# Optimal transportation between unequal dimensions

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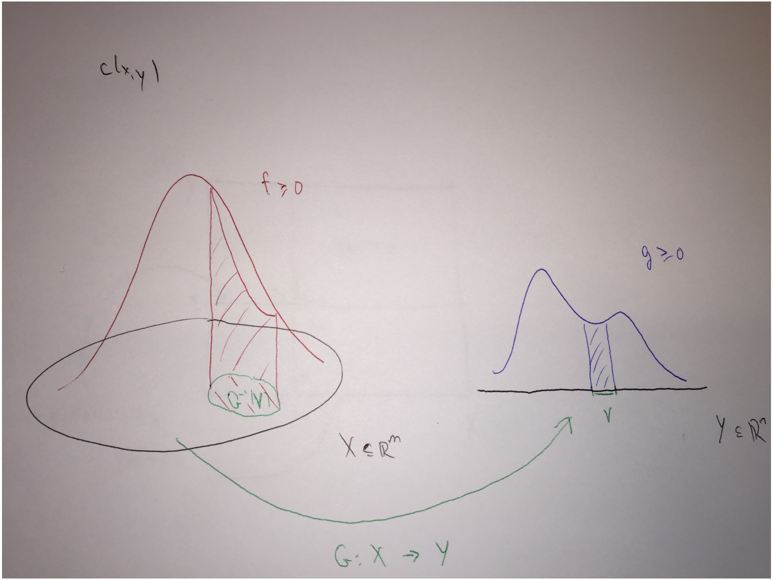
# Outline


c(x,y)
f ≥ 0
G⁻¹(V)
g ≥ 0
V
X ⊆ ℝⁿ
Y ⊆ ℝⁿ
G: X → Y

# Monge-Kantorovich optimal transport

$c(x, y) =$ 'cost' per unit mass transported from $x \in X$ to $y \in Y$

$X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ bounded open sets of dimension $m \geq n$

densities of supply $f(x) \geq 0$ on $X$ and demand $g(y) \geq 0$ on $Y$

$\int_X f = \int_Y g = 1$; normalization: probability densities / measures ('pdfs' )

Seek $\gamma \in \Gamma := \Gamma(f, g)$ where

$$\Gamma = \left\{ \begin{array}{c} 0 \leq \gamma \text{ on} \\ X \times Y \end{array} \;\middle|\; \begin{array}{lll} \int_U f(x)dx & = \gamma(U \times Y) & \forall U \subset X \\ & \gamma(X \times V) = \int_V g(y)dy & \forall V \subset Y \end{array} \right\}$$

such that...

seek $\gamma \in \Gamma = \Gamma(f,g)$ attaining

$$\inf_{\gamma \in \Gamma} \gamma[c]$$

where

$$\gamma[c] := \int_{X \times Y} c(x,y) d\gamma(x,y)$$

QUESTIONS:

- is the infimum attained? uniquely?
- can optimizers be characterized? (e.g. using PDE?)
- Monge: must $\mathrm{spt}\,\gamma \subset_{\gamma-a.e.} Graph(G)$ for some map $G : X \longrightarrow Y$?
(in which case we write $\gamma = (id \times G)_\# f$)
- what are their geometric and analytical properties?
- how do these depend on the choice of cost-benefit $c(x,y) = -b(x,y)$?
- applications?

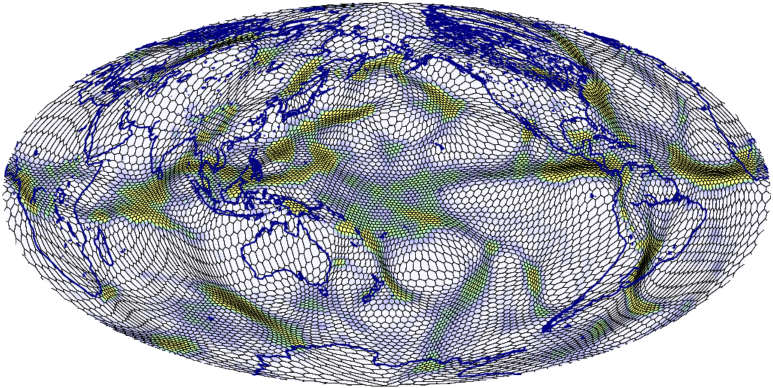
# Applications (a very incomplete sampler)

• Image processing (Delon, Kaijser, Peyre, Rumpf, Tannenbaum . . . )
(medicine, movies, and data compression)

Monge

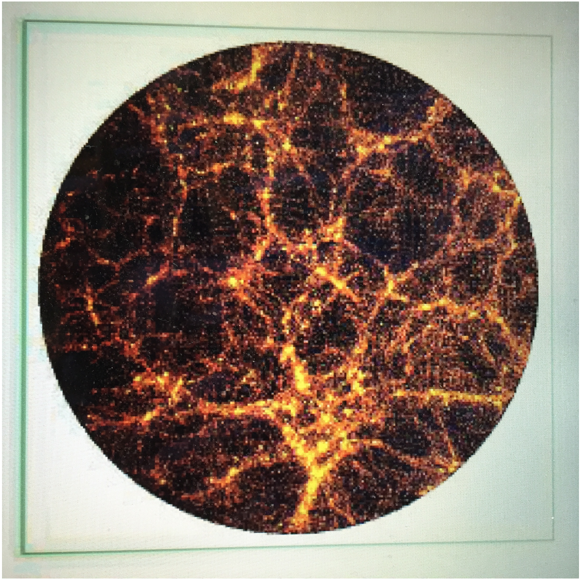
Kantorovich

• Weather prediction, mesh generation



from Weller, Browne, Budd and Cullen (2015 preprint)

