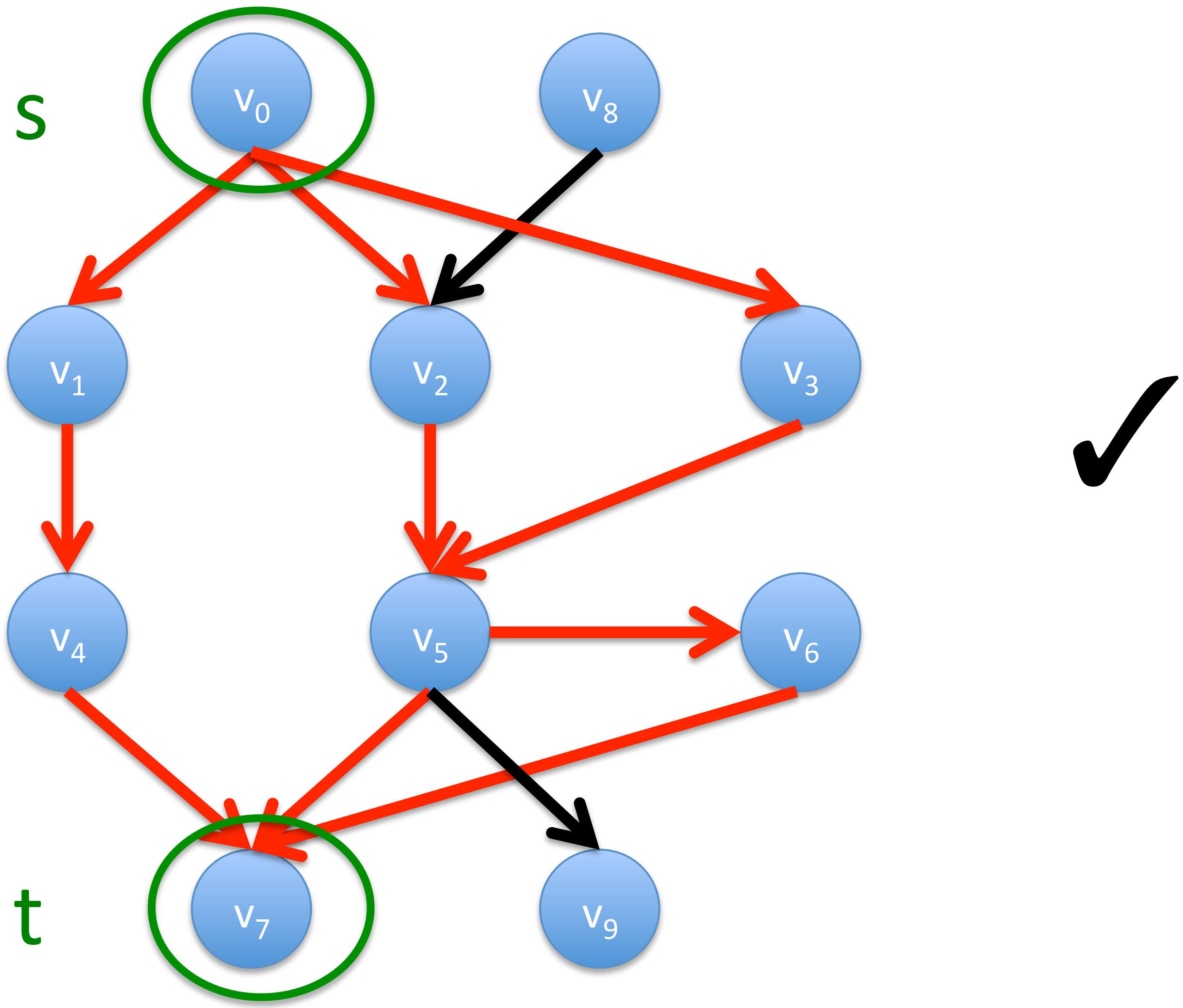
Set of s-t Paths with no common arcs

Arc Disjoint s-t Paths



Set of s-t Paths with no common arcs

Arc Disjoint s-t Paths

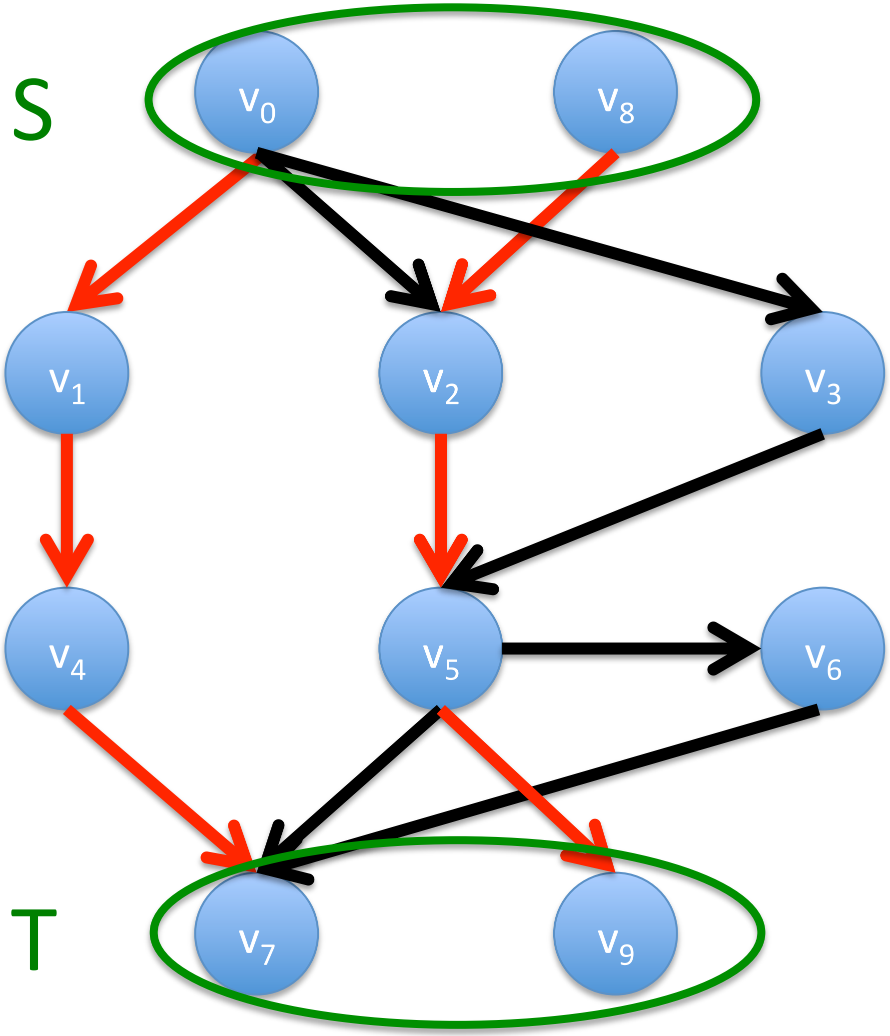


Set of s-t Paths with no common arcs

Outline

- Preliminaries
- Menger's Theorem for Disjoint Paths
  - **Vertex Disjoint S-T Paths**
  - Internally Vertex Disjoint s-t Paths
  - Arc Disjoint s-t Paths
- Path Packing

Vertex Disjoint S-T Paths

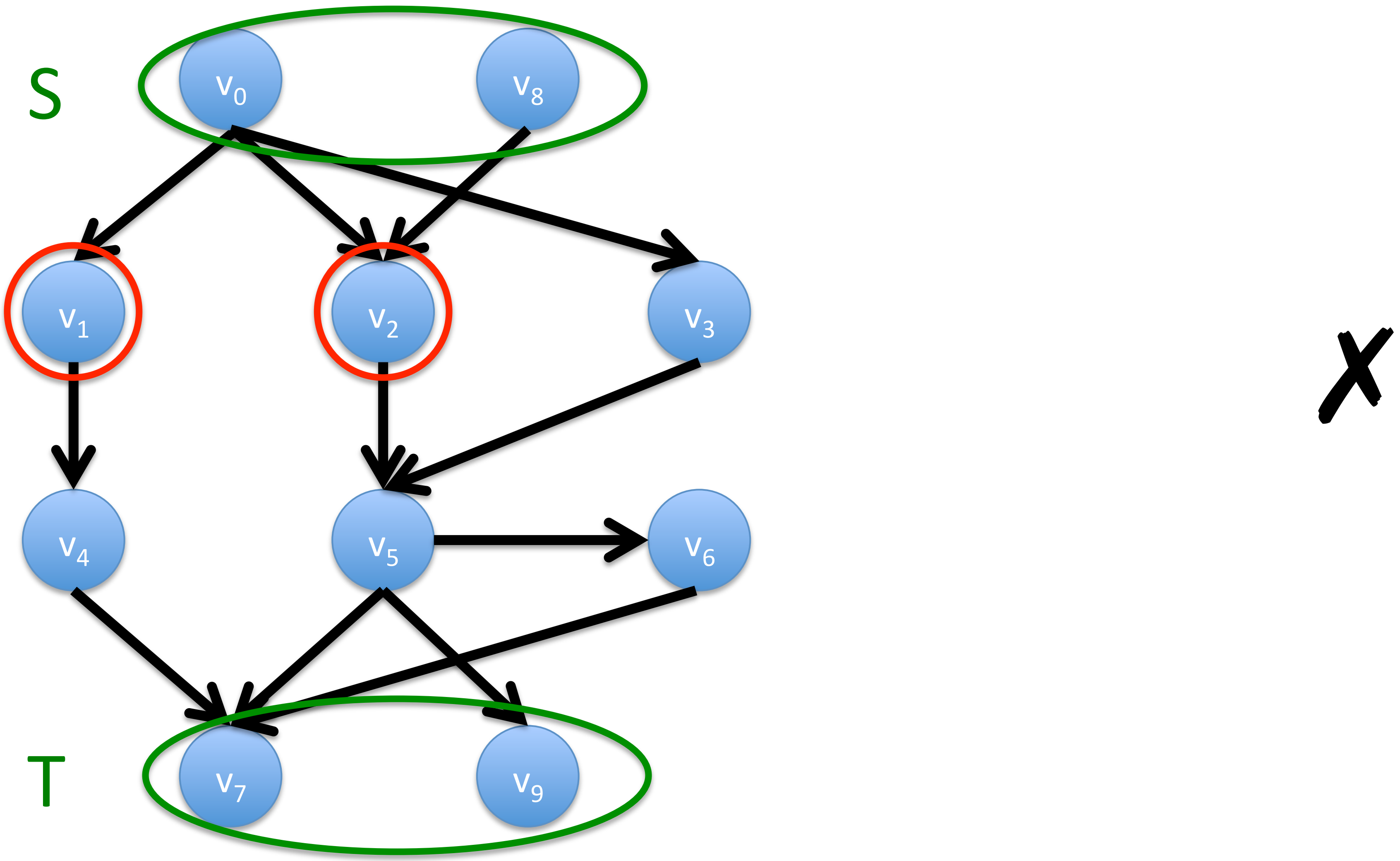


Maximum number of disjoint paths?

Minimum size of S-T disconnecting vertex set !!

