# Discrete Optimization

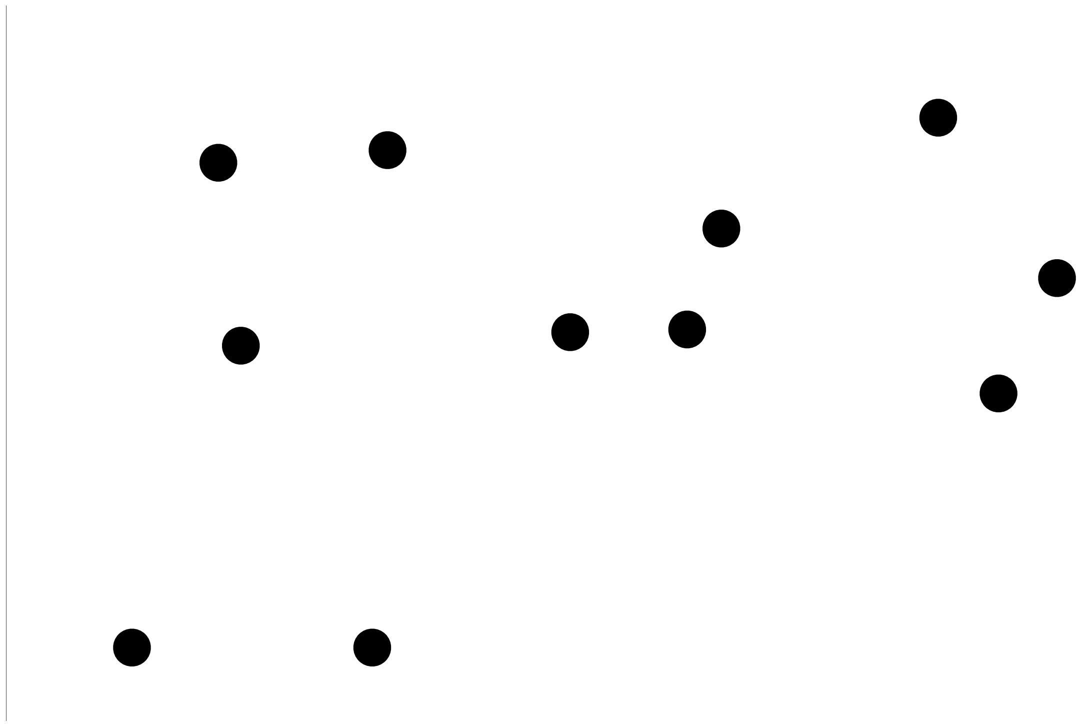
MA2827

*Fondements de l'optimisation discrète*

https://project.inria.fr/2015ma2827/

Material from P. Van Hentenryck's course

# Traveling salesman problem



# Traveling salesman problem

- MIP for TSP
  - Decision variables, constraints, objectives?
- Decision variables
  - Is an edge part of the tour or not?
- Constraints
  - Degree constraints: Each node has exactly two edges selected

# MIP for TSP

- Decision variables
  - $x_e$ is 1 if edge e is selected
- Notation
  - V: set of vertices
  - E: set of edges
  - δ(v): edges adjacent to vertex v
  - δ(S): edges with exactly one vertex in S (subset of V)
  - γ(S): edges with both vertices in S
  - $x_{\{e_1,\ldots,e_n\}} = x_{e_1} + \ldots + x_{e_n}$