- Early universe reconstruction



Brenier, Frisch, Hénon, Loeper, Matarrese, Mohayee, Sobolevskii (2003)

• Price equilibration of supply with demand; asymmetric information (Ekeland, Carlier, McCann, . . . )

• 'Stable marriage' problem (Shapley, Shubik, . . . )
(National Medical Residency Matching Program)

. . .

REMARKS:

- SETTING: $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ open and bounded with $m \geq n$
- $b \in C(\overline{X \times Y})$ the Banach space of cts fns, normed by supremum
- CONVEX: $\Gamma$ convex & wk-$*$ compact in the dual space of measures
- NON-EMPTY $d\gamma(x,y) = f(x)g(y)dxdy \in \Gamma$ (product measure)
- LINEAR: $\gamma[b]$ is a cts linear functional on $\Gamma$, hence maximum attained

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- LINEAR: $\gamma[b]$ is a cts linear functional on $\Gamma$, hence maximum attained (at an extreme point)
- EXTREMAL: $\gamma$ is *extremal* in $\Gamma$ unless it is midpoint of a segment in $\Gamma$

e.g. $(id \times G)_\# f$ is extremal, but not all extreme points take this form

# Unidimensional (very classical)

Lorentz '53 Mirrlees '71 Becker '73 Spence '73 ($b \in C^2(\mathbf{R}^2)$, $m = 1 = n$)

- if $\frac{\partial^2 b}{\partial x \partial y} > 0$ (supermodular) the maximizer $\gamma$ is uniquely characterized by
- a non-decreasing map $G : \mathbf{R} \longrightarrow \mathbf{R}$ of producer to consumer such that

$$\gamma[\mathbf{R}^2 \setminus Graph(G)] = 0$$

where

$$Graph(G) := \{(x, G(x)) \mid x \in \mathbf{R}^m\}$$

- from formulas like

$$\int_{-\infty}^{x} f(\bar{x}) d\bar{x} = \int_{-\infty}^{G(x)} g(y) dy$$

or

$$f(x)/g(G(x)) = G'(x)$$

we deduce $G \in C^\infty$ where $0 < f, g \in C^\infty$ smooth and positive.

# Differential criteria for uniqueness and maps ($mn > 1$)

Assume $b \in C^2(\overline{X \times Y})$ is *twisted* and *non-degenerate* (ND), meaning

- (twist): for all $y \neq y' \in \overline{Y}$, the function

$$x \in X \mapsto b(x, y) - b(x, y')$$

has no critical points.

Equivalently, for each $x \in X$, map $y \in \overline{Y} \mapsto D_x b(x, y)$ is one-to-one.

- (ND): the matrix $D^2_{xy} b := [\frac{\partial^2 b}{dx^i dy^j}]$ has full rank $\forall (x, y) \in \overline{X \times Y}$

THM: (Gangbo '95, Levin '99) Twist implies the optimal $\gamma \in \Gamma(f, g)$ is unique, and supported on the graph of a map $G : X \longrightarrow \overline{Y}$ which acts as a change of variables between $f$ and $g$

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THM: (Gangbo '95, Levin '99) Twist implies the optimal $\gamma \in \Gamma(f, g)$ is unique, and supported on the graph of a map $G : X \longrightarrow \overline{Y}$ which acts as a change of variables between $f$ and $g$, a.e. $|\det DG(x)| = f(x)/g(G(x))$ if $m = n$.

# Partial differential equations and smoothness ($m = n$)

e.g. (Brenier '87 (p=2), Gangbo & M. '95, Caffarelli, Rüschendorf '96 )
$X = Y = \mathbf{R}^n$ with $b(x, y) = \pm|x - y|^p$ for $0 \neq p \neq 1$

$$|\det DG(x)| = \frac{f(x)}{g(G(x))} \qquad a.e.$$

- for $p = 2$ characterized $G = Du$ with $u : \mathbf{R}^n \longrightarrow \mathbf{R} \cup \{+\infty\}$ convex