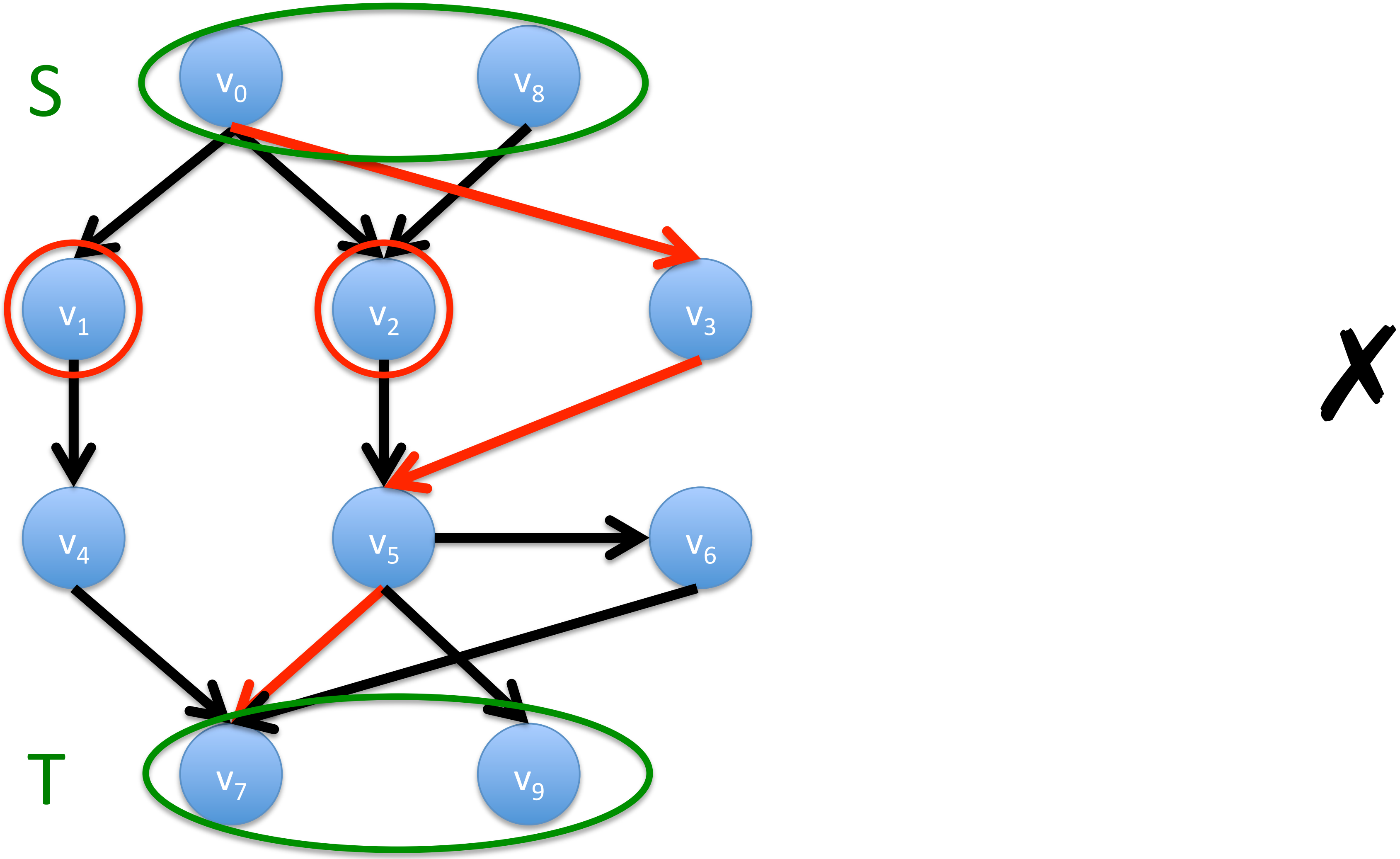
Set of S-T Paths with no common vertex

S-T Disconnecting Vertex Set



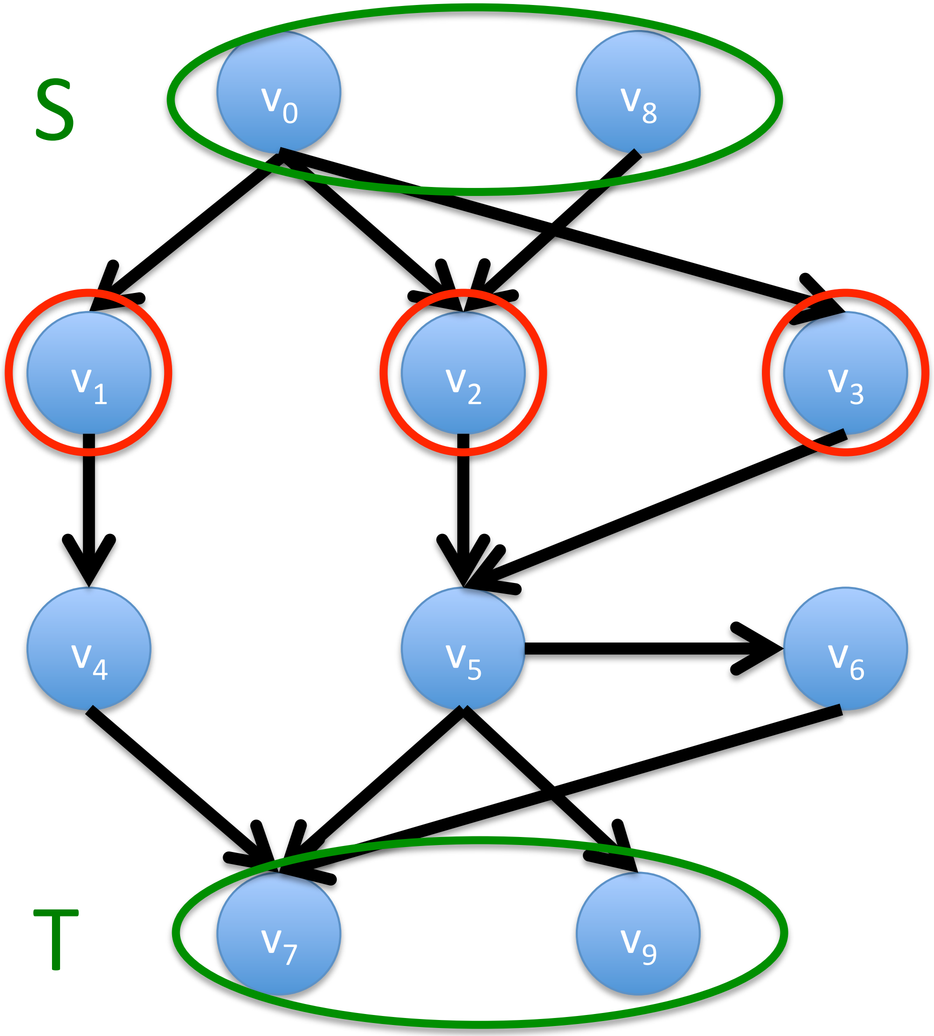
Subset U of V which intersects with all S-T Paths

S-T Disconnecting Vertex Set



Subset U of V which intersects with all S-T Paths

S-T Disconnecting Vertex Set



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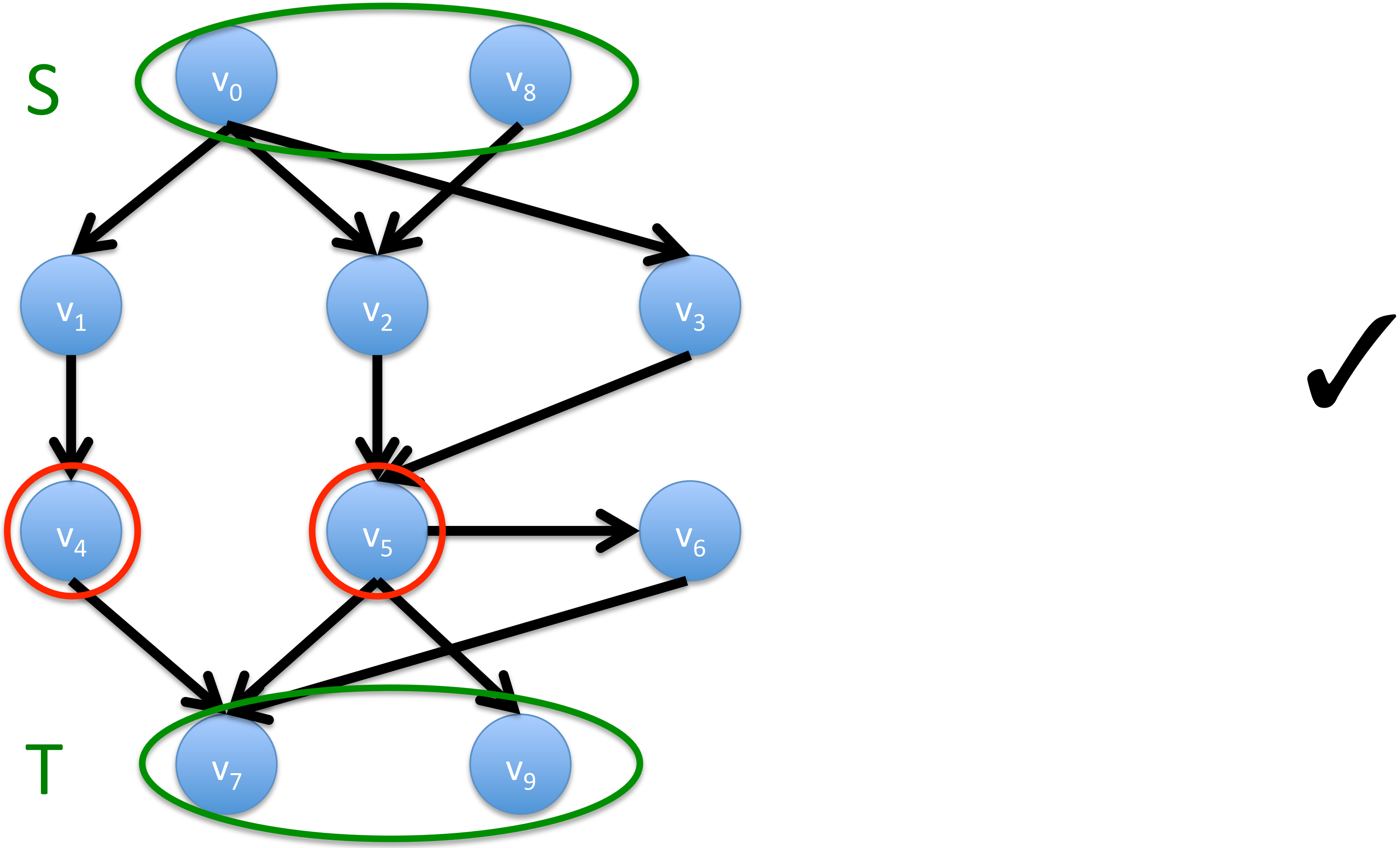
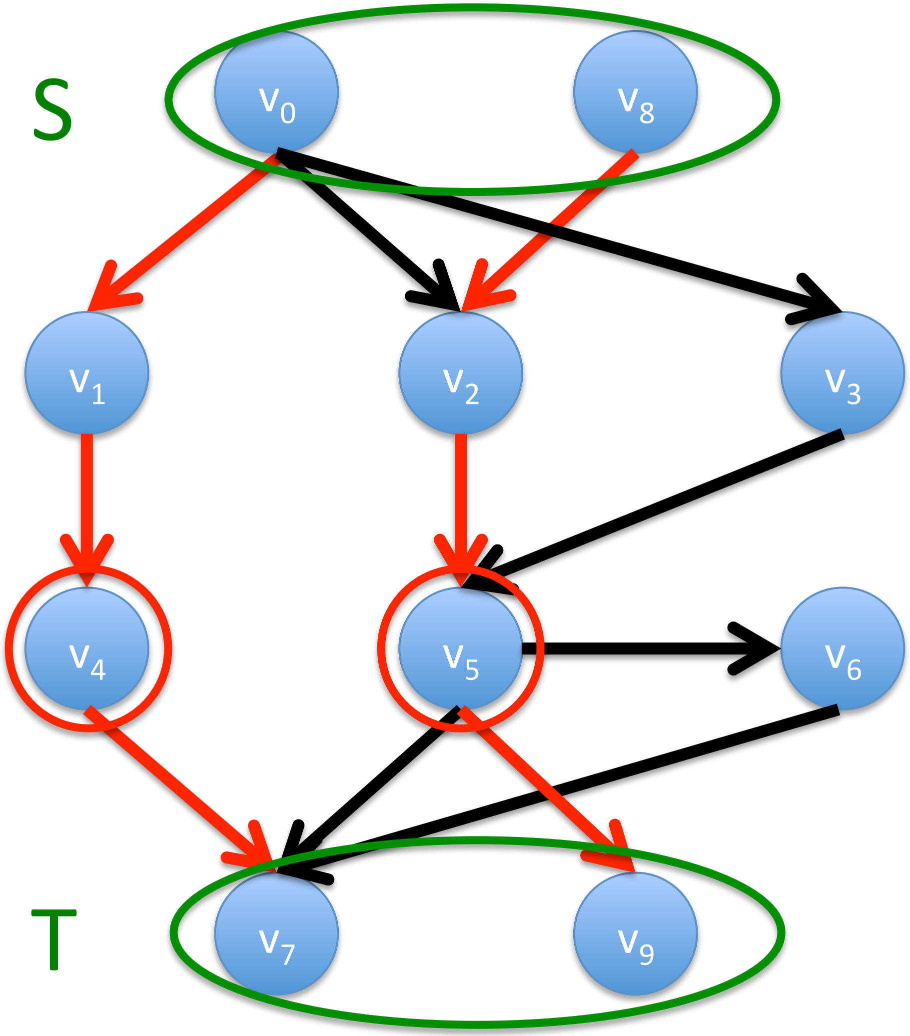
Subset U of V which intersects with all S-T Paths

S-T Disconnecting Vertex Set



Subset U of V which intersects with all S-T Paths

Connection

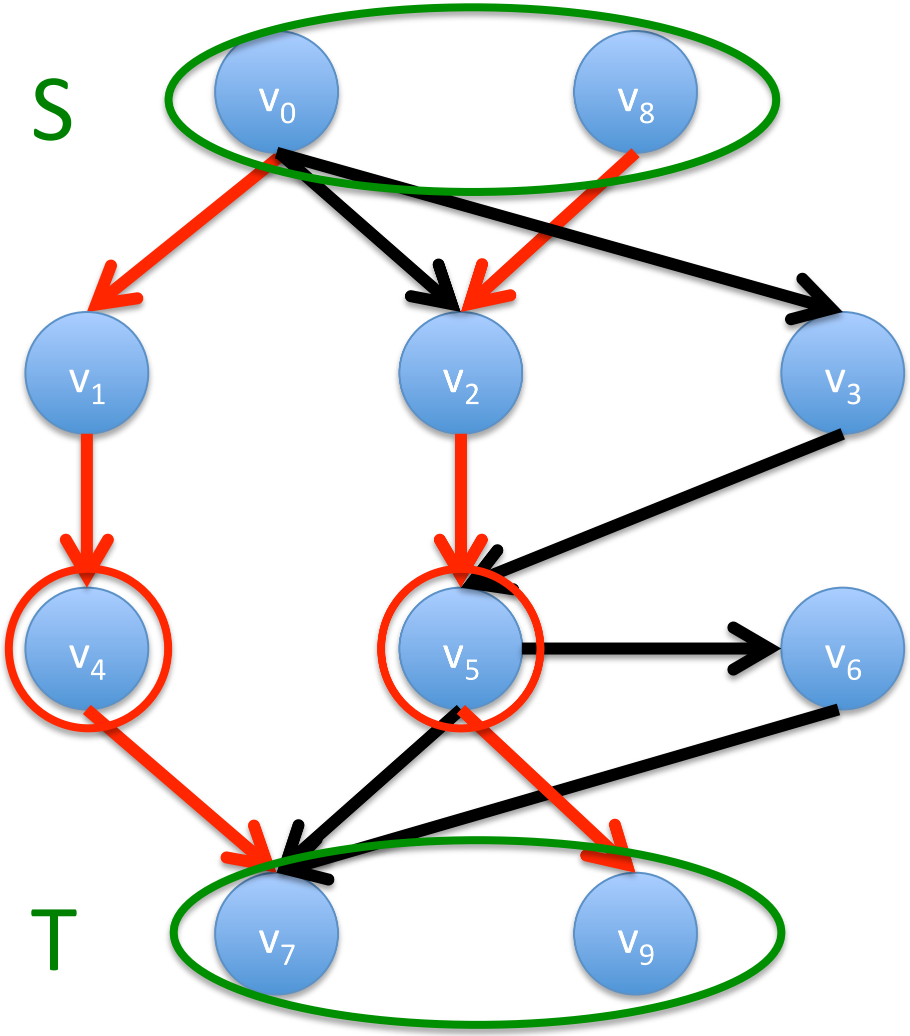


Maximum number of disjoint paths

≤

Minimum size of S-T disconnecting vertex set !!

Menger's Theorem



Maximum number of disjoint paths

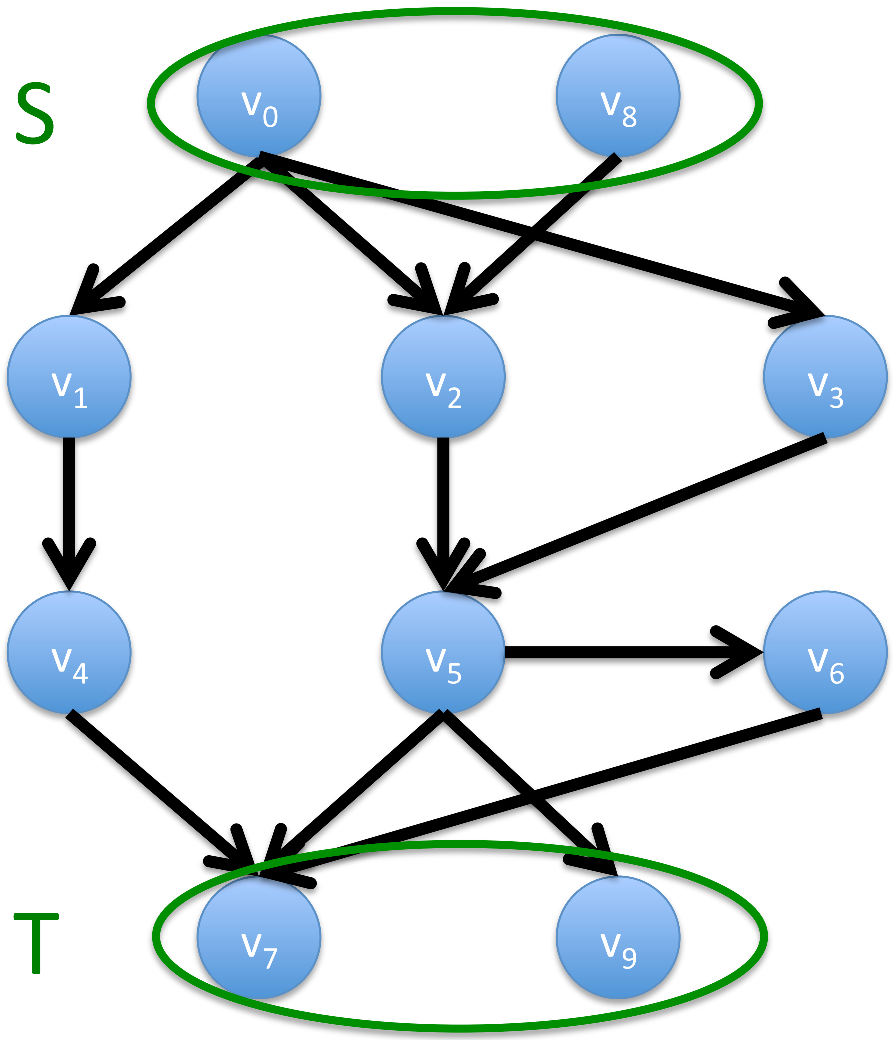
=

Minimum size of S-T disconnecting vertex set !!