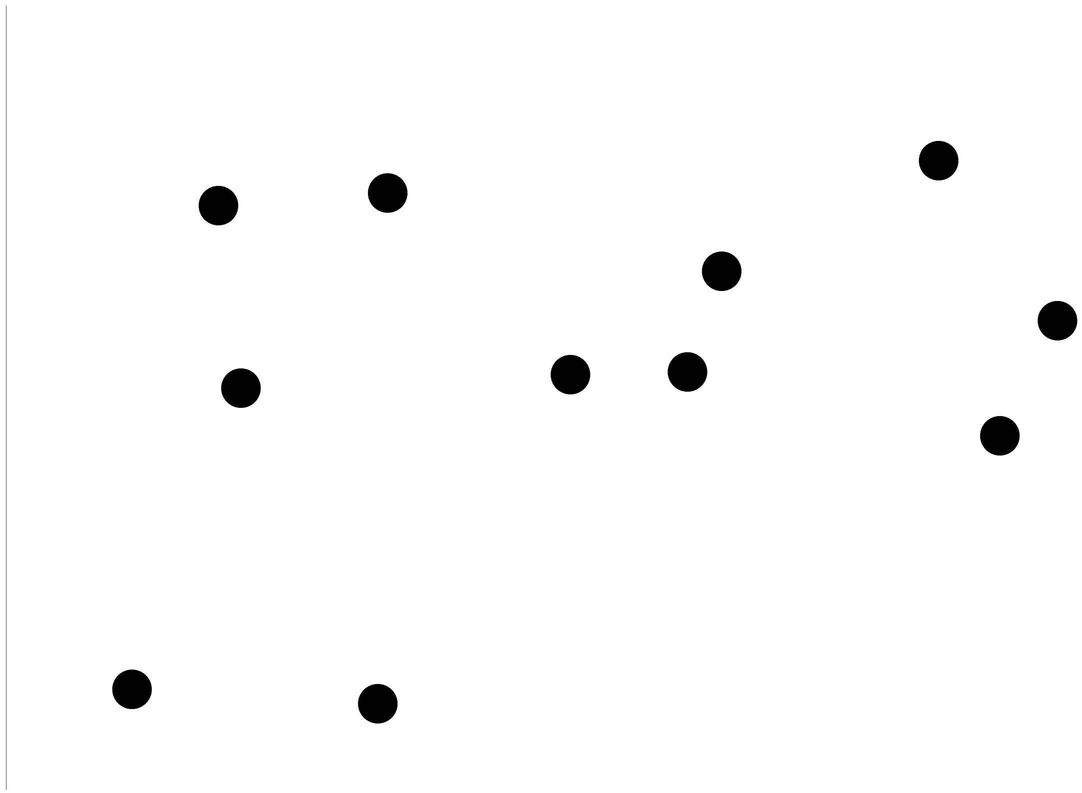
# MIP for TSP

flow conservation

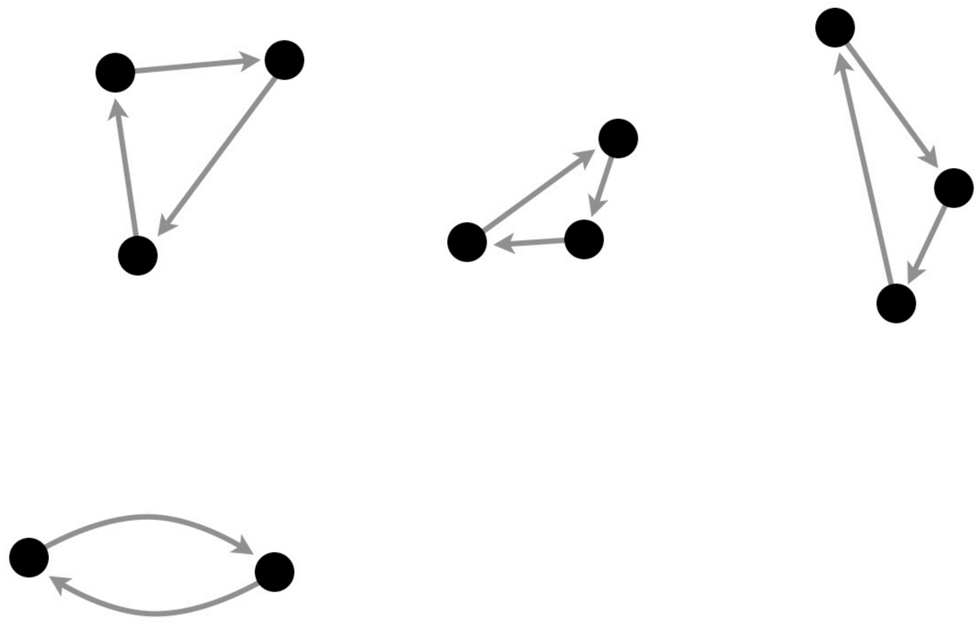
$$
\begin{aligned}
\text{min} \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \\
& x_{\delta(v)} = 2 \qquad v \in V \\
& x_e \in \{0, 1\} \quad e \in E
\end{aligned}
$$

minimize cost

# MIP for TSP

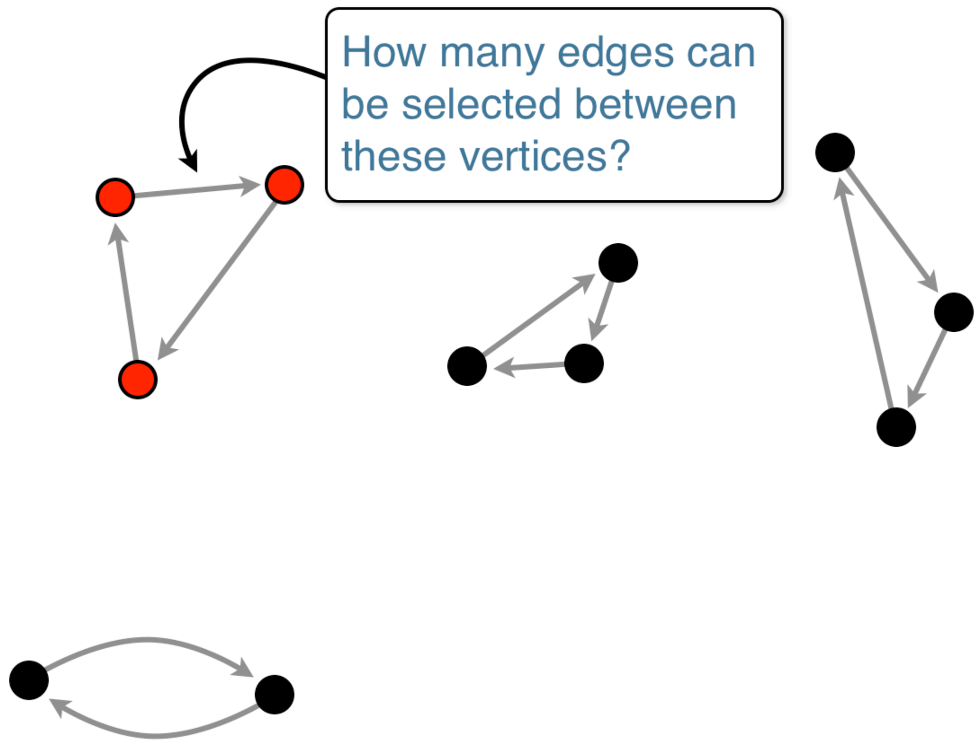


# MIP for TSP



# MIP for TSP

- Eliminate these subtours



# Subtour elimination

$$\begin{aligned}
\min \quad & \sum_{e \in E} c_e x_e \\
\text{subject to} \quad & \\
& x_{\delta(v)} = 2 && v \in V \\
& x_{\gamma(S)} \leq |S| - 1 && S \subset V \\
& x_e \in \{0, 1\} && e \in E
\end{aligned}$$