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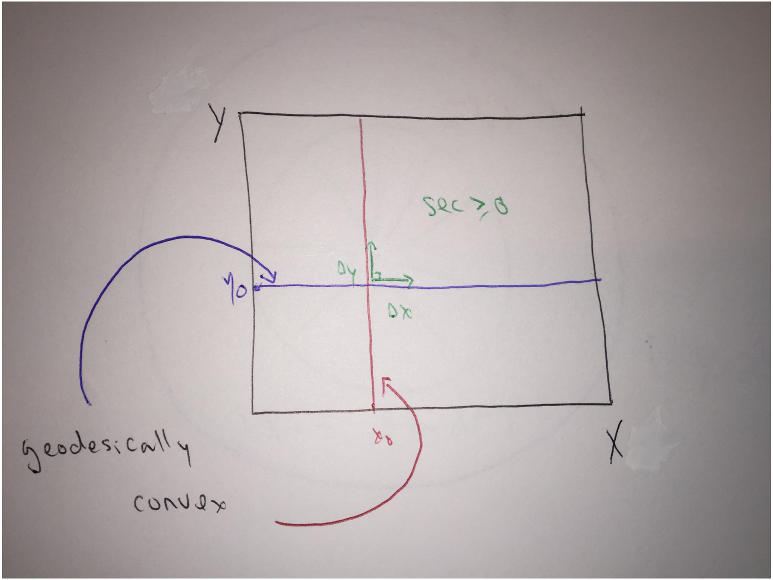
- for $p = 2$ characterized $G = Du$ with $u : \mathbf{R}^n \longrightarrow \mathbf{R} \cup \{+\infty\}$ convex
- PDE becomes elliptic Monge-Ampère equation: $\det D^2 u = \frac{f}{g \circ Du}$
- Caffarelli '92: $u \in C^{k,\alpha}(X)$ if $Y$ convex & $\log f, \log g \in L^\infty \cap C^{k-2,\alpha}$
- Ma-Trudinger-Wang '05: $G$ as smooth for other costs, if $\overline{X \times Y}$

has good geometry when metrized (Kim & M.'10) by $\begin{bmatrix} 0 & D^2_{xy} b \\ D^2_{xy} b^\dagger & 0 \end{bmatrix}$


Y
sec ≥ 0
Dy
$y_0$
Dx
$x_0$
X
geodesically
convex

# What if dimensions unequal: $(dimX = m > n = dimY)$?

Pass '12: regularity cannot hold for all $\log f, \log g \in L^\infty \cap C^\infty$, except in the (pseudo-)indicial case: $b(x,y) = \tilde{b}(I(x), y) + n(x)$ for some $I : \mathbf{R}^m \longrightarrow \mathbf{R}^n$, and $\tilde{b}$ satisfying the MTW '05 conditions on $\mathbf{R}^n \times \mathbf{R}^n$

But if $b$ is not pseudo-indicial, might regularity hold for certain $f$ and $g$?

Co-area formula suggests the mass balance condition

$$g(y) = \int_{G^{-1}(y)} \frac{f(x)}{JG(x)} d\mathcal{H}^{m-n}(x)$$

where Jacobian $JG(x) := \sqrt{\det DG(x) DG(x)^\dagger}$

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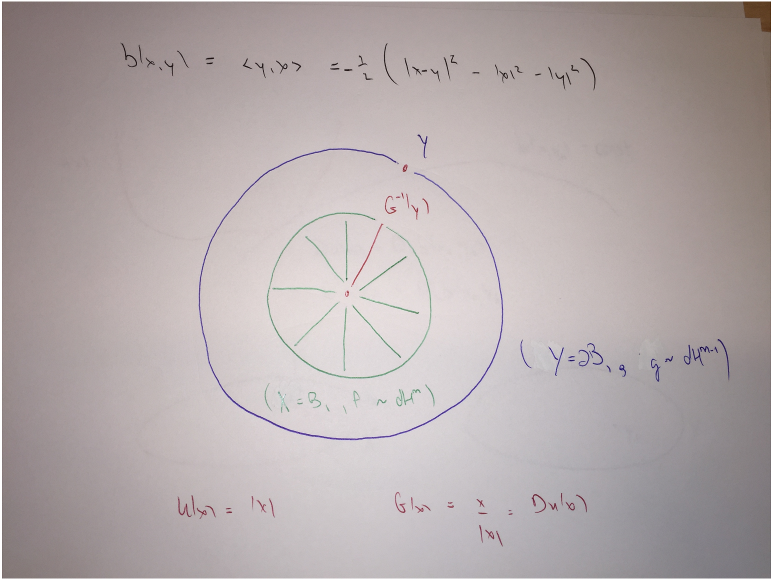
e.g. (disk to circle)

$$X = B_R(0) \subset \mathbf{R}^2, \qquad Y = \partial B_1(0) \subset \mathbf{R}^2$$

$b(x,y) = x \cdot y \qquad f = \frac{1}{\pi R^2} \qquad g = \frac{1}{2\pi}$

$$G(x) = Du(x) = \frac{x}{|x|} \quad \text{where} \quad u(x) = |x|$$

$$G^{-1}(\hat{y}) = \{\lambda \hat{y} \mid \lambda > 0\}$$


$b(x,y) = \langle y,x \rangle = -\frac{1}{2}\left(|x-y|^2 - |x|^2 - |y|^2\right)$
$Y$
$G^{-1}(y)$
$(X = B_1,\ f \sim dH^m)$
$(Y = \partial B_1,\ g \sim dH^{m-1})$
$u(x) = |x|$
$G(x) = \frac{x}{|x|} = Du(x)$