Proof ?

Mathematical Induction on |A|

Menger's Theorem

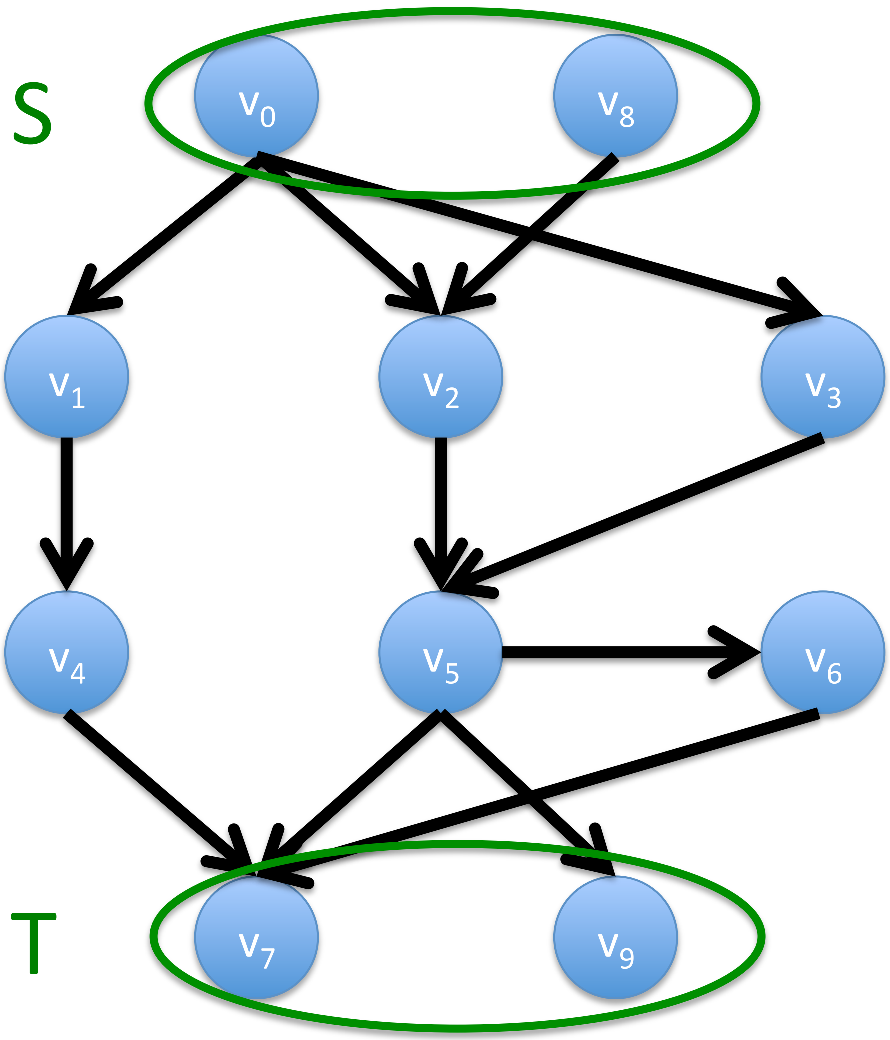


True for $|A| = 0$

Assume it is true for $|A| < m$

Mathematical Induction on $|A|$

Menger's Theorem



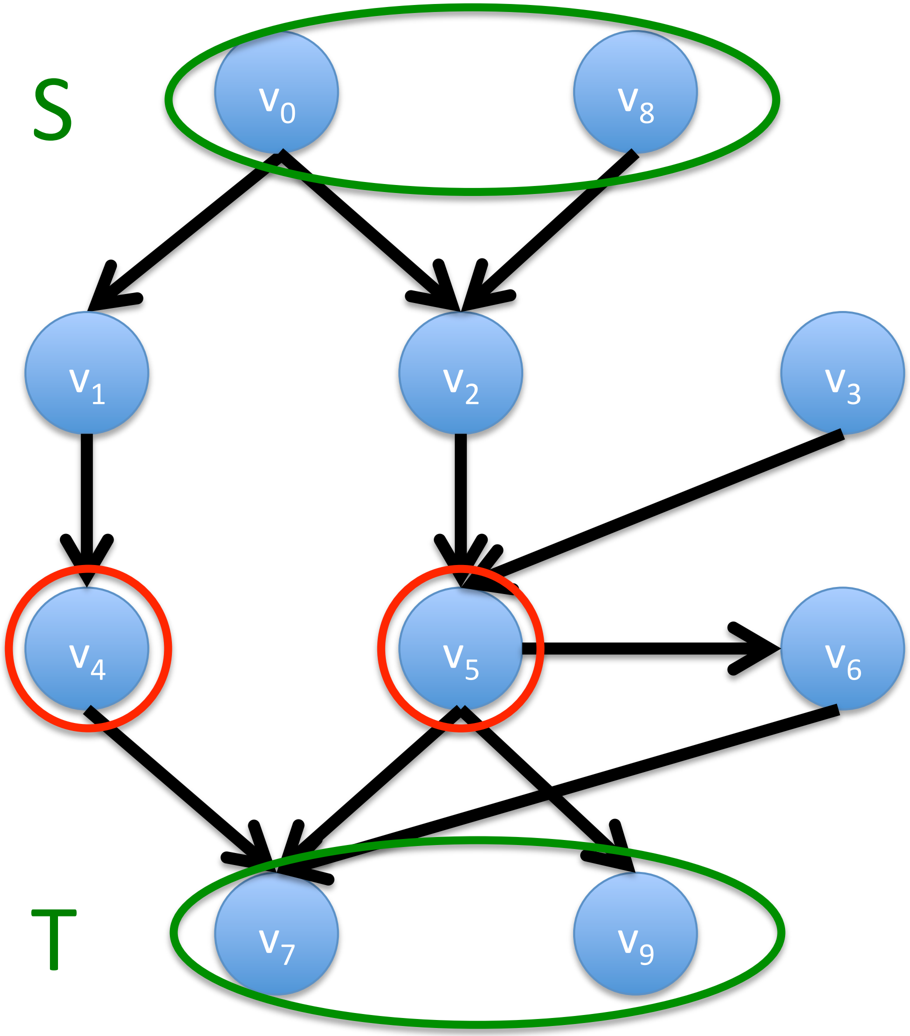
Consider |A| = m

Let minimum size disconnecting vertex set = k

Remove an arc (u,v)

Mathematical Induction on |A|

Menger's Theorem

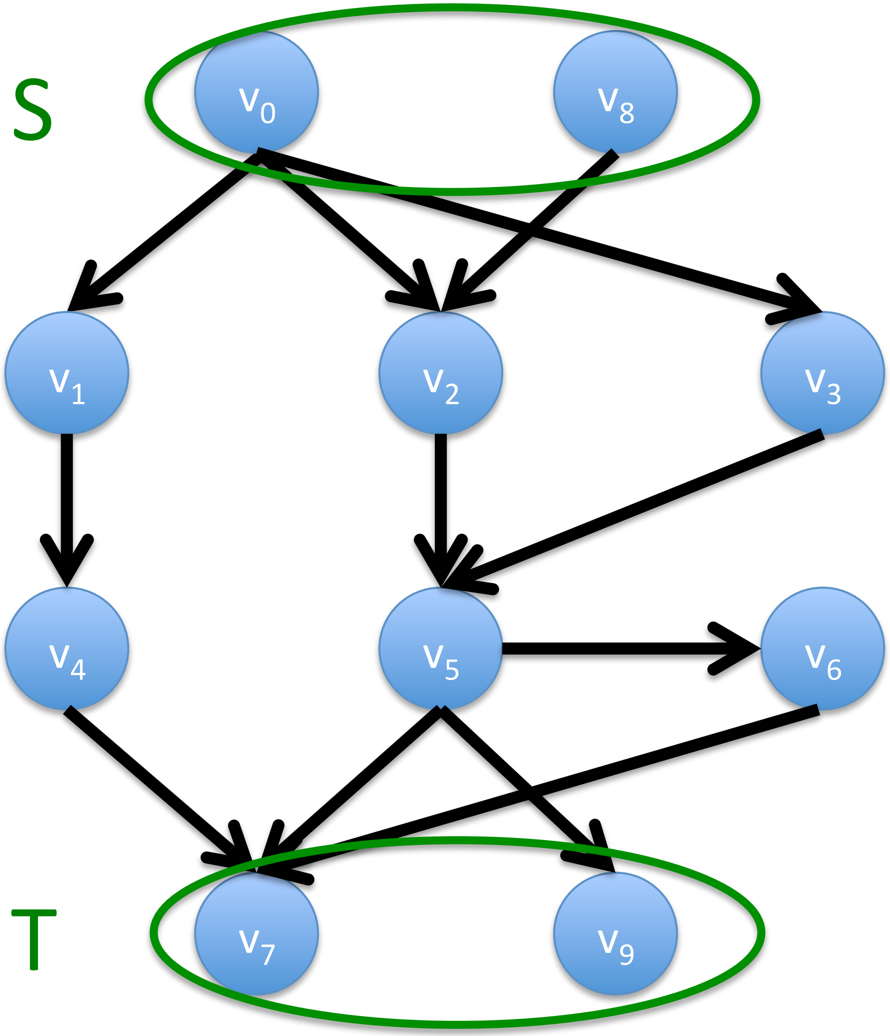


Minimum disconnecting vertex set = U. |U| = k

Proved by induction

Mathematical Induction on |A|

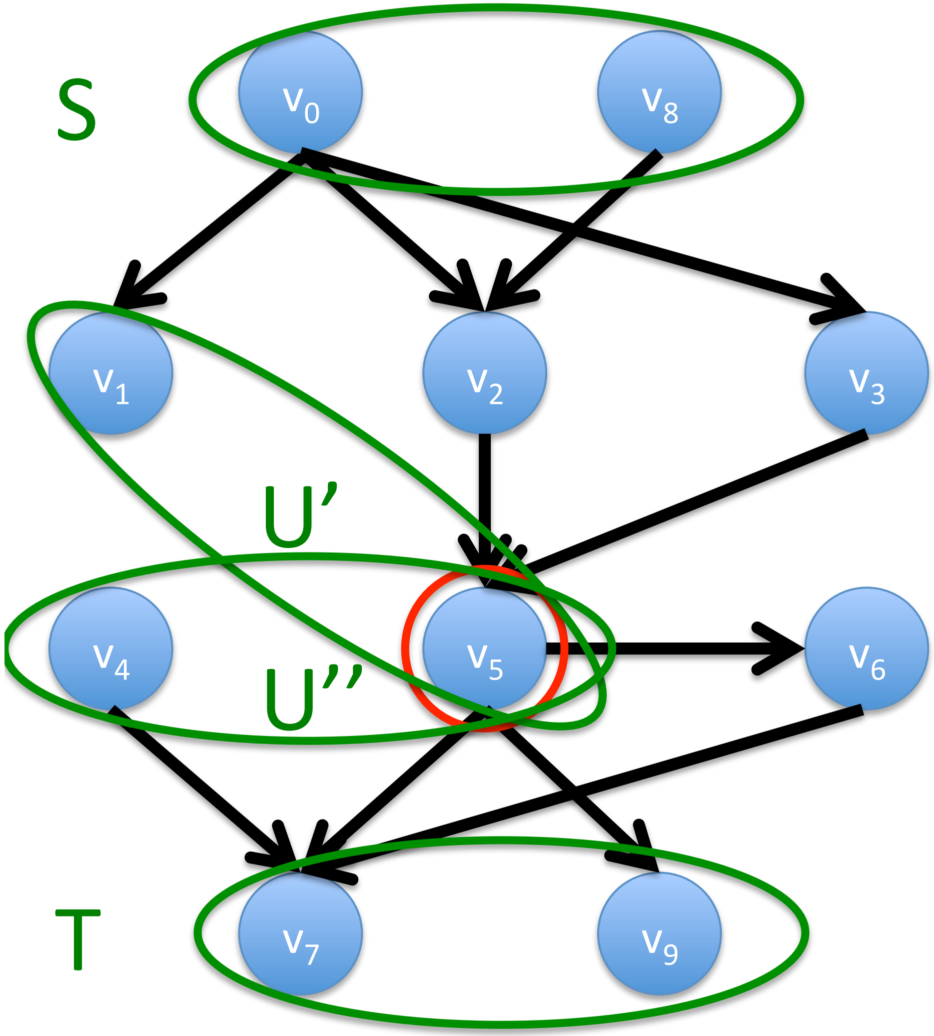
Menger's Theorem



Remove an arc (u,v)

Mathematical Induction on |A|

Menger's Theorem



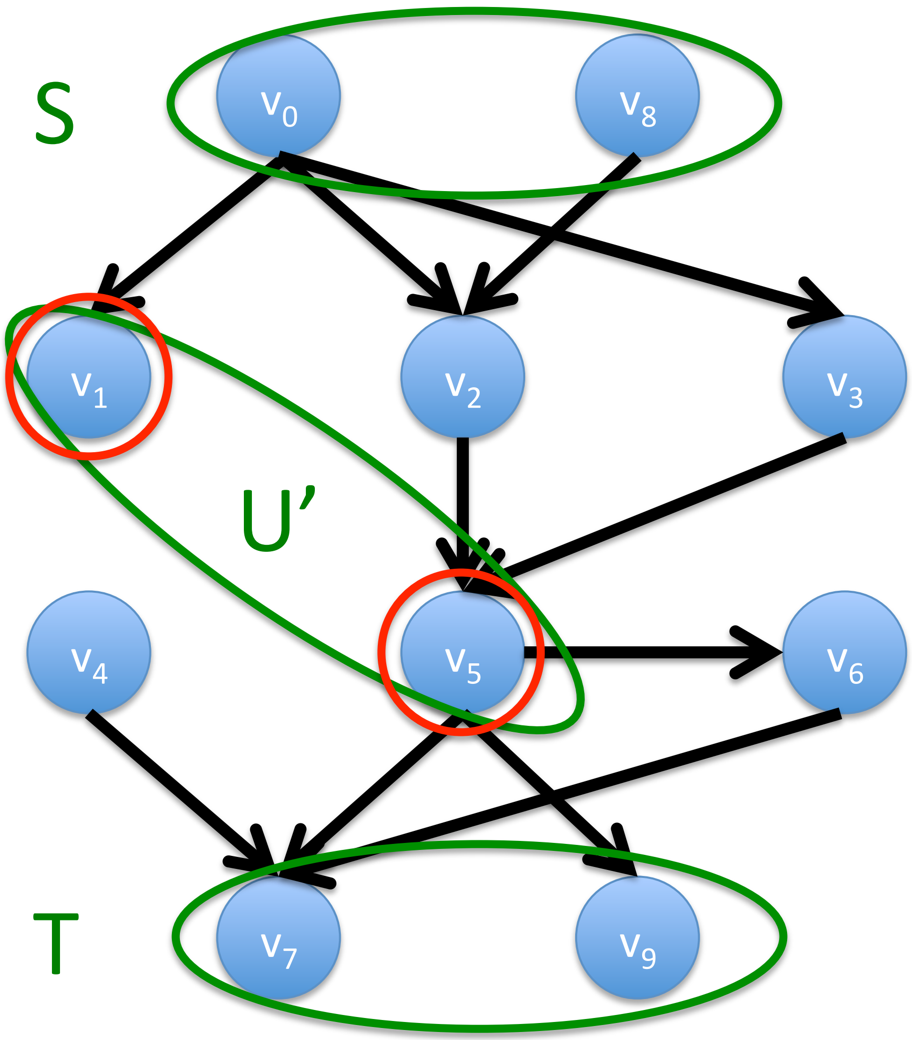
Minimum disconnecting vertex set U. |U| = k-1