- Great, but too many (exponential no.) of them
- Branch and cut
  - Generate them on demand

# Subtour elimination

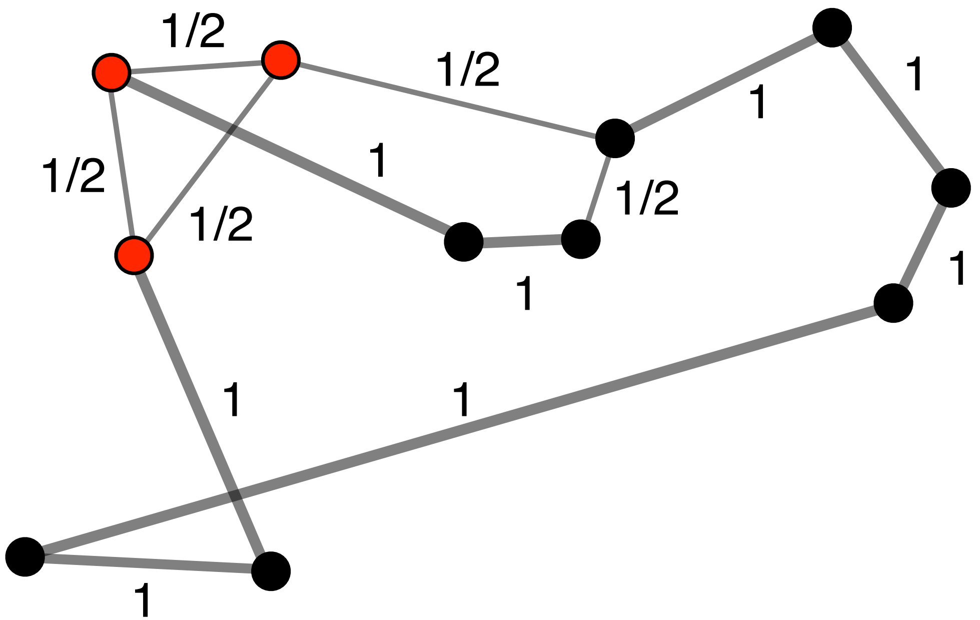
$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \\ & x_{\delta(v)} = 2 \qquad v \in V \\ & x_{\delta(S)} \geq 2 \qquad S \subset V \\ & x_e \in \{0,1\} \qquad e \in E \end{aligned}$$

- How to separate subtour constraints?

# Separation of subtour constraints

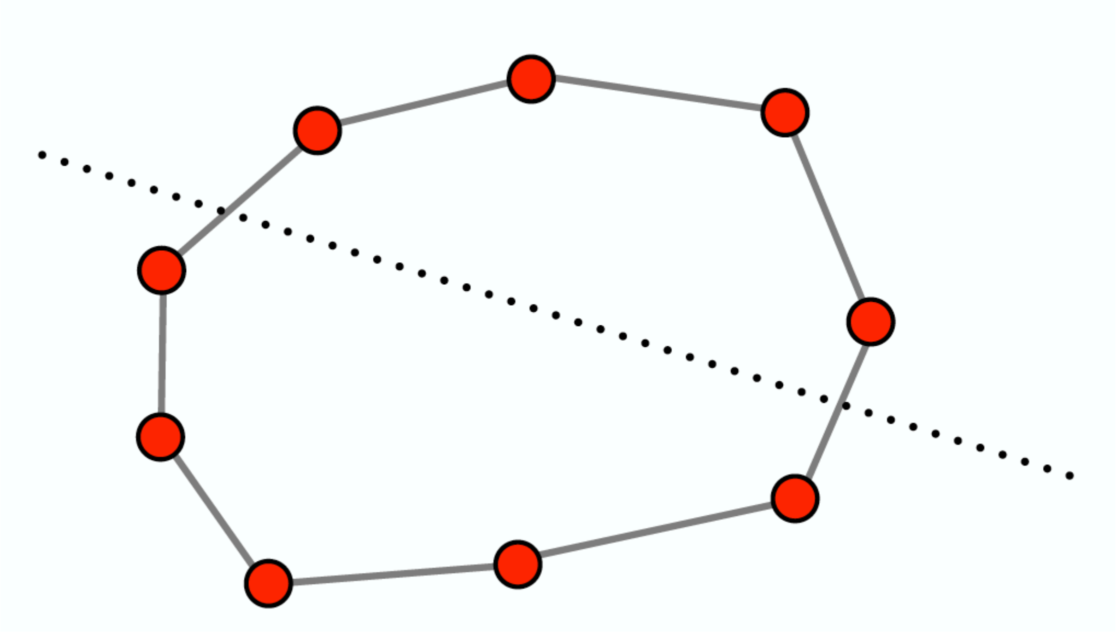
- Build a graph G* = (V, E) where
  - weight of an edge e, $w(e) = x_e^*$
- Finding a separation is equivalent to finding
  - A minimum cut in G*
  - If the cost of the cut is less than 2, then we have isolated a subtour constraint violated by the linear relaxation
  - Recall: Finding the cut takes polynomial time