# Dual linear program and stable marriage problem

$L := \{u \in L^1(f), v \in L^1(g) \mid u(x) + v(y) \geq b(x,y) \text{ on } X \times Y\}$ implies

$$\begin{aligned} (\textit{Kantorovich}, \ 1942) \ \textit{primal} \ P \quad &:= \quad \inf_{(u,v)\in L} \int_X uf + \int_Y vg \\ &= \quad \max_{\gamma \in \Gamma} \gamma[b] \\ &=: \quad \textit{dual} \ (\textit{Monge}, \ 1781) \end{aligned}$$

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• this primal program is a key tool for analysis,

• duality shows equivalence of the transport and stable matching problems, where $u(x)$ and $v(y)$ are the payoffs to wife $x$ and husband $y$ respectively

• If $P$ attained, any optimizer $\gamma$ vanishes outside the zeros of $u + v - b \geq 0$ i.e. $(Du - D_x b, Dv - D_y b) = (0, 0)$ holds $\gamma$-a.e., (and similarly $Hess \geq 0$)

- $b \in C^1$ implies primal $P$ attained by $(u, v) = (v^b, u^{\tilde{b}})$ where

$$v^b(x) := \sup_{y \in \overline{Y}} b(x, y) - v(y), \qquad u^{\tilde{b}}(y) := \sup_{x \in \overline{X}} b(x, y) - u(x)$$

- here $v = (v^b)^{\tilde{b}} =: v^{b\tilde{b}}$ is called $b$-convex, where $\tilde{b}(y, x) = b(x, y)$

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• inherits upper bounds on $|Dv|$ and $-D^2 v$ from $-b \in C^2$

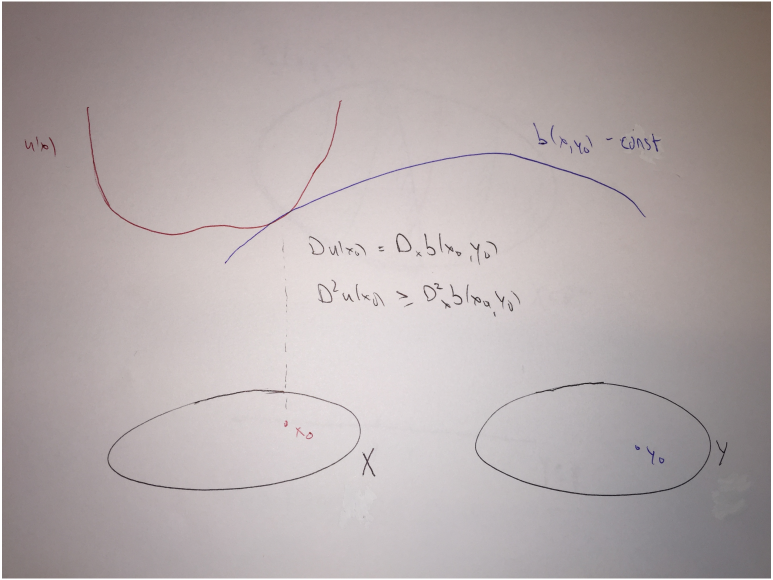
hence is twice differentiable Lebesgue a.e.

• optimal map $G$ is defined by

$$Du(x) = D_x b(x, G(x))$$

using twist (i.e. invertibility of $y \in \overline{Y} \mapsto D_x b(x, y)$) and similarly...

ASIDE: twist can now be interpreted as meaning husband's identity determined from wife's by his marginal willingness to pay for variations in her qualities


u(x)
b(x, y_0) - const
D u(x_0) = D_x b(x_0, y_0)
D^2 u(x_0) ≥ D^2_x b(x_0, y_0)
x_0
X
y_0
Y

# Towards a partial differential equation

satisfies $Dv(G(x)) = D_y b(x, G(x))$ and $D^2 v(G(x)) \geq D^2_{yy} b(x, G(x))$

- thus

$$(D^2 v - D^2_{yy} b) DG = D^2_{xy} b(x, G(x)) = \text{full rank by assumption}$$

giving the Jacobian

$$JG(x) := \sqrt{\det(DG)(DG)^\dagger}$$

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$$JG(x) := \sqrt{\det(DG)(DG)^\dagger} = \frac{\sqrt{\det(D^2_{xy} b)(D^2_{xy} b)^\dagger}}{\det[D^2 v - D^2_{yy} b]}$$

- the mass balance (co-area) formula becomes

$$g(y) = \int_{G^{-1}(y)} \frac{\det[D^2 v(y) - D^2_{yy} b(x, y)]}{\sqrt{\det D^2_{xy} b(x, y) D^2_{xy} b(x, y)^\dagger}} f(x) d\mathcal{H}^{m-n}(x)$$

- were it not for the domain of the integral, this would be a PDE for $v$!