U' = U union {u}

U'' = U union {v}

Mathematical Induction on |A|

Menger's Theorem



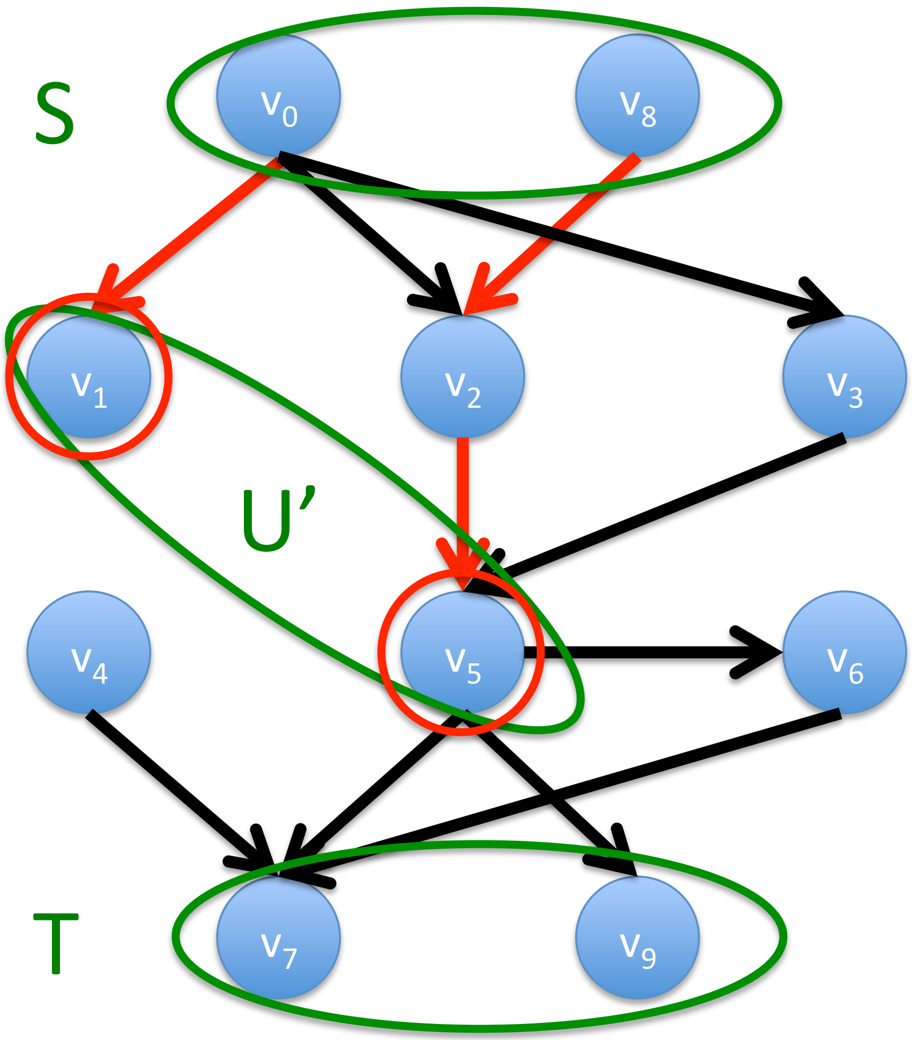
U' is an S-T disconnecting vertex set of size k in D

Any S-U' disconnecting vertex set has size ≥ k

Graph contains k disjoint S-U' paths (induction)

Mathematical Induction on |A|

Menger's Theorem



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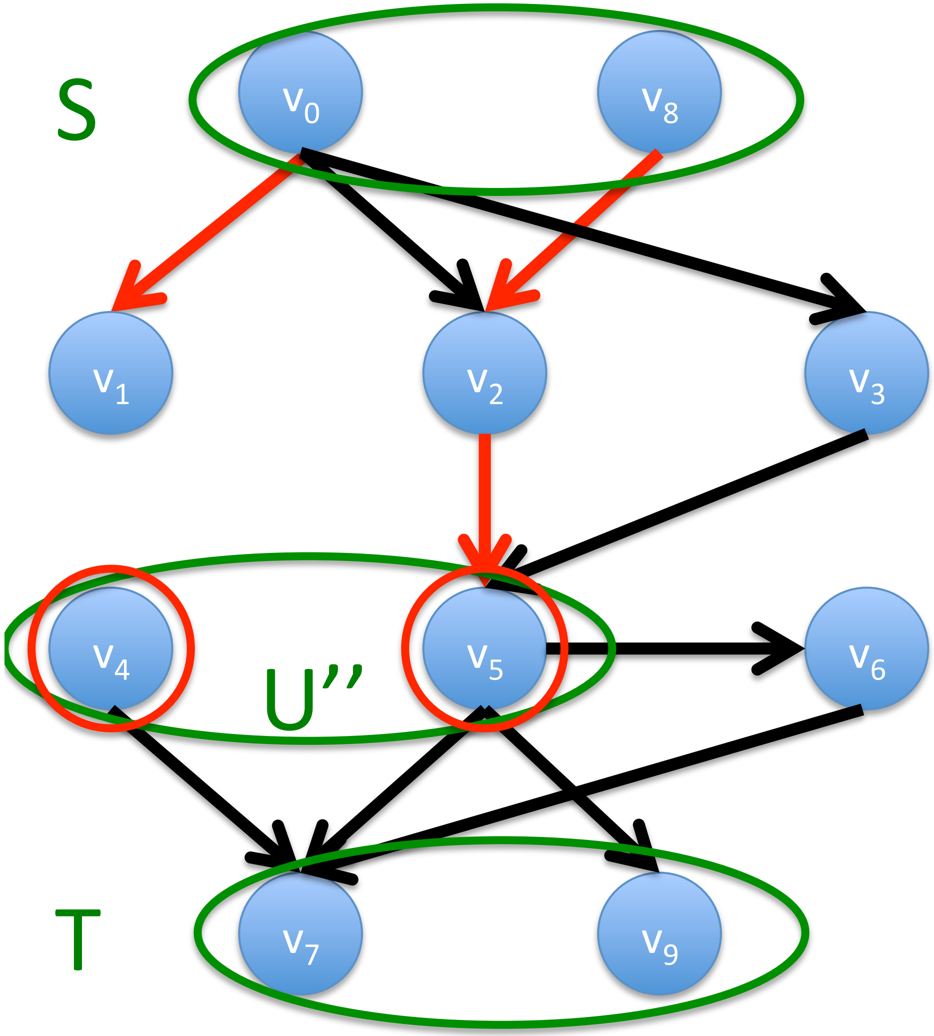


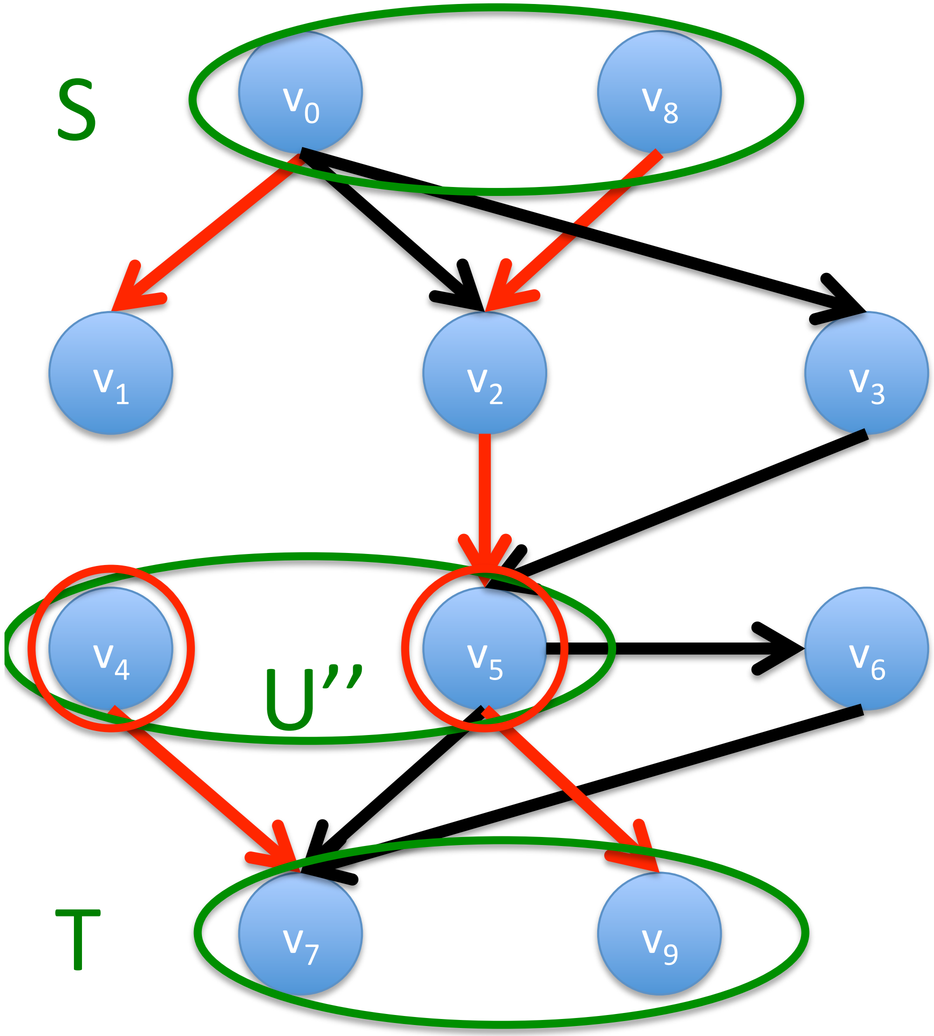
U'' is an S-T disconnecting vertex set of size k in D

Any U''-T disconnecting vertex set has size ≥ k

Graph contains k disjoint U''-T paths (induction)

Mathematical Induction on |A|

Menger's Theorem



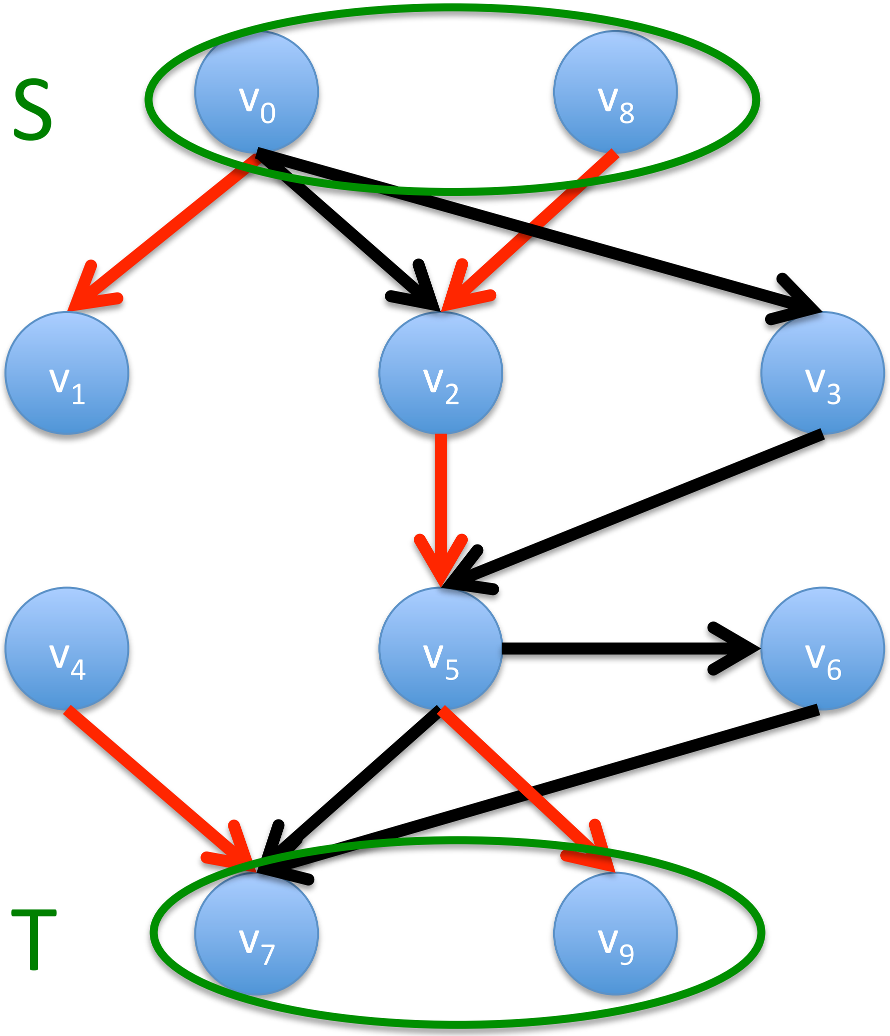
U'' is an S-T disconnecting vertex set of size k in D

Any U''-T disconnecting vertex set has size ≥ k

Graph contains k disjoint U''-T paths (induction)