# MIP for TSP



# MIP for TSP

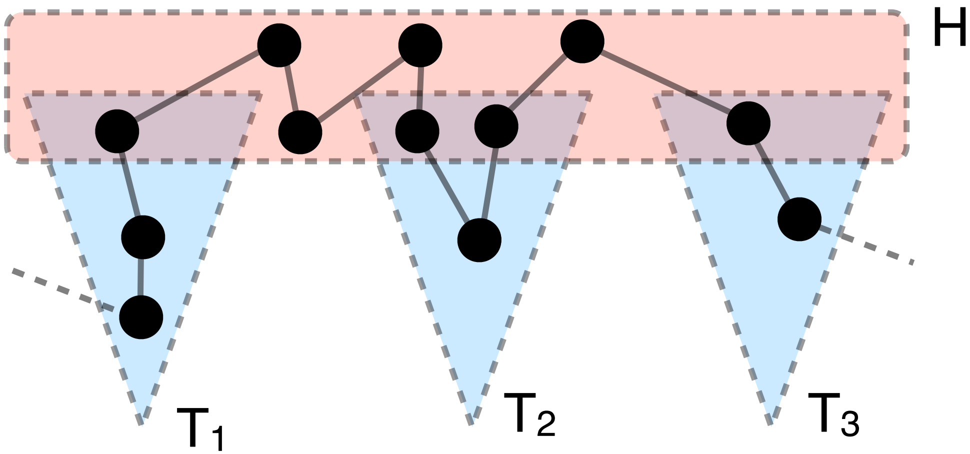
- Comb constraints



- Number of edges crossed?

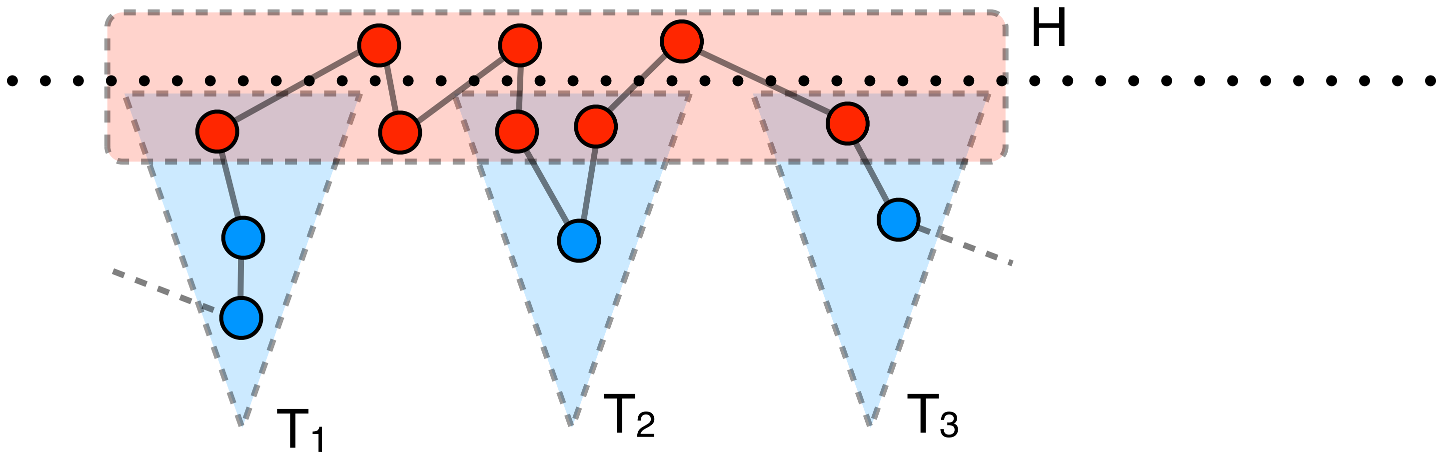
# MIP for TSP

- Comb constraints



# MIP for TSP

- Comb constraints




‣ comb inequalities

$$x_{\gamma(H)} + \sum_{i=1}^{t} x_{\gamma(T_i)} \leq |H| + \sum_{i=1}^{k} |T_i| - \lceil \frac{3k}{2} \rceil$$

# Branch and cut for TSP

- On the TSPLIB benchmark
  - Subtour elimination: 2% of optimality gap
  - Subtour + comb cuts: 0.5% of optimality gap
- Other constraints are needed for large instances

# Practicalities

- How to find min cut?
  - Stoer-Wagner minimum cut algorithm
- How to run MIP on this?
  - Use lazy constraint generation
    - https://techblog.aimms.com/2015/05/26/solving-a-tsp-using-lazy-constraints/

# TSP

- Approximation methods
  – Using MST, Perfect matchings and Euler circuit

# Outline

- Mixed Integer Program
- Examples
  - Warehouse location
  - Knapsack
- Branch and Bound
- Branch and Cut
- TSP
- Bonus! (Duality)