Neglecting the set of zero volume where differentiability of $v(y)$ fails:

$$G^{-1}(y) \subset \{x \in X \mid v^b(x) + v(y) - b(x, y) = 0\}$$

$$=: X_3(v; y) \qquad \text{(badly nonlocal)}$$

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$$\subset X_2(v; y) := X_2(y, Dv(y), D^2v(y)) \qquad \text{(both}$$

$$\subset X_1(v; y) := X_1(y, Dv(y)) \qquad \text{local!)}$$

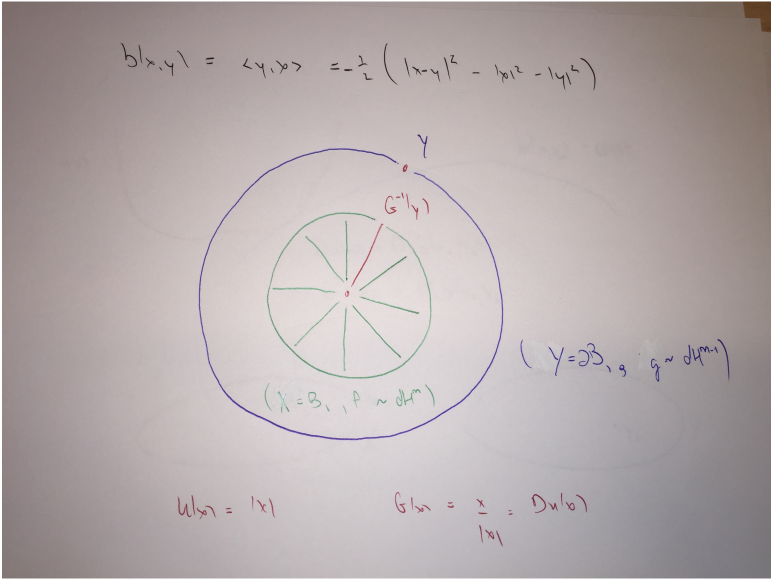
where

$$X_1(y, q) := \{x \in X \mid D_y b(x,y) = q\} \leftarrow \text{submanifold}$$

codimension $n$

$$X_2(y, q, Q) := \{x \in X_1(y,q) \mid D^2_{yy} b(x,y) \leq Q\}$$

↑ with boundary and corners


b(x,y) = <y,x> = -1/2 ( |x-y|^2 - |x|^2 - |y|^2 )
Y
G^-1(y)
( Y = ∂B1 , g ~ dH^m-1 )
( X = B1 , f ~ dH^m )
u(x) = |x|
G(x) = x/|x| = Du(x)

$$F_i(v;y) := \int_{X_i(v;y)} \frac{\det[D^2 v(y) - D^2_{yy} b(x,y)]}{\sqrt{\det D^2_{xy} b(x,y) D^2_{xy} b(x,y)^\dagger}} f(x) d\mathcal{H}^{m-n}(x)$$

$$g(y) \le F_3(v,y) \le F_2(y, Dv(y), D^2 v(y))$$

THM: (Nonlocal characterization of optimizers) Fix pdfs $f$ and $g$ on bounded open subsets $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}^n$ with $m \ge n$, $b \in C^2(\overline{X \times Y})$ twisted non-degenerate, and $v = v^{b\tilde{b}}$. Then $(v^b, v)$ minimizes the Kantorovich primal problem if and only if $F_3(v;y) = g(y)$ holds a.e.

THM: (Local characterization of smooth optimizers) Fix pdfs $f$ on $X \subset \mathbf{R}^m$ and $g$ on $Y \subset \mathbf{R}^n$ bounded and open sets with $m \geq n$, $b \in C^2(\overline{X \times Y})$ twisted non-degenerate, and $v = v^{b\tilde{b}} \in C^2(Y)$.

- If $F_2(v; y) = g(y)$ on $Y$ and $v \in C^2\left(\overline{Y}\right)$ then $(v^b, v)$ minimizes primal
- Conversely,

THM: (Local characterization of smooth optimizers) Fix pdfs $f$ on $X \subset \mathbf{R}^m$ and $g$ on $Y \subset \mathbf{R}^n$ bounded and open sets with $m \geq n$, $b \in C^2(\overline{X \times Y})$ twisted non-degenerate, and $v = v^{b\tilde{b}} \in C^2(Y)$.

- If $F_2(v; y) = g(y)$ on $Y$ and $v \in C^2\left(\overline{Y}\right)$ then $(v^b, v)$ minimizes primal
- Conversely, if $(v^b, v)$ minimizes primal, $v^b \in C^2(X)$ and $X_2(v, y)$ is connected for a.e. $y$, then $F_2(v; y) = g(y)$ on $Y$.

RMK:

$$F_2(v; y) = \int_{X_2(y, Dv(y), D^2v(y))} \frac{\det[D^2 v(y) - D^2_{yy} b(x, y)]}{\sqrt{\det D^2_{xy} b(x, y) D^2_{xy} b(x, y)^\dagger}} f(x) d\mathcal{H}^{m-n}(x)$$

is degenerate elliptic: $P = P^T \geq 0$ implies $F_2(y, q, Q) \leq F_2(y, q, Q + P)$

- thus $v \in C^{2,\alpha}$ inherits the regularity of $F_2(y, q, Q)$ and $g$

# Multi- to one-dimension: $m > n = 1$? (with Chiappori)

• Since (ND) implies $|D_x b_y| \neq 0$, $g$-a.s. $spt\gamma \cap (X \times \{y\})$ lies in a $C^1$ hypersurface $X_1(y, Dv(y))$ splitting $spt\, f$ in two parts.