Mathematical Induction on |A|

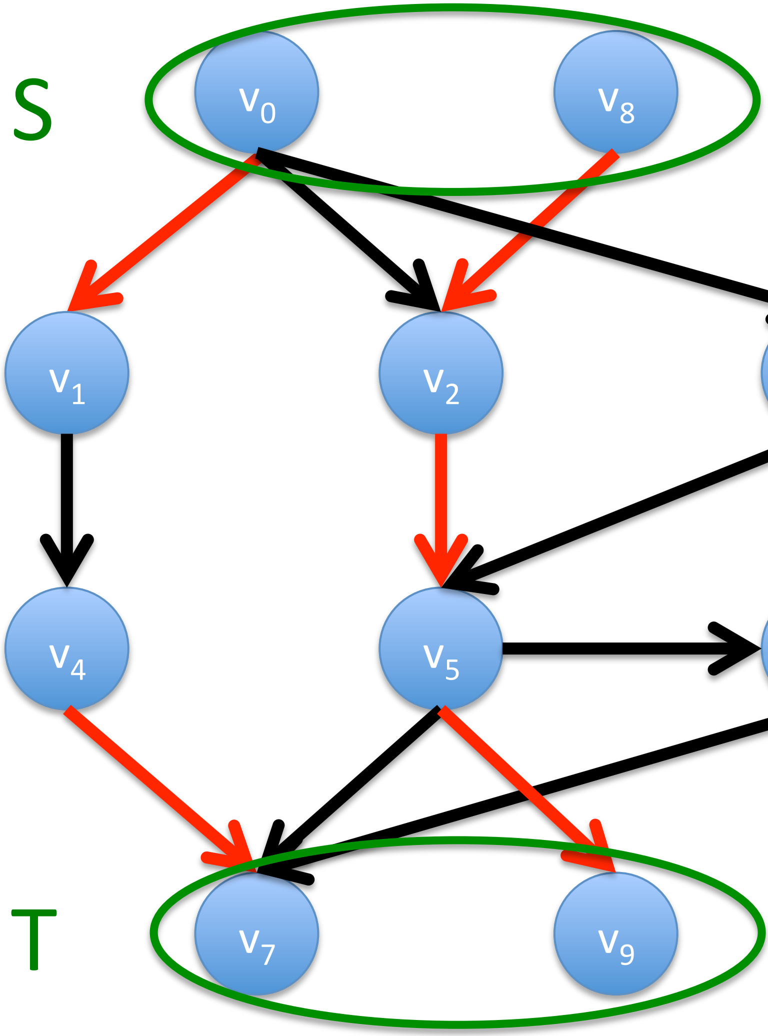
Menger's Theorem



Add arc (u,v) back

Mathematical Induction on |A|

Menger's Theorem



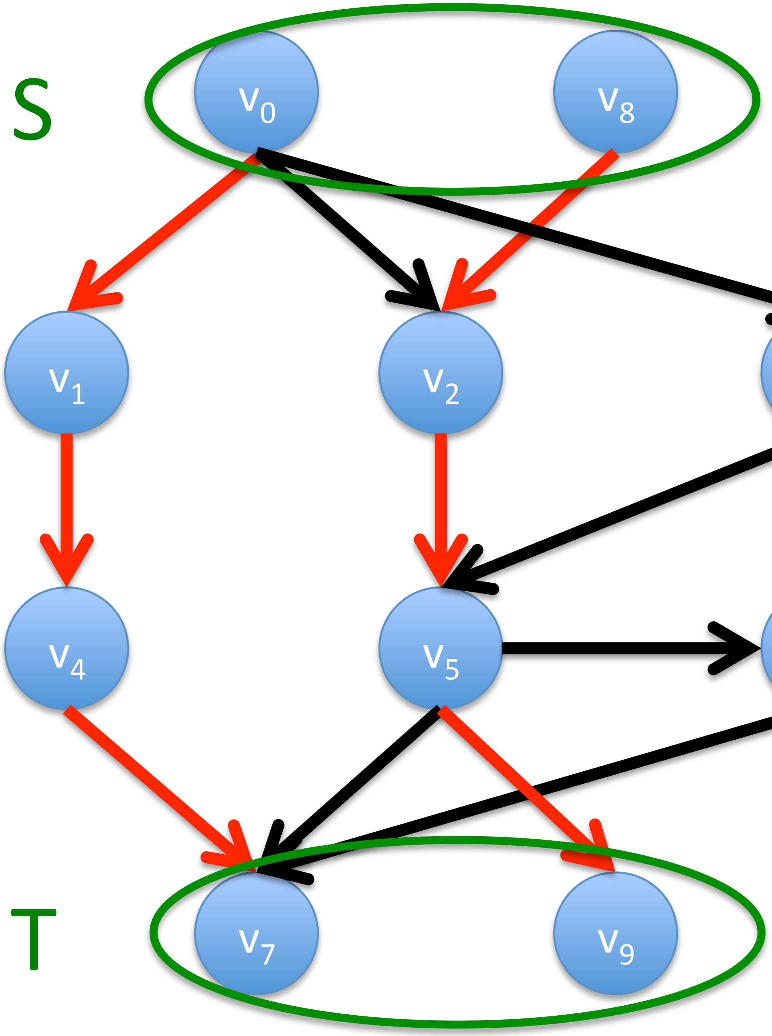
Add arc (u,v) back

(k-1) disjoint pairs of paths intersecting in U

1 disjoint pair of paths now connected by (u,v)

Mathematical Induction on |A|

Menger's Theorem



Total k disjoint paths

Hence proved

Mathematical Induction on |A|

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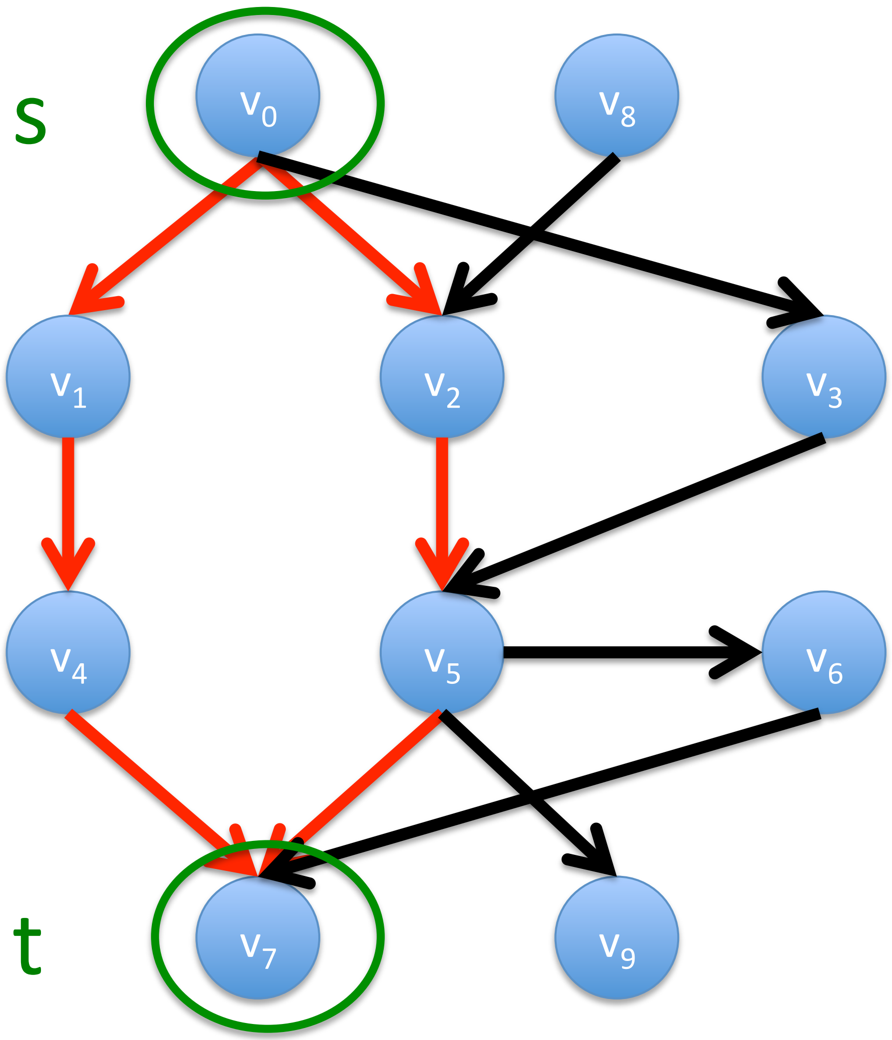
  - Vertex Disjoint S-T Paths
  - **Internally Vertex Disjoint s-t Paths**
  - Arc Disjoint s-t Paths
- Path Packing

Internally Vertex Disjoint s-t Paths

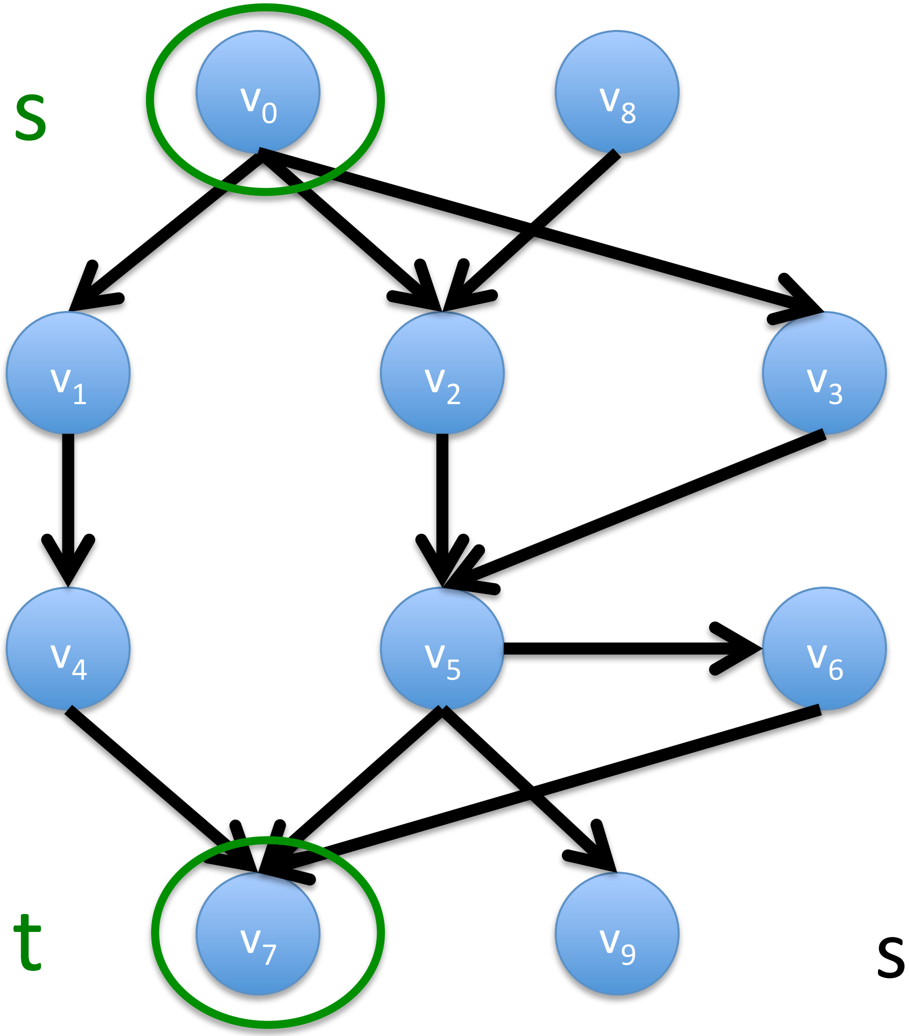


Maximum number of disjoint paths?

Minimum size of s-t vertex-cut !!

Set of s-t Paths with no common internal vertex

s-t Vertex-Cut

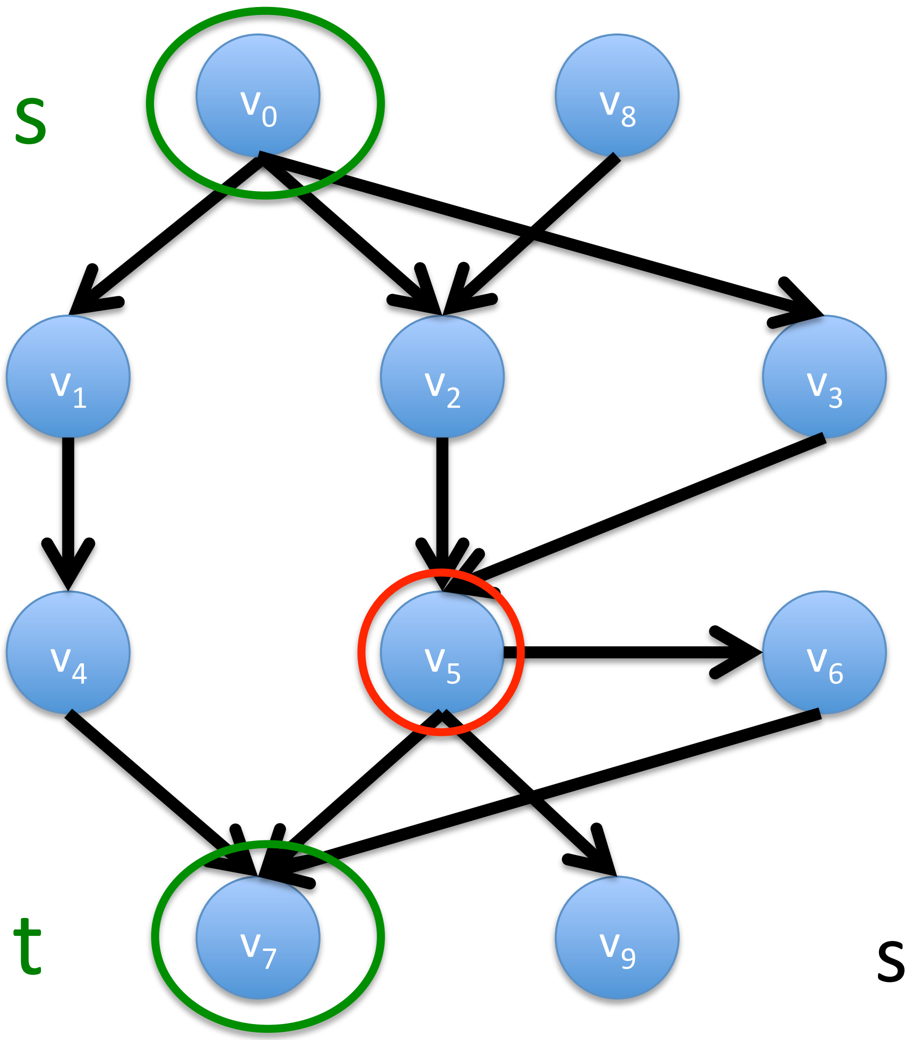


s not in U

t not in U

Subset U of V which intersects with all s-t Paths

s-t Vertex-Cut



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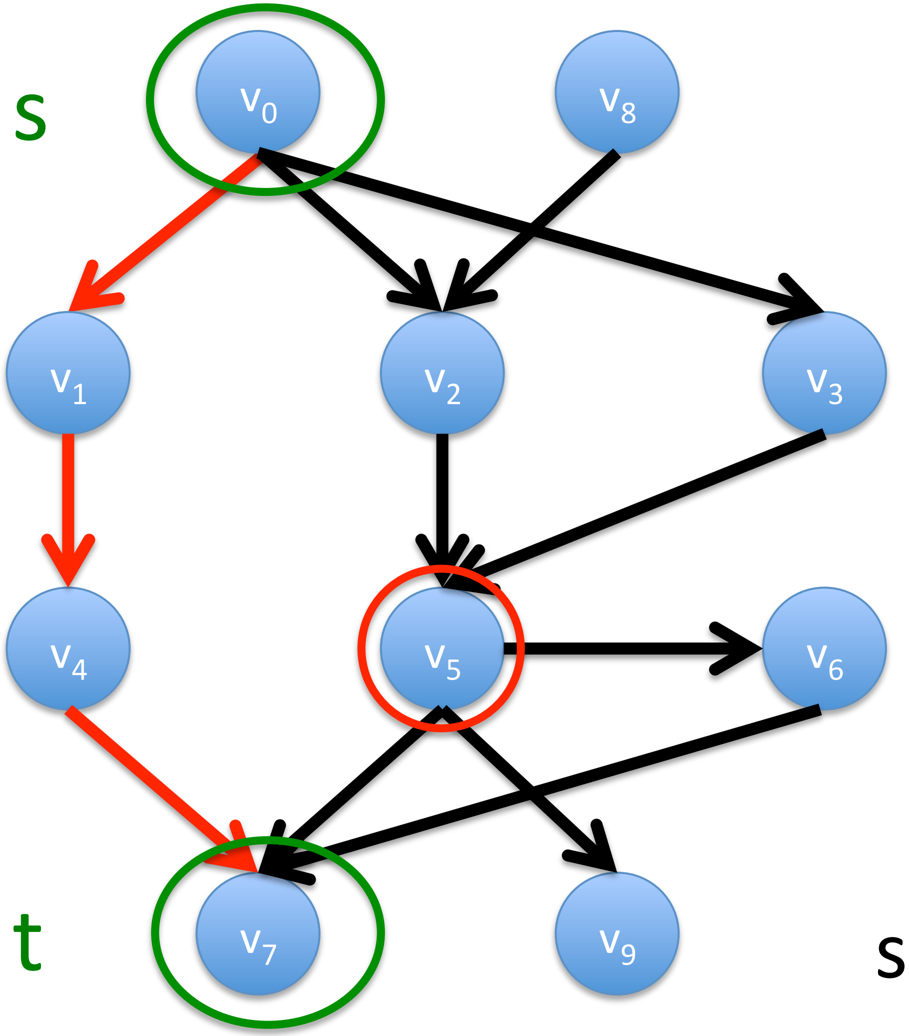
Subset U of V which intersects with all s-t Paths