# Duality

https://en.wikipedia.org/wiki/File:German_postcard_from_1888.png

# Duality

$$\begin{array}{ll} \min & c\ x \\ \text{subject to} & \\ & Ax \geq b \\ & x_j \geq 0 \end{array}$$

primal

$$\begin{array}{ll} \max & y\ b \\ \text{subject to} & \\ & yA \leq c \\ & y_i \geq 0 \end{array}$$

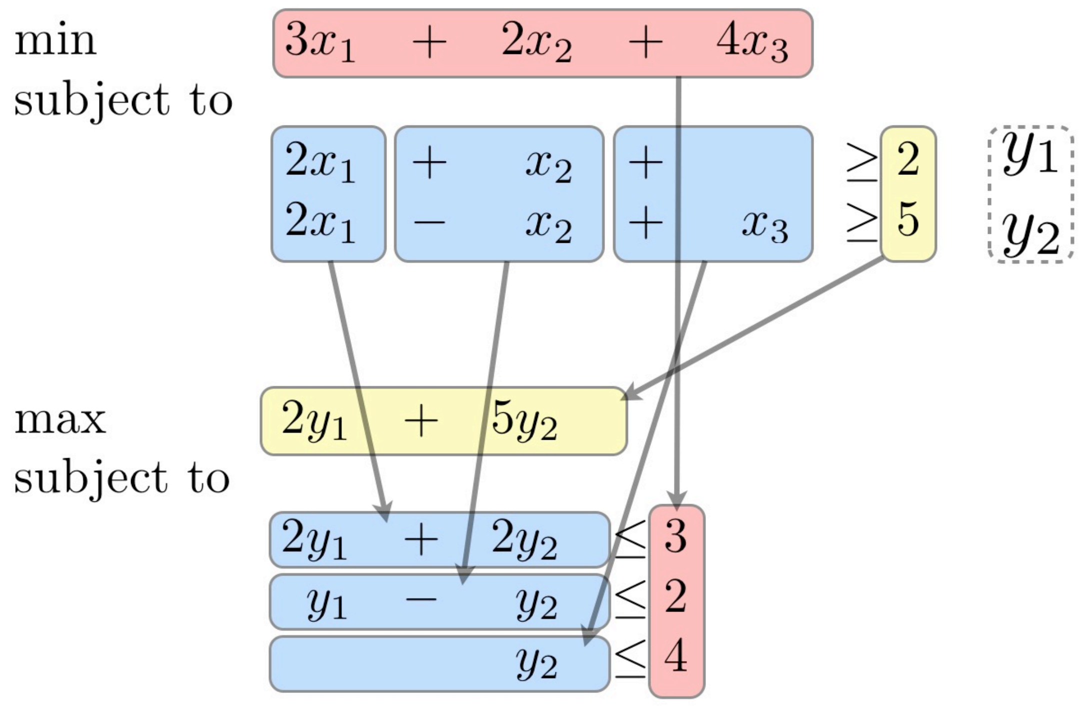
dual

# Duality

$$
\begin{array}{lrcrcrl}
\text{min} & 3x_1 & + & 2x_2 & + & 4x_3 & \\
\text{subject to} & & & & & & \\
 & 2x_1 & + & x_2 & + & & \geq 2 \\
 & 2x_1 & - & x_2 & + & x_3 & \geq 5
\end{array}
$$

$$
\begin{array}{lrcrl}
\text{max} & 2y_1 & + & 5y_2 & \\
\text{subject to} & & & & \\
 & 2y_1 & + & 2y_2 & \leq 3 \\
 & y_1 & - & y_2 & \leq 2 \\
 & & & y_2 & \leq 4
\end{array}
$$

# Duality


min
subject to
max
subject to
$3x_1 + 2x_2 + 4x_3$
$2x_1 + x_2 \geq 2$
$2x_1 - x_2 + x_3 \geq 5$
$y_1$
$y_2$
$2y_1 + 5y_2$
$2y_1 + 2y_2 \leq 3$
$y_1 - y_2 \leq 2$
$y_2 \leq 4$

# Duality

$$\min \quad \begin{bmatrix} 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\max \quad \begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

subject to

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Theorem:
If the primal has an optimal solution, the dual has an optimal solution with the same cost

# Bounding

$$\begin{array}{lrcrcrcrcr}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}$$

- can we find an upper bound?

$$10x_1 \quad + \quad 2x_2 \quad + \quad 6x_3 \quad + \quad 16x_4 \quad \leq \quad 110$$

# Bounding

$$
\begin{array}{lrcrcrcrcr}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}
$$

- can we find an upper bound?

$$
10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110
$$

# Bounding

$$\begin{array}{lrcrcrcrcr}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}$$

- can we find an upper bound?

$$10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$$

# Bounding

$$
\begin{array}{lrcrcrcrcr}
\text{max} & 4x_1 & + & \phantom{0}x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}
$$

- can we find an upper bound?

$$10x_1 + 2x_2 + 6x_3 + 16x_4 \leq 110$$

# Bounding

$$
\begin{array}{lrcrcrcrcr}
\text{max} & \boxed{4}x_1 & + & \boxed{\phantom{0}}\,x_2 & + & \boxed{5}x_3 & + & \boxed{3}x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}
$$

- can we find an upper bound?

$$
\begin{array}{rcrcrcrcr}
\boxed{10}x_1 & + & \boxed{2}x_2 & + & \boxed{6}x_3 & + & \boxed{16}x_4 & \leq & \boxed{110} \\
4x_1 & + & 3x_2 & + & 6x_3 & + & 3x_4 & \leq & 58
\end{array}
$$