• For each fixed $y \in Y \subset \mathbf{R}^1$, motivated by $v'(y) = b_y(x, y)$, define

$$X_{\leq}(y,q) := \begin{cases} X & \text{if } q = +\infty \\ \{x \in X \subset \mathbf{R}^m \mid b_y(x,y) := \frac{\partial b}{\partial y}(x,y) \leq q\} & \text{else} \\ \emptyset & \text{if } q = -\infty; \end{cases}$$

it depends monotonically on $q \in \mathbf{R} \cup \{\pm\infty\}$

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IDEA: Choose $q = q(y)$ to "split the masses proportionately", i.e. so that

$$0 = \int_{X_{\leq}(y,q)} f(x)dx - \int_{-\infty}^{y} g(\bar{y})d\bar{y} =: F(y, q)$$

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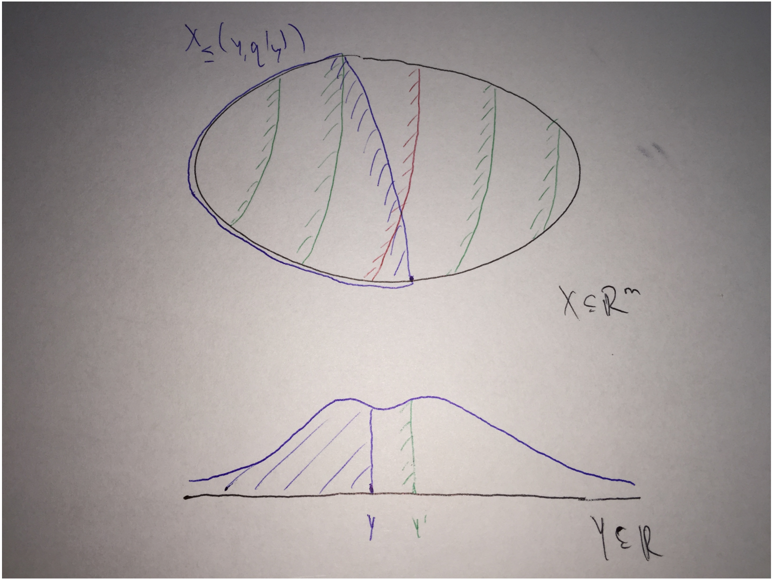
- this choice is unique for $g$-a.e. $y$
- inherits smoothness from $F$ by implicit function theorem if $F_q := \frac{\partial F}{\partial q} \neq 0$
- try to define $G : X \longrightarrow \overline{Y}$ so $G(x) = y \iff x \in \partial X_{\leq}(y, q(y))$


X≤(y,q|y)
X ⊆ ℝ^m
y
y'
Y ⊆ ℝ

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(for any $q : M^- \longrightarrow \mathbf{R}$ chosen to split the masses proportionately).

THM 1: $X \subset \mathbf{R}^n$, $Y \subset \mathbf{R}$ open connected, with probability densities $f$ & $g$. $b \in C^{1,1}(X \times Y)$ non-degenerate. If $(b, f, g)$ nested then $G : X \longrightarrow \overline{Y}$ is well-defined $f$-a.e., and $\gamma[b]$ uniquely maximized on $\Gamma(f, g)$.

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$b \in C^{1,1}(X \times Y)$ non-degenerate. If $(b, f, g)$ nested then $G : X \longrightarrow \overline{Y}$
is well-defined $f$-a.e., and $\gamma[b]$ uniquely maximized on $\Gamma(f, g)$.
The maximizer $\gamma$ is determined by $G$ and supported on $Graph(G)$
(namely $\gamma = (id \times G)_{\#} f$).

If in addition $\overline{\{g > 0\}}$ is connected, $G$ agrees $f$-a.e. with some continuous map $\overline{G} : X \longrightarrow \overline{Y}$.

Problem: if $x \in \partial X_{\leq}(y, q(y)) \cap \partial X_{\leq}(y', q(y'))$, then $G(x)$ not well-defined

DEFN: $(b, f, g)$ is nested if $\int_y^{y'} g > 0 \Rightarrow X_{\leq}(y, q(y)) \subset X_{<}(y', q(y'))$
(for any $q : M^- \longrightarrow \mathbf{R}$ chosen to split the masses proportionately).

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Proof strategy: Motivated by $Dv(y) = D_y b(x,y)$, use $q(y)$ to define $v(y) := \int^y q(\bar{\mathbf{y}}) d\bar{y}$ and

$$u(x) := \sup_{y \in Y} b(x,y) - v(y)$$

Then $(u, v) \in L$. Moreover, the measure $\gamma := (id \times G)_{\#} f \in \Gamma(f, g)$ then vanishes outside the zero set of $u(x) + v(y) - b(x,y) \geq 0$, showing $(u, v)$ and $\gamma$ optimize the primal and dual problems respectively. Indeed the zero set of $u + v - b$ is essentially a graph, hence the dual optimizer $\gamma$ is unique.