s-t Vertex-Cut



✗

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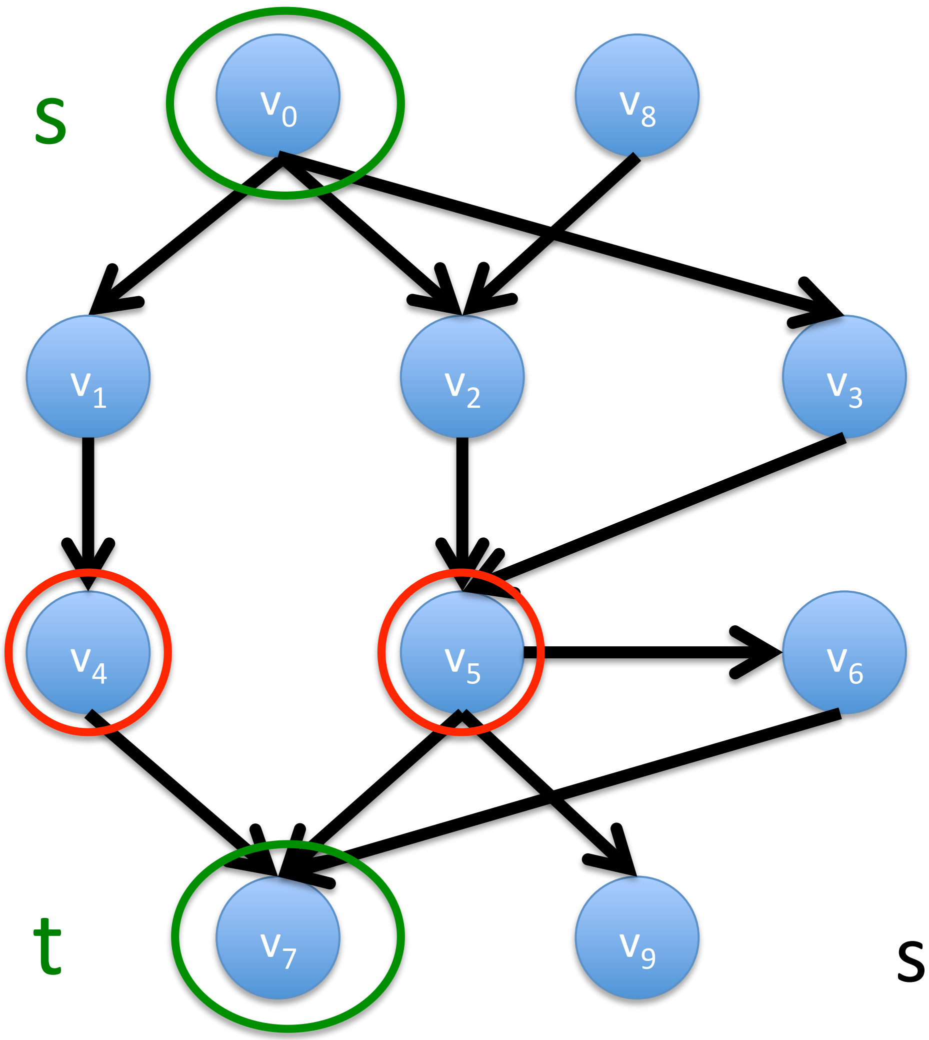
Subset U of V which intersects with all s-t Paths

s-t Vertex-Cut



✓

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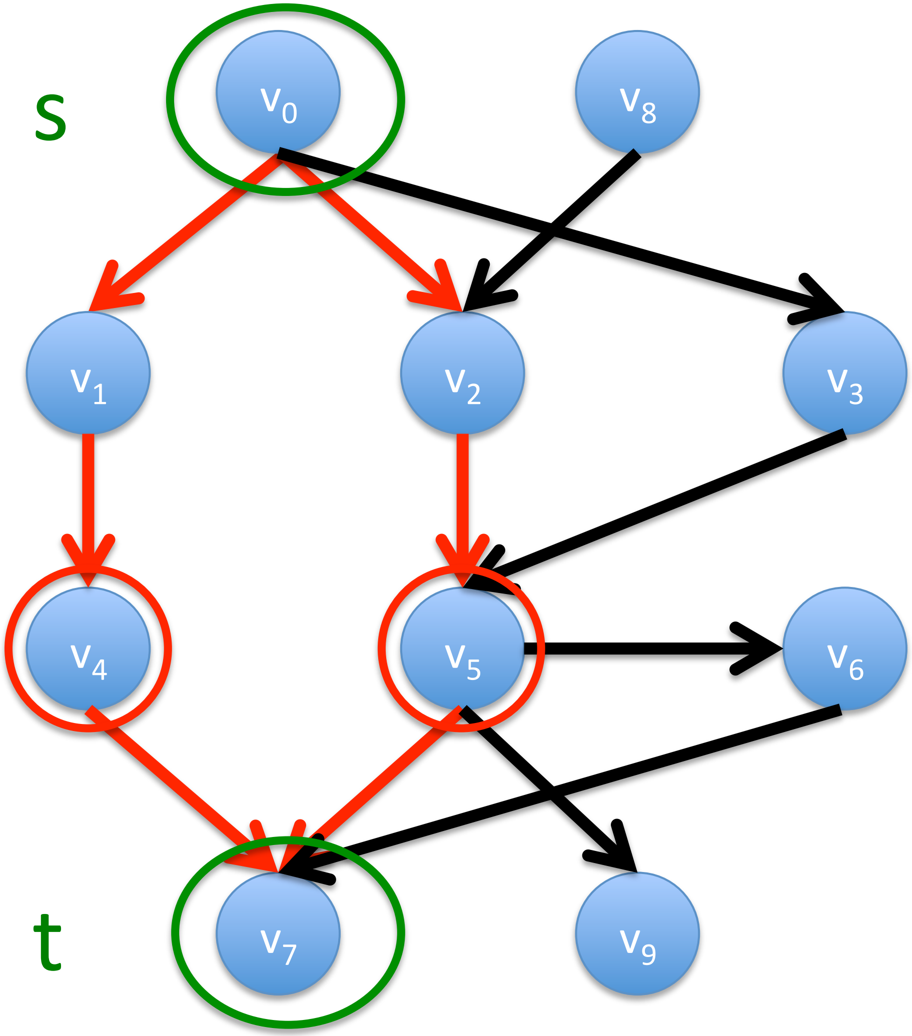
Subset U of V which intersects with all s-t Paths

Connection

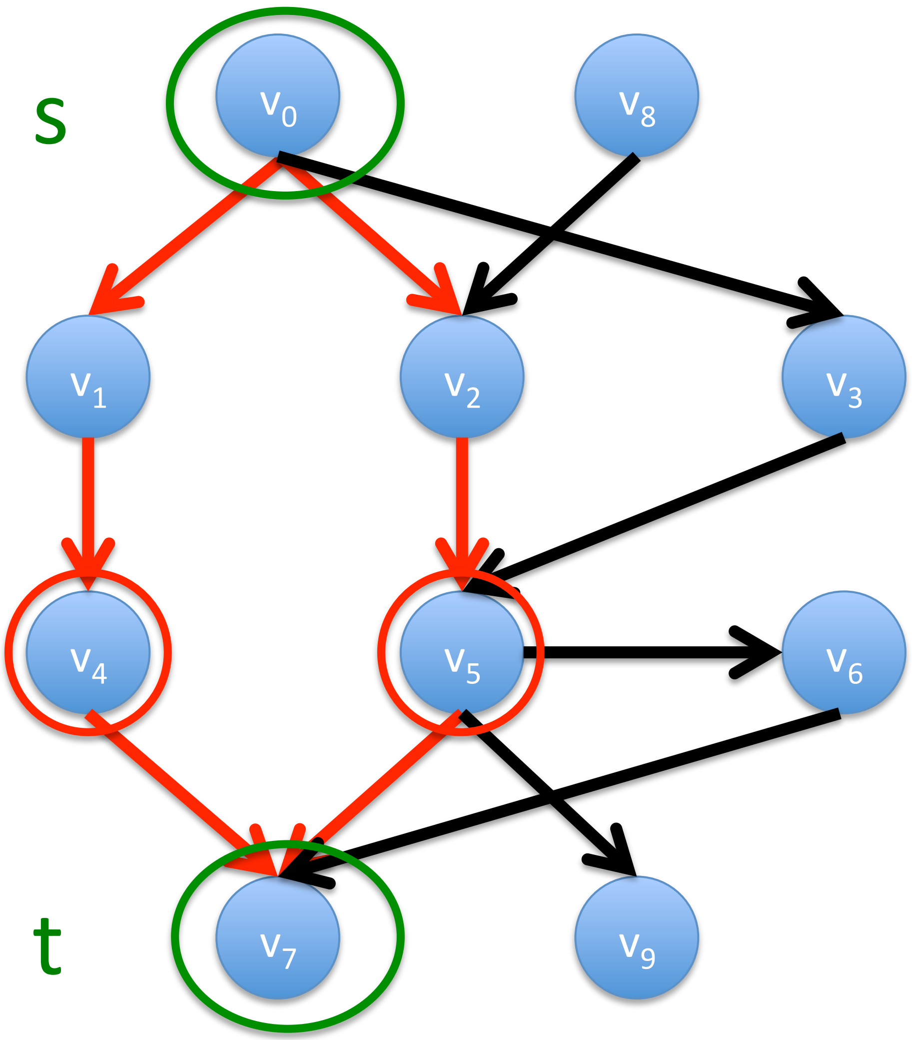


Maximum number of disjoint paths?

≤

Minimum size of s-t vertex-cut !!

Menger's Theorem



Maximum number of disjoint paths?

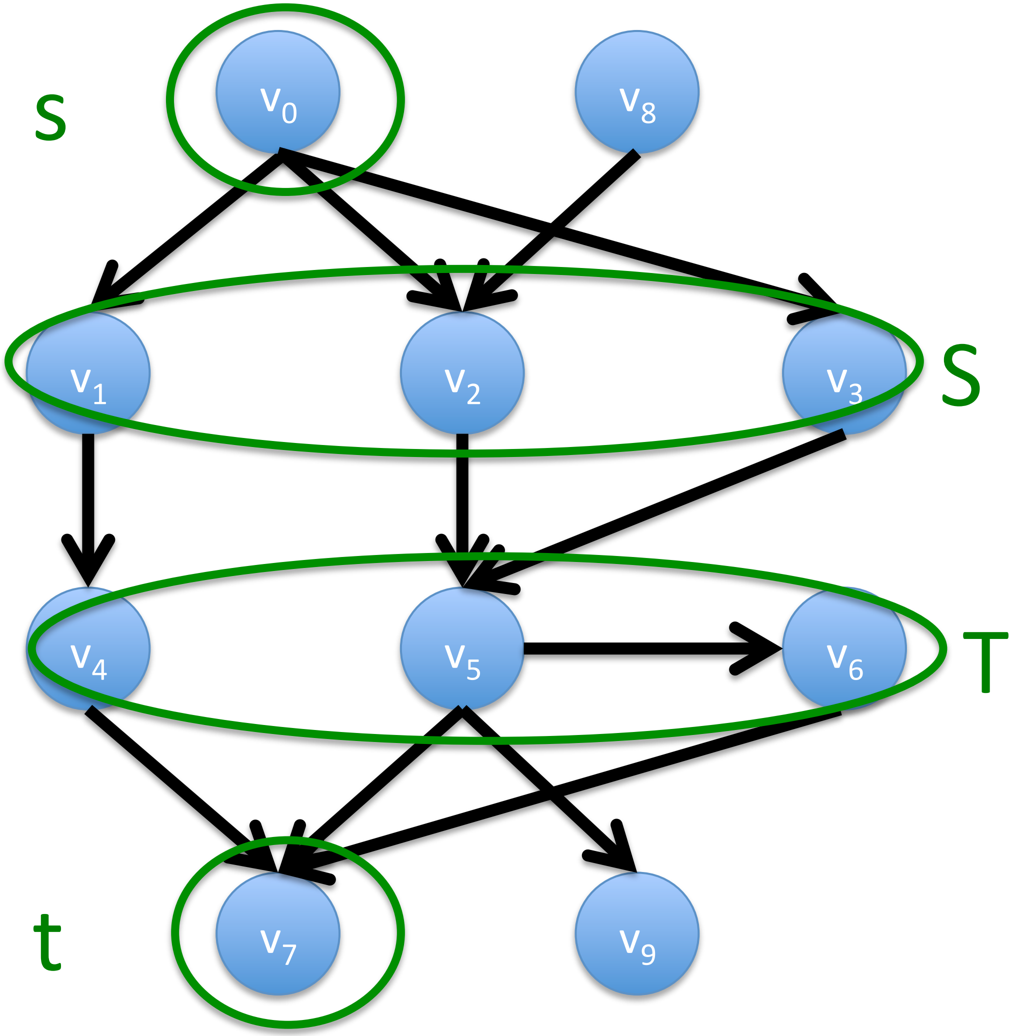
=

Minimum size of s-t vertex-cut !!

Proof ?