# Bounding

$$
\begin{array}{lrcrcrcrcr}
\text{max} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}
$$

- can we find an upper bound?

$$
\begin{array}{rcrcrcrcr}
10x_1 & + & 2x_2 & + & 6x_3 & + & 16x_4 & \leq & 110 \\
4x_1 & + & 3x_2 & + & 6x_3 & + & 3x_4 & \leq & 58
\end{array}
$$

# Bounding

$$
\begin{array}{lrcrcrcrcr}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}
$$

- can we find an upper bound?

$$
\begin{array}{rcrcrcrcr}
10x_1 & + & 2x_2 & + & 6x_3 & + & 16x_4 & \leq & 110 \\
4x_1 & + & 3x_2 & + & 6x_3 & + & 3x_4 & \leq & 58
\end{array}
$$

# Bounding

$$
\begin{array}{lrcrcrcrcr}
\text{max} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}
$$

- can we find an upper bound?

$$
\begin{array}{rcrcrcrcr}
10x_1 & + & 2x_2 & + & 6x_3 & + & 16x_4 & \leq & 110 \\
4x_1 & + & 3x_2 & + & 6x_3 & + & 3x_4 & \leq & 58
\end{array}
$$

# Bounding

$$\begin{array}{lrcrcrcrcr}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & \\
\text{subject to} & & & & & & & & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3
\end{array}$$

- positive combinations of the constraints

# Bounding

$$
\begin{array}{lrcrcrcrcrl}
\text{max} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & & \\
\text{subject to} & & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 & y_1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 & y_2 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 & y_3
\end{array}
$$

- positive combinations of the constraints

# Bounding

$$
\begin{array}{lrcrcrcrcrl}
\max & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & & \\
\text{subject to} & & & & & & & & & & \\
 & x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 & y_1 \\
 & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 & y_2 \\
 & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 & y_3
\end{array}
$$

- positive combinations of the constraints

$$
\begin{array}{llrcrcrcrll}
y_1 & ( & x_1 & - & x_2 & - & x_3 & + & 3x_4 & ) & + \\
y_2 & ( & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & ) & + \\
y_3 & ( & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & ) & \\
 & & & & \leq & & & & & & \\
 & & y_1 & + & 55y_2 & + & 3y_3 & & & &
\end{array}
$$

# Bounding

$$\begin{array}{lrcrcrcrcrl}
\text{max} & 4x_1 & + & x_2 & + & 5x_3 & + & 3x_4 & & & \\
\text{subject to} & & & & & & & & & & \\
& x_1 & - & x_2 & - & x_3 & + & 3x_4 & \leq & 1 & y_1 \\
& 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & \leq & 55 & y_2 \\
& -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & \leq & 3 & y_3
\end{array}$$

- positive combinations of the constraints

$$\begin{array}{lcrcrcrcrcc}
y_1 & ( & x_1 & - & x_2 & - & x_3 & + & 3x_4 & ) & + \\
y_2 & ( & 5x_1 & + & x_2 & + & 3x_3 & + & 8x_4 & ) & + \\
y_3 & ( & -x_1 & + & 2x_2 & + & 3x_3 & - & 5x_4 & ) &
\end{array}$$

$$\leq$$

minimize $y_1 + 55y_2 + 3y_3$

# Vehicle routing problem

- The most important extension of TSP
- One has to schedule a number of vehicles, of limited capacity, around a number of customers
- **Sequence the customers (TSP) + decide which vehicles visit which customers**
- MANY applications: waste collection, street cleaning, school bus routing, …

# VRP

- We assume
  - Each customer apart from the depot (customer 0) is visited by exactly 1 vehicle
  - There are $m$ vehicles
  - Vehicle $k$ has capacity $Q_k$
  - Customer $i$ has requierement $q_i$
  - Time taken between customers $i$ and $j$ is $d_{ij}$
  - Customer $i$ must take delivery between times $a_i$ and $b_i$

# VRP

Let $\delta_{ijk} = 1$ iff vehicle $k$ goes directly from customer $i$ to customer $j$,

$\gamma_{ik} = 1$ iff vehicle $k$ visits customer $i$, (apart from the depot),

$\tau_i =$ time at which customer $i$ is visited.

- Possible objectives:
  - Minimize total cost
  - Minimize the number of vehicles needed
  - Minimize maximum time taken by a vehicle

# VRP with min total cost objective

$$\text{Minimize} \quad \sum_{ijk} c_{ij}\delta_{ijk},$$

subject to $\sum_j \delta_{ijk} = \sum_j \delta_{jik} = \gamma_{ik}$, if a vehicle $k$ visits customer $i$, it is entered once and left once by $k$.

$\sum_k \gamma_{ik} = 1$, if customer $i$ (not 0) is visited by exactly one vehicle.

$\sum_i q_i\gamma_{ik} \leq Q_k$, the combined requirements of customers visited by vehicle $k$ must lie within the capacity of $k$.

$\tau_i - \tau_j \leq M(1 - \delta_{ijk}) - d_{ij}$, if customer $j$ is visited immediately after customer $i$, then this must be at a time at least $d_{ij}$ greater than the time at which customer $i$ is visited.