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Continuity of $T$ follows from the fact that $\overline{\{g > 0\}} = Y$ and nestedness force strict monotonicity of sequence $y \in Y \subset \mathbf{R} \longrightarrow X_\leq(y, q(y))$.

By contrast, $\overline{\{f > 0\}} = X$ would preclude jumps in this sequence, and is related instead to the continuity of $q = dv/dy$ on $Y$

Nestedness can be interpreted to mean there is a matching $\gamma \in \Gamma(f, g)$ in which the women's preferences are compatible, in the sense that for each pair of matched couples $(\bar{x}, \bar{y}), (\underline{x}, \underline{y}) \in spt\gamma$, the wife $\bar{x}$ of the higher type husband $\bar{y} > \underline{y}$ has a greater marginal willingness to pay for variations in the quality of either husband than the second woman $\underline{x}$ does.

# Criteria for nestedness

Outward normal velocity at $x \in \partial X_{\leq}(y, q)$ of

$$X_{\leq}(y, q) = \{x \in X \subset \mathbf{R}^n \mid b_y(x, y) \leq q\}$$

with respect to changes in $y$ (or $q$) is given by $\frac{b_{yy}(x,y)}{|D_x b_y|}$ (or $\frac{1}{|D_x b_y|}$)

Thus outward normal velocity of $X_{\leq}(y, q(y))$ wrt $y$ should be

$$\frac{q'(y) - b_{yy}}{|D_x b_y|}$$

- this expectation can be made rigorous under suitable hypotheses
- nestedness implies non-negativity of this normal velocity;
- global positivity of this normal velocity implies nestedness

Suitable hypotheses:

- $X \subset \mathbf{R}^m$ and $Y \subset \mathbf{R}$ open, connected, finite perimeter
- $b \in C^{2,1}$ non-deg., $\log f \in (C \cap W^{1,1})(X)$ and $\log g \in C^0_{loc}(Y)$
- a mild form of transversality: $Z = \emptyset$, where

$$Z := \{y \in M^- \mid Area[\overline{X}_1(y, q(y)) \cap \partial^* X] > 0\}$$

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COROLLARY (Unique splitting criterion for nestedness) $(b, f, g)$ satisfying the hypotheses above with $Z = \emptyset$ is nested if and only if each $x \in X$ corresponds to a unique $y \in Y$ such that

$$0 = F(y, b_y(x, y)) := \int_{X_{\leq}(y, b_y(x,y))} f(\bar{x}) d\bar{x} - \int_{-\infty}^{y} g(\bar{y}) d\bar{y}$$

In this case, $G(x) = y$.

# Lipschitz (and Hölder) classes

For each integer $k \geq 0$, and exponent $0 < \alpha \leq 1$ we denote by $C^{k,\alpha}(X)$ the class of functions which are $k$ times continuously differentiable, and whose $k$-th derivatives are all Lipschitz continuous functions with respect to the distance function $|x - x'|^\alpha$ on $X$ (in which case both properties extend to the closure $\overline{X}$ of $X$.) We norm this space by

$$\|f\|_{C^{k,\alpha}(X)} := \sum_{i=0}^{k} \sum_{|\beta|=i} \|D^\beta f\|_\infty + \sup_{x \neq x' \in X} \sum_{|\beta|=k} \frac{|D^\beta f(x') - D^\beta f(x)|}{|x' - x|^\alpha}$$

where $D^\beta f = \frac{\partial^{|i|} f}{\partial x_1 \cdots \partial x_i}$ and the sums are over multi-indices $\beta$ of degree $|\beta|$.

THM 3: Fix $k \geq 1$. Under the hypotheses of THM 2, suppose $\underline{Y' := (y_0, y_1)} \subset Y$ is an interval on which $\underline{\partial X \in C^1(X')}$ and intersects $\overline{\partial X_{\leq}(y, q(y))}$ transversally. If $X' := \cup_{y \in Y'} \overline{\partial X_{\leq}(y, q(y))}$ then $\|q\|_{C^{k,1}(Y')}$ is locally controlled by the following quantities, assumed positive and finite:

- $\| \log f / \log g \|_{C^{k-1,1}}, \quad \|b\|_{C^{k+1,1}}, \quad Area(\partial X), \quad \|\hat{n}_X\|_{C^{k-2,1} \cap W^{1,1}},$
- $$\inf_{y \in Y'} Area[X_1(y, q(y))]$$ (proximity to ends of $Y$)
- $$\inf_{(x,y) \in X' \times Y'} |D_x b_y(x, y)|$$ (non-degeneracy)
- $$\inf_{(x,y) \in (X' \cap \partial X) \times Y'} 1 - [\hat{n}_X \cdot \hat{n}_{X_{\leq}(x,y)}]^2$$ (transversality)
- and $$\mathcal{H}^{m-2} \left[ \overline{\partial X_{\leq}(y_0, q(y_0))} \cap \partial X \right]$$

Proof: Use Riemannian level set techniques to establish smoothness of

$$F(y,q) := \int_{X_{\leq}(y,q)} f(\bar{x})d\bar{x} - \int_{-\infty}^{y} g(\bar{y})d\bar{y}.$$

Then conclude smoothness of $q(y)$ using implicit function theorem.