Menger's Theorem for vertex disjoint S-T paths

Menger's Theorem



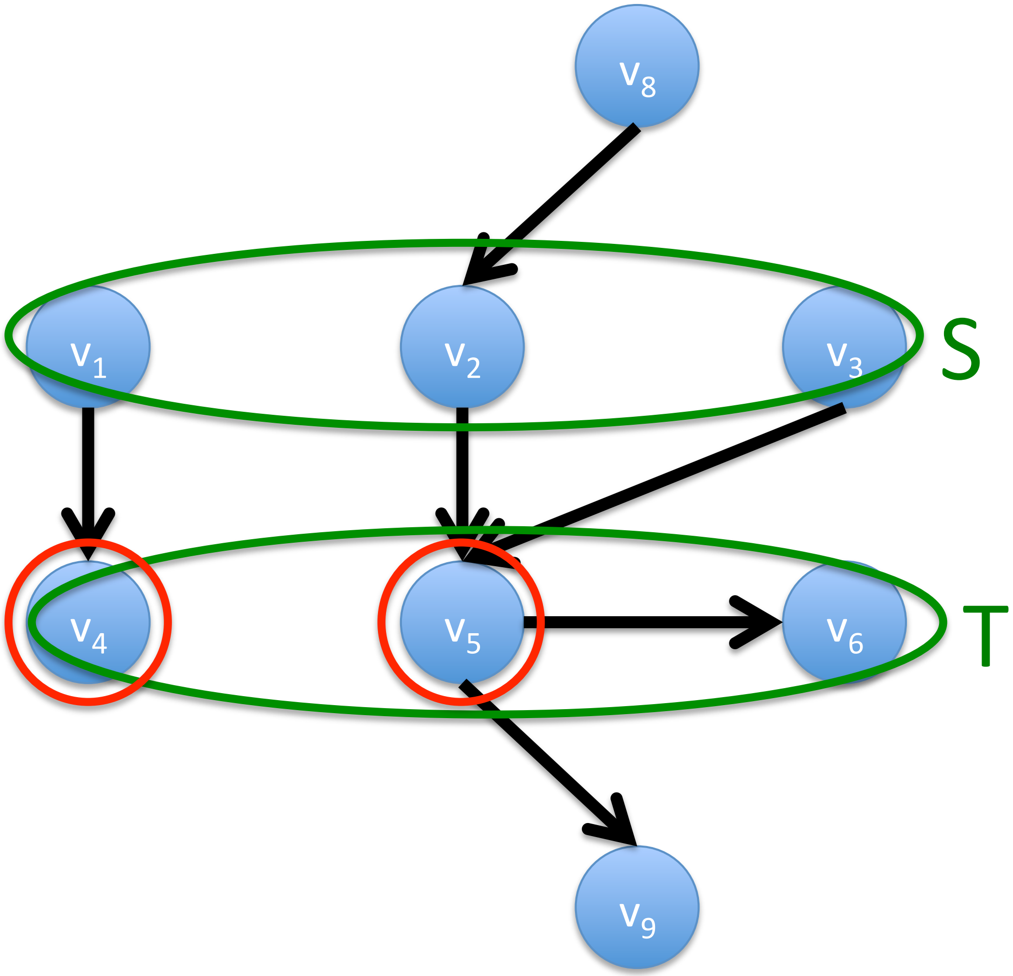
S = outneighbors of s

T = inneighbors of t

Remove s and t

Menger's Theorem for vertex disjoint S-T paths

Menger's Theorem



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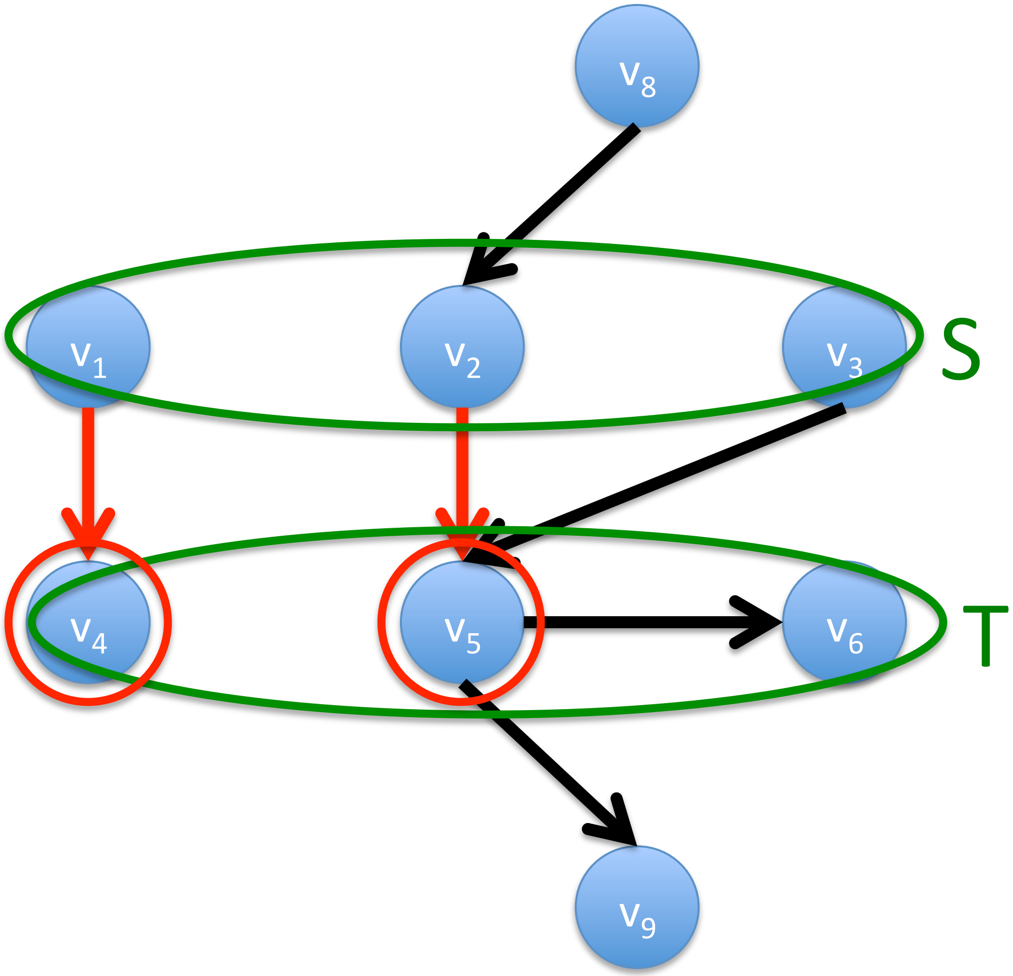
Menger's Theorem for vertex disjoint S-T paths

Menger's Theorem



S = outneighbors of s

T = inneighbors of t

Remove s and t

Hence proved

Menger's Theorem for vertex disjoint S-T paths

Menger's Theorem

Theorem for vertex disjoint S-T paths

implies

Theorem for internally vertex disjoint s-t paths

Menger's Theorem

Theorem for vertex disjoint S-T paths

is implied by

Theorem for internally vertex disjoint s-t paths

Left As Exercise !!!

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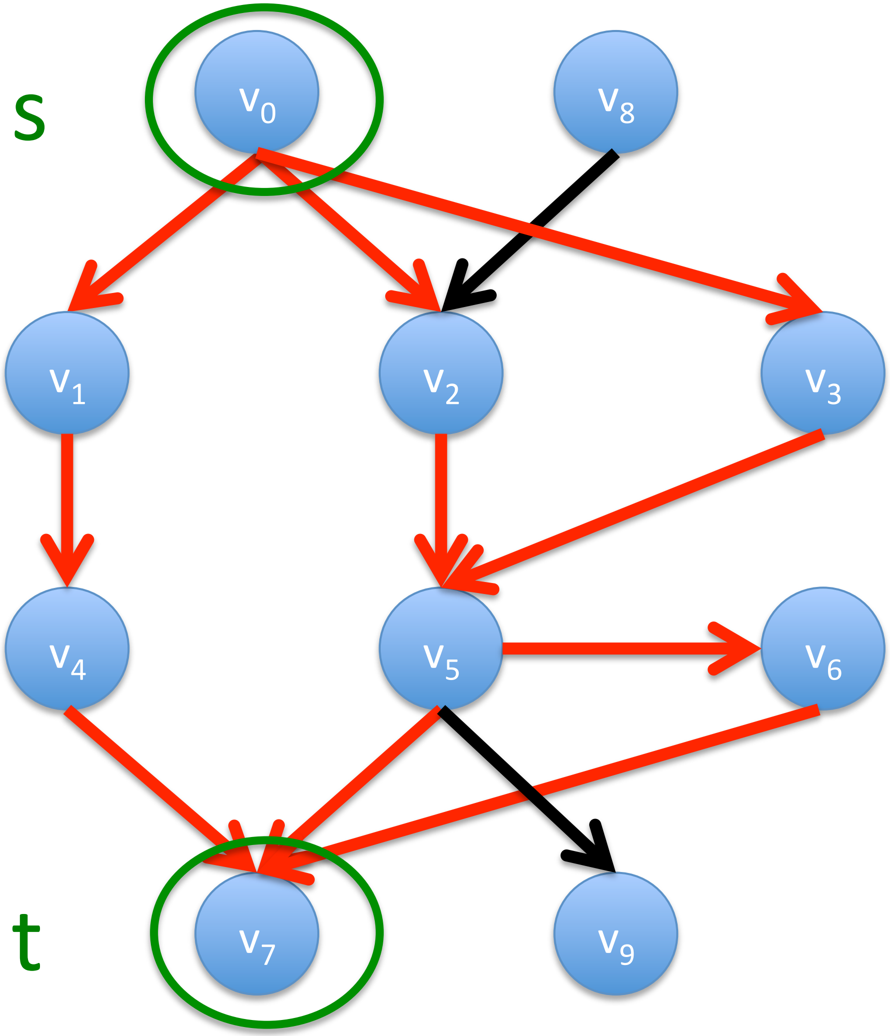
  - Internally Vertex Disjoint s-t Paths
  - **Arc Disjoint s-t Paths**
- Path Packing

Arc Disjoint s-t Paths

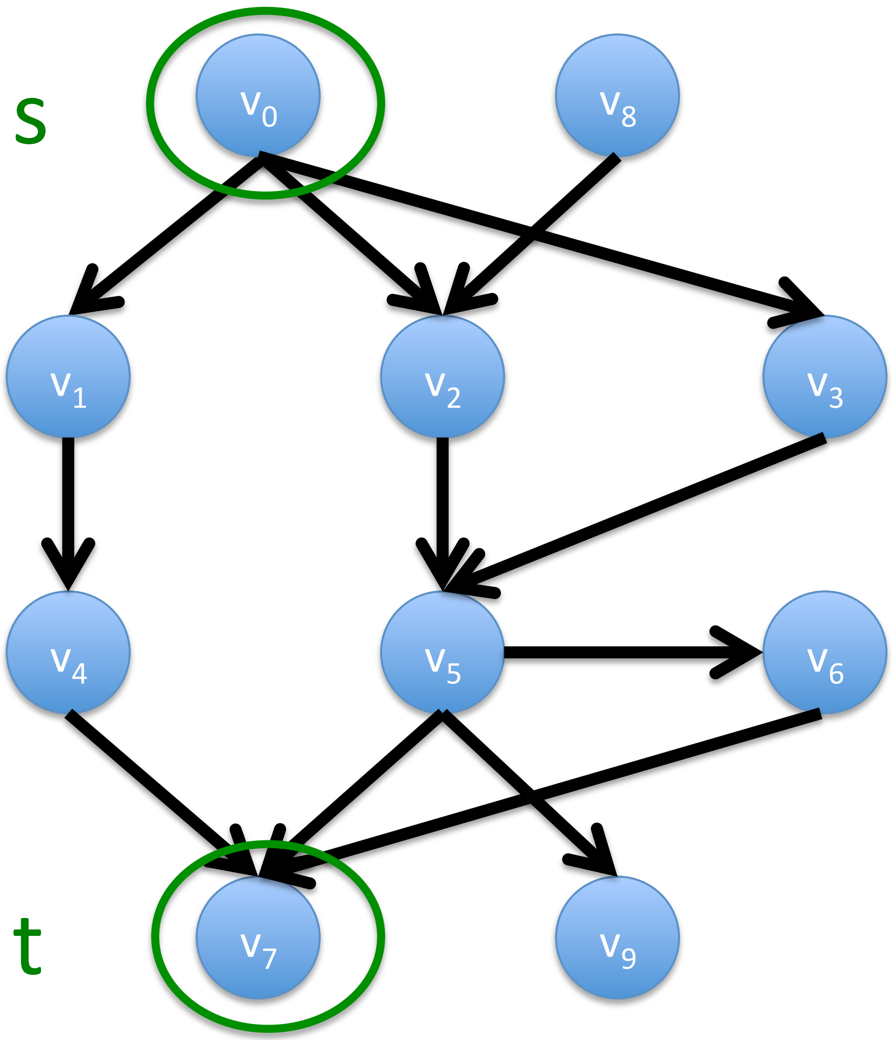


Maximum number of disjoint paths?

Minimum size of s-t cut !!