$a_i \leq \tau_i \leq b$, each customer must be visited within its time window.

# VRP

- Separation of constraints to make the solution easier
- …

# Exo 1 – Formulate IP

The CALIFORNIA MANUFACTURING COMPANY is considering expansion by building a new factory in either Los Angeles or San Francisco, or perhaps even in both cities. It also is considering building at most one new warehouse, but the choice of location is restricted to a city where a new factory is being built. The *net present value* (total profitability considering the time value of money) of each of these alternatives is shown in the fourth column of Table 12.1. The rightmost column gives the capital required (already included in the net present value) for the respective investments, where the total capital available is $10 million. The objective is to find the feasible combination of alternatives that maximizes the total net present value.

**TABLE 12.1** Data for the California Manufacturing Co. example

| Decision Number | Yes-or-No Question | Decision Variable | Net Present Value | Capital Required |
|---|---|---|---|---|
| 1 | Build factory in Los Angeles? | $x_1$ | $9 million | $6 million |
| 2 | Build factory in San Francisco? | $x_2$ | $5 million | $3 million |
| 3 | Build warehouse in Los Angeles? | $x_3$ | $6 million | $5 million |
| 4 | Build warehouse in San Francisco? | $x_4$ | $4 million | $2 million |
| | | | Capital available: | $10 million |

The Research and Development Division of the GOOD PRODUCTS COMPANY has developed three possible new products. However, to avoid undue diversification of the company's product line, management has imposed the following restriction.

> **Restriction 1:** From the three possible new products, *at most two* should be chosen to be produced.

Each of these products can be produced in either of two plants. For administrative reasons, management has imposed a second restriction in this regard.

> **Restriction 2:** Just one of the two plants should be chosen to be the sole producer of the new products.

The production cost per unit of each product would be essentially the same in the two plants. However, because of differences in their production facilities, the number of hours of production time needed per unit of each product might differ between the two plants. These data are given in Table 12.2, along with other relevant information, including marketing estimates of the number of units of each product that could be sold per week if it is produced. The objective is to choose the products, the plant, and the production rates of the chosen products so as to maximize total profit.

| | **Production Time Used for Each Unit Produced** | | | **Production Time Available per Week** |
|---|---|---|---|---|
| | **Product 1** | **Product 2** | **Product 3** | |
| Plant 1 | 3 hours | 4 hours | 2 hours | 30 hours |
| Plant 2 | 4 hours | 6 hours | 2 hours | 40 hours |
| Unit profit | 5 | 7 | 3 | (thousands of dollars) |
| Sales potential | 7 | 5 | 9 | (units per week) |

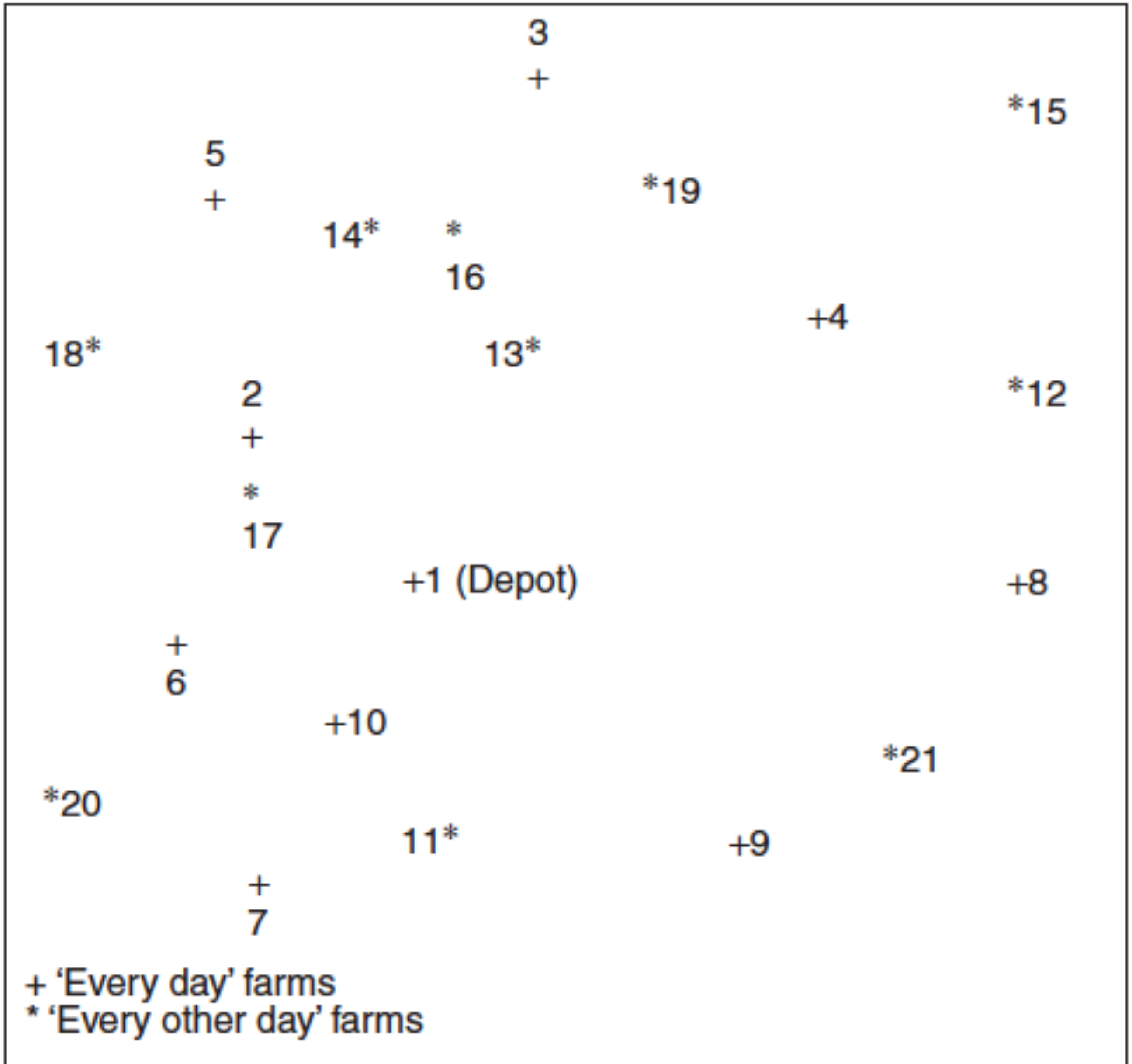
SOUTHWESTERN AIRWAYS needs to assign its crews to cover all its upcoming flights. We will focus on the problem of assigning three crews based in San Francisco to the flights listed in the first column of Table 12.4. The other 12 columns show the 12 feasible sequences of flights for a crew. (The numbers in each column indicate the order of the flights.) Exactly three of the sequences need to be chosen (one per crew) in such a way that every flight is covered. (It is permissible to have more than one crew on a flight, where the extra crews would fly as passengers, but union contracts require that the extra crews would still need to be paid for their time as if they were working.) The cost of assigning a crew to a particular sequence of flights is given (in thousands of dollars) in the bottom row of the table. The objective is to minimize the total cost of the three crew assignments that cover all the flights.