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e.g. $F_q = \int_{X_1(y,q)} f V \cdot \hat{n}_1 d\mathcal{H}^{m-1}(x) > 0$ ,

where $V(x,y) = \frac{\hat{n}_1}{|D_x b_y|}$ and $\hat{n}_1(x,y) = \frac{D_x b_y}{|D_x b_y|}$ .

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thus $F_q = \int_{X_{\leq}(y,q)} \nabla \cdot (fV) dx - \int_{X_{\leq}(y,q) \cap \partial X} fV \cdot \hat{n}_X d\mathcal{H}^{m-1}$ ,

and

$$F_{qq} = \int_{X_1(y,q)} \nabla \cdot (fV) V \cdot \hat{n}_1 d\mathcal{H}^{m-1} - \int_{X_1(y,q) \cap \partial X} fV \cdot \hat{n}_X V_\partial \cdot \hat{n}_\partial d\mathcal{H}^{m-2} ,$$

where $V_\partial = \frac{V \cdot \hat{n}_1}{\sqrt{1 - (\hat{n}_1 \cdot \hat{n}_X)^2}} \hat{n}_\partial$ and $\hat{n}_\partial = \frac{\hat{n}_1 - (\hat{n}_1 \cdot \hat{n}_X)\hat{n}_X}{\sqrt{1-(\hat{n}_1 \cdot \hat{n}_X)^2}}$ .

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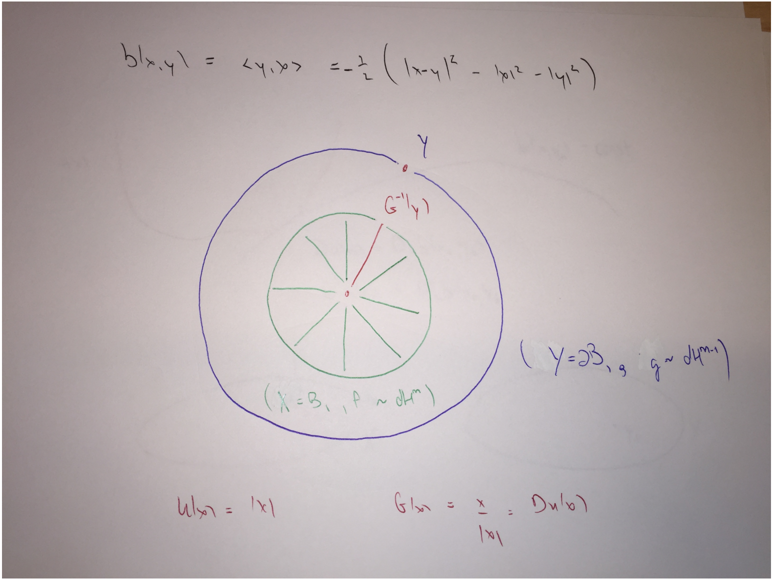
b(x,y) = <y,x> = -1/2 ( |x-y|^2 - |x|^2 - |y|^2 )
Y
G^-1(y)
(Y = ∂B1 , g ~ dH^m-1)
(X = B1 , f ~ dH^m)
u(x) = |x|
G(x) = x/|x| = Du(x)

# What about the map? (and the wives' payoffs u?)

PROP: If $(b, f, g)$ is nested and satisfies the preceding hypotheses then

- $u \in C^1(X)$,
- $G$ and $D_x b_y(\cdot, G(\cdot)) \in BV_{loc} \cap C(X)$,
- $G \in dom\, Dq$ holds $|DG|$-a.s.
- differentiating $q(G(x)) = v'(G(x)) = b_y(x, G(x))$ at such points yields

$$[q'(G(\cdot)) - b_{yy}(\cdot, G(\cdot))]DG(\cdot) = D_x b_y(\cdot, G(\cdot))$$

- on any open $X' \subset X$ obeying a speed limit

$$\ell := \inf_{x \in X'} \frac{q'(G(x)) - b_{yy}(x, G(x))}{|D_x b_y(x, G(x)|} > 0$$

$G$ is Lipschitz: $\|DG\|_{L^\infty(X')} \leq \ell^{-1}$

- higher regularity of $u$ and $G$ then follows from that of $v$

# CONCLUSIONS

with Pass

- optimal transport equivalent to solving a new, nonlocal 'PDE'
- however a local equation satisfied iff the optimal potentials are smooth
- depends subtly on the interaction of $b$ with $(X, f)$ and $(Y, g)$

with Chiappori and Pass

- nestedness is a key criterion singling out tractable matching problems, generalizing the Lorentz-Spence-Mirrlees-Becker (supermodularity) condition to the case where only one side of the market is unidimensional
- guarantees existence, uniqueness and regularity of husband's payoff
- smoothness of wife's payoff and map follows provided speed limit $\ell > 0$

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THANK YOU!