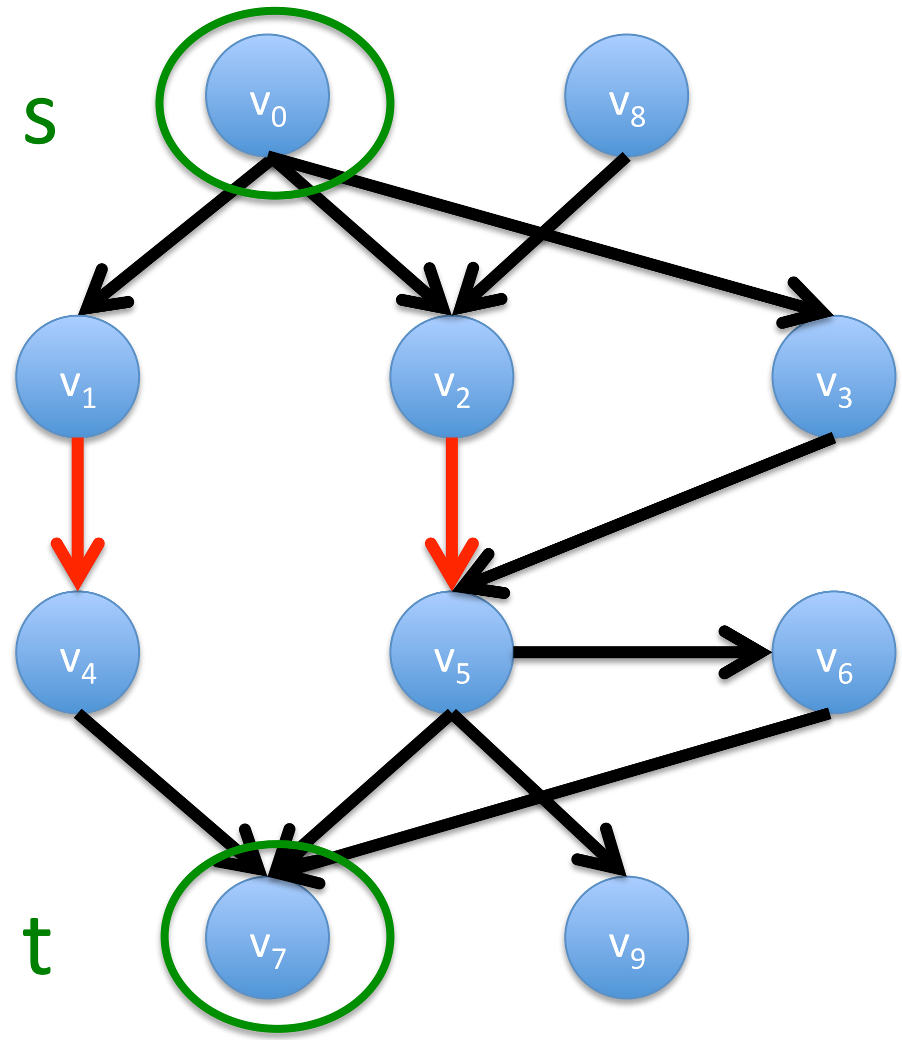
Set of s-t Paths with no common arcs

s-t Cut



C = out-arcs(U) s in U t not in U

s-t Cut



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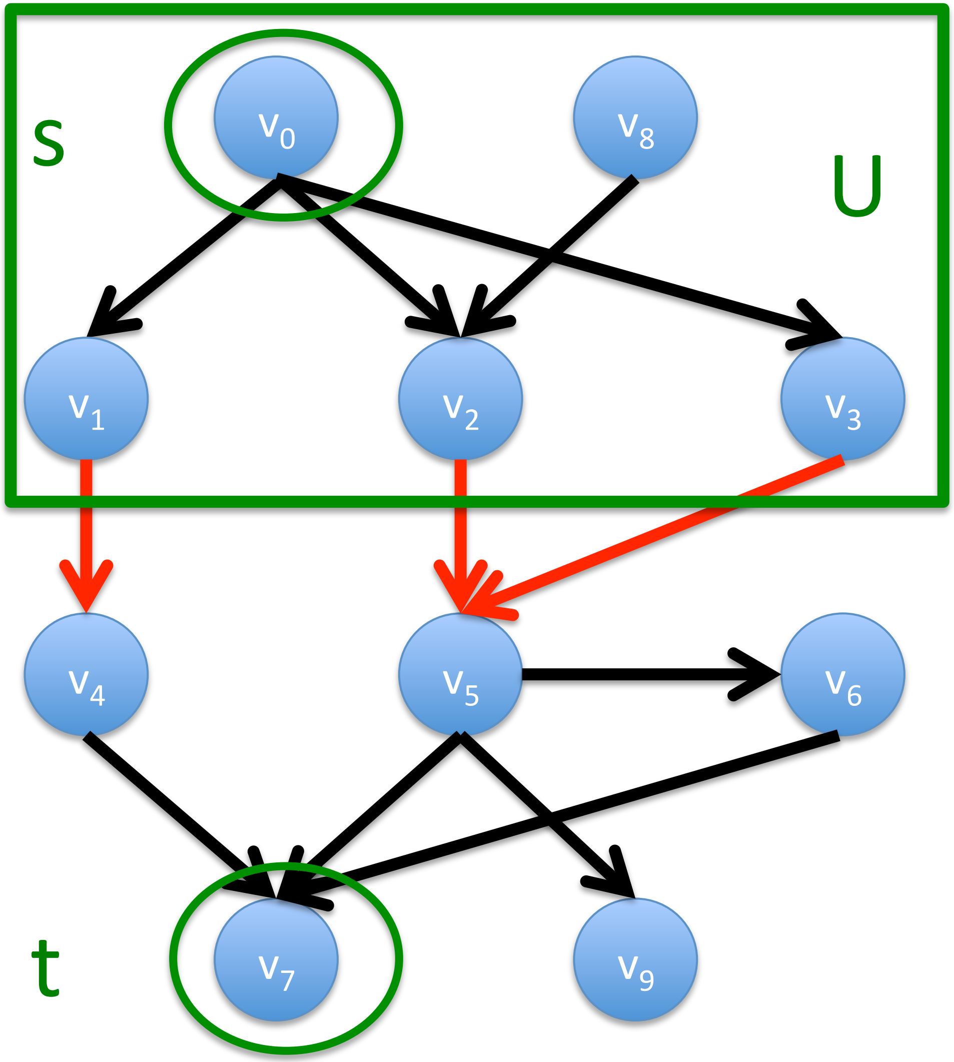
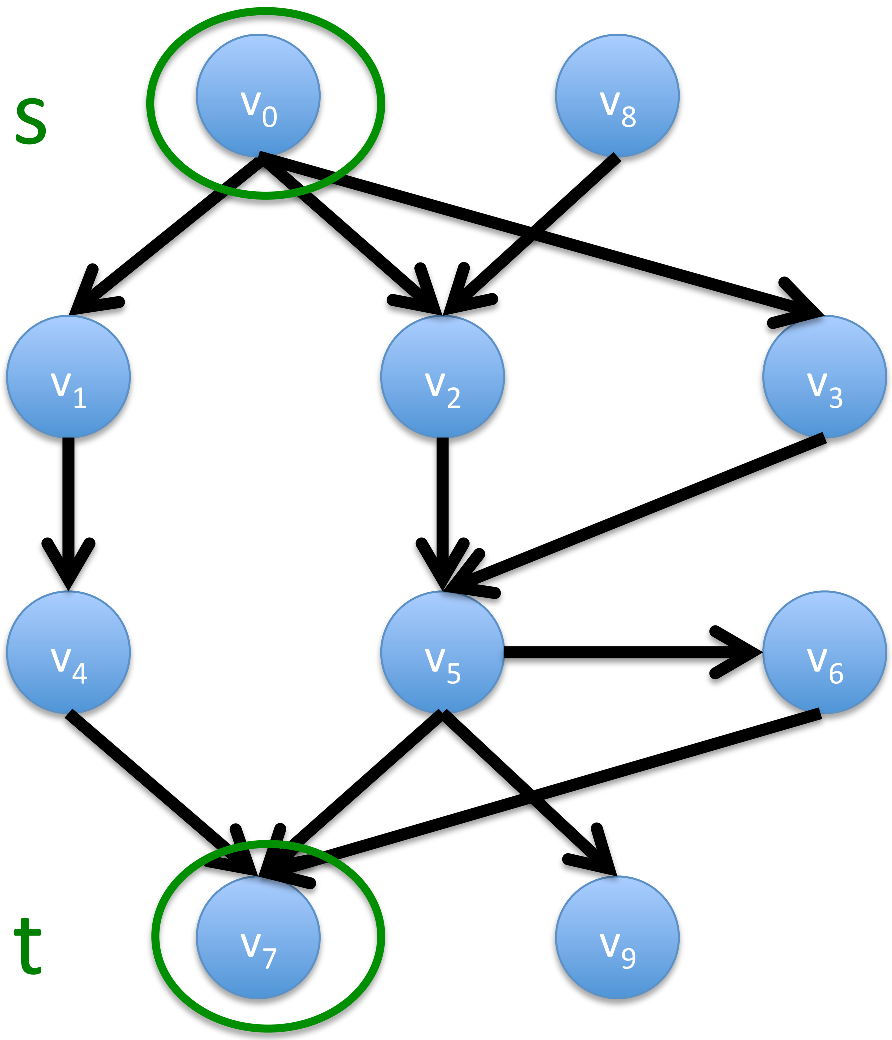
s-t Cut



✓

C = out-arcs(U)   s in U   t not in U

Connection



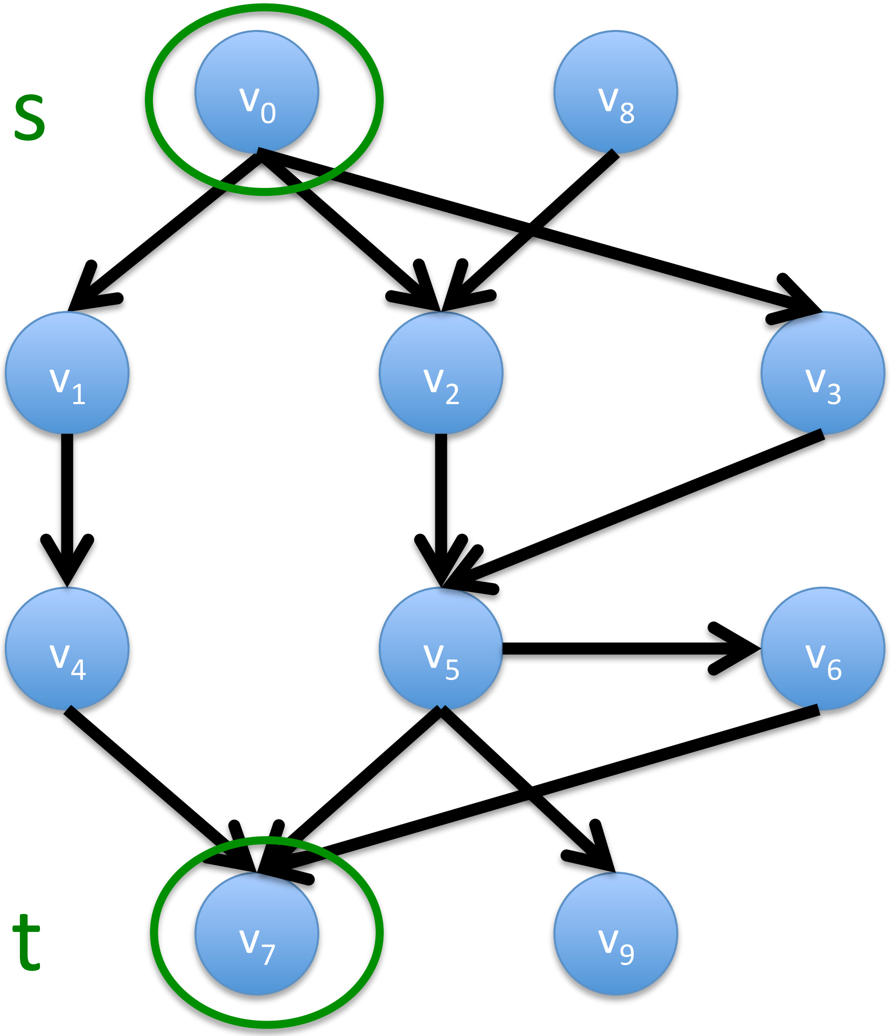
Maximum number of disjoint paths?

≤

Minimum size of s-t cut !!

Minimum set of arcs intersecting all s-t paths is a cut

Menger's Theorem



Maximum number of disjoint paths?

=

Minimum size of s-t cut !!

Proof ?

Menger's Theorem for vertex disjoint S-T paths

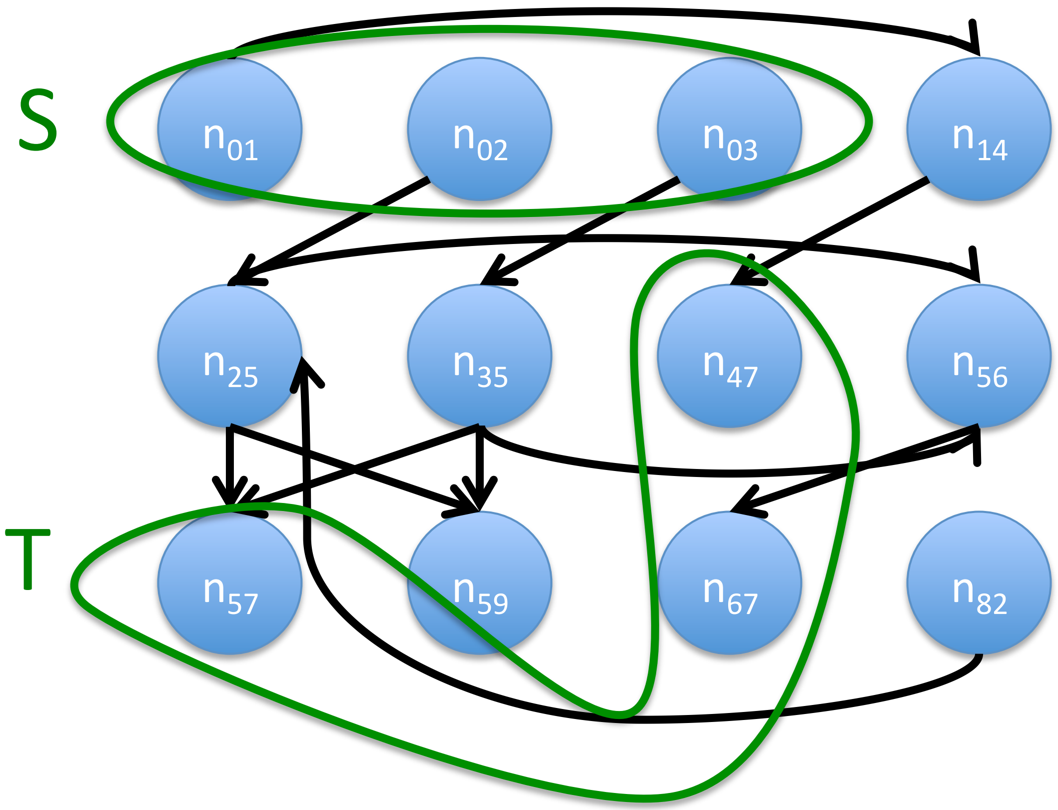
Line Digraph

L(D) is a line digraph of D

Node n_{ij} of L(D) corresponds to arc (v_i,v_j) in D

Arc (n_{ij},n_{kl}) exists if and only if $j = k$

Menger's Theorem



S = out-arcs(s)

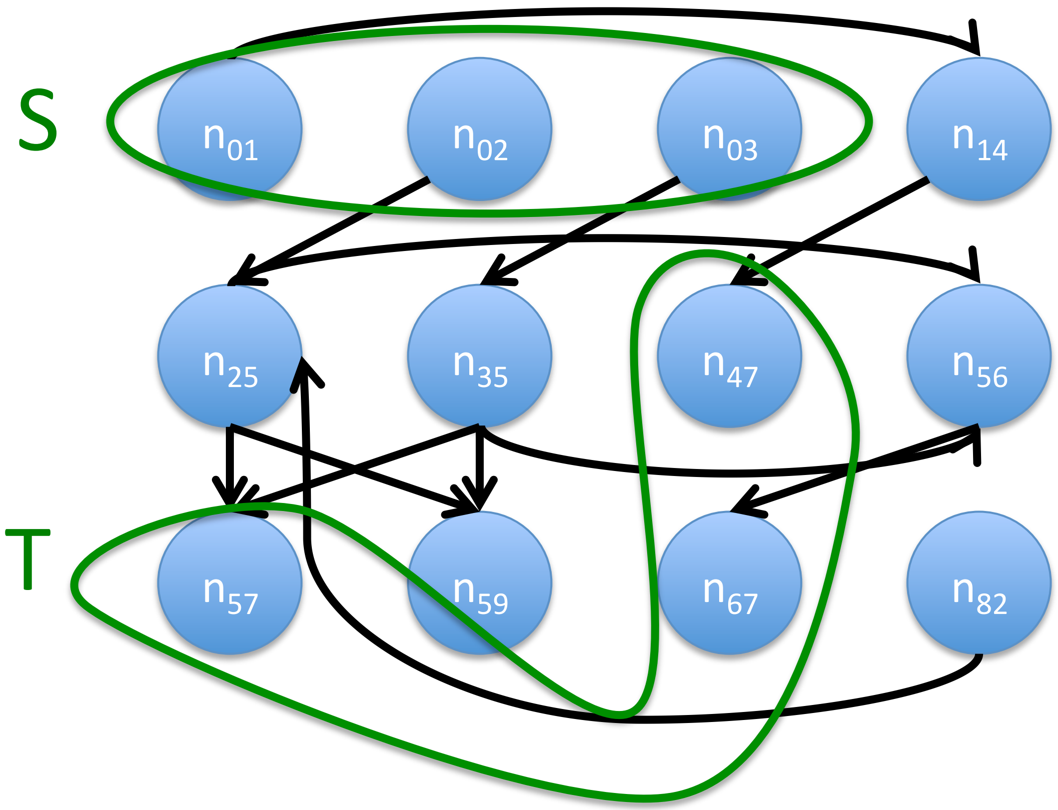
T = in-arcs(t)

Vertex disjoint S-T path in L(D)

implies

Arc disjoint s-t path in D

Menger's Theorem



S = out-arcs(s)

T = in-arcs(t)

Hence proved

Minimum size S-T disconnecting vertex set

implies

s-t cut

Menger's Theorem

Theorem for vertex disjoint S-T paths

implies

Theorem for arc disjoint s-t paths

Menger's Theorem

Theorem for vertex disjoint S-T paths

is implied by

Theorem for arc disjoint s-t paths

Left As Exercise !!!

Outline

- Preliminaries
- Menger's Theorem for Disjoint Paths
- **Path Packing (todo)**
  - Description of the Algorithm
  - Analysis of the Algorithm