| | Feasible Sequence of Flights | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Flight** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** | **12** |
| 1. San Francisco to Los Angeles | 1 | | | 1 | | | 1 | | | 1 | | |
| 2. San Francisco to Denver | | 1 | | | 1 | | | 1 | | | 1 | |
| 3. San Francisco to Seattle | | | 1 | | | 1 | | | 1 | | | 1 |
| 4. Los Angeles to Chicago | | | | 2 | | | 2 | | 3 | 2 | | 3 |
| 5. Los Angeles to San Francisco | 2 | | | | | 3 | | | | 5 | 5 | |
| 6. Chicago to Denver | | | | 3 | 3 | | | | 4 | | | |
| 7. Chicago to Seattle | | | | | | | 3 | 3 | | 3 | 3 | 4 |
| 8. Denver to San Francisco | | 2 | | 4 | 4 | | | | 5 | | | |
| 9. Denver to Chicago | | | | | 2 | | | 2 | | | 2 | |
| 10. Seattle to San Francisco | | | 2 | | | | 4 | 4 | | | | 5 |
| 11. Seattle to Los Angeles | | | | | | 2 | | | 2 | 4 | 4 | 2 |
| Cost, $1,000's | 2 | 3 | 4 | 6 | 7 | 5 | 7 | 8 | 9 | 9 | 8 | 9 |

A small milk processing company is committed to collecting milk from 20 farms and taking it back to the depot for processing. The company has one tanker lorry with a capacity for carrying 80 000 litres of milk. Eleven of the farms are small and need a collection only every other day. The other nine farms need a collection every day. The positions of the farms in relation to the depot (numbered 1) are given in Table 12.16 together with their collection requirements.

Find the optimal route for the tanker lorry on each day, bearing in mind that it has to (i) visit all the 'every day' farms, (ii) visit some of the 'every other day' farms and (iii) work within its capacity. On alternate days, it must again visit the 'every day' farms and also visit the 'every other day' farms not visited on the previous day.

| Farm | Position 10 miles | | Collection frequency | Collection requirement (1000 l) |
|---|---|---|---|---|
| | East | North | | |
| 1 (Depot) | 0 | 0 | – | – |
| 2 | −3 | 3 | Every day | 5 |
| 3 | 1 | 11 | Every day | 4 |
| 4 | 4 | 7 | Every day | 3 |
| 5 | −5 | 9 | Every day | 6 |
| 6 | −5 | −2 | Every day | 7 |
| 7 | −4 | −7 | Every day | 3 |
| 8 | 6 | 0 | Every day | 4 |
| 9 | 3 | −6 | Every day | 6 |
| 10 | −1 | −3 | Every day | 5 |
| 11 | 0 | −6 | Every other day | 4 |
| 12 | 6 | 4 | Every other day | 7 |
| 13 | 2 | 5 | Every other day | 3 |
| 14 | −2 | 8 | Every other day | 4 |
| 15 | 6 | 10 | Every other day | 5 |
| 16 | 1 | 8 | Every other day | 6 |
| 17 | −3 | 1 | Every other day | 8 |
| 18 | −6 | 5 | Every other day | 5 |
| 19 | 2 | 9 | Every other day | 7 |
| 20 | −6 | −5 | Every other day | 6 |
| 21 | 5 | −4 | Every other day | 6 |


3
*15
5
*19
14*
16
+4
18*
13*
*12
2
17
+1 (Depot)
+8
6
+10
*21
*20
11*
+9
7
+ 'Every day' farms
* 'Every other day' farms