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Joint works with Jean-Marie Mirebeau et Thomas Gallouët

Brenier60: Calculus of Variations & Optimal Transport / January 2017 / IHP

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(. . . and borrowing many ideas from Yann. . . )

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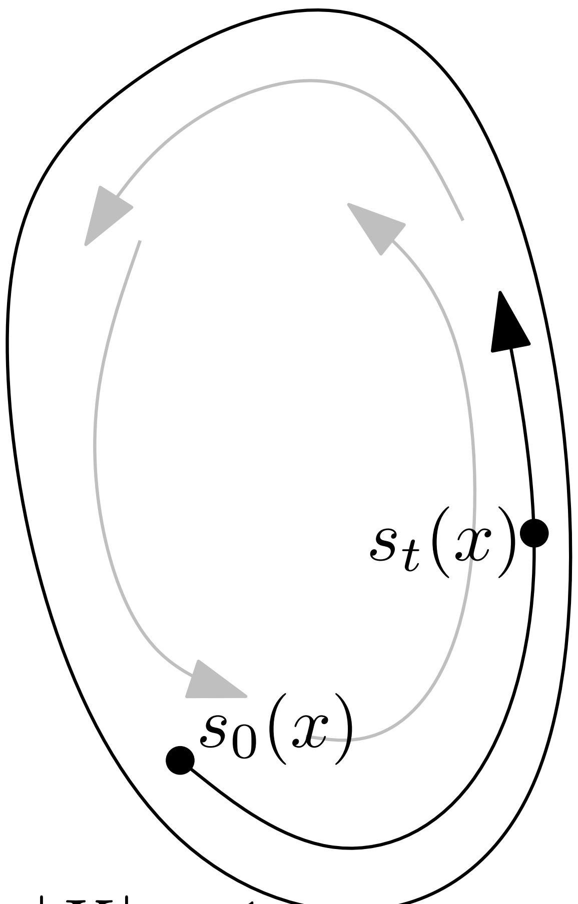
# Solutions to Euler's equations as geodesics in $\mathbb{S}\mathrm{Diff}$

$\mathbb{S}\mathrm{Diff}$ = measure-preserving diffeomorphisms from $X$ to itself $\subseteq \mathrm{L}^2(X, \mathbb{R}^d)$

[Arnold 1966]

$X \subseteq \mathbb{R}^d$ bounded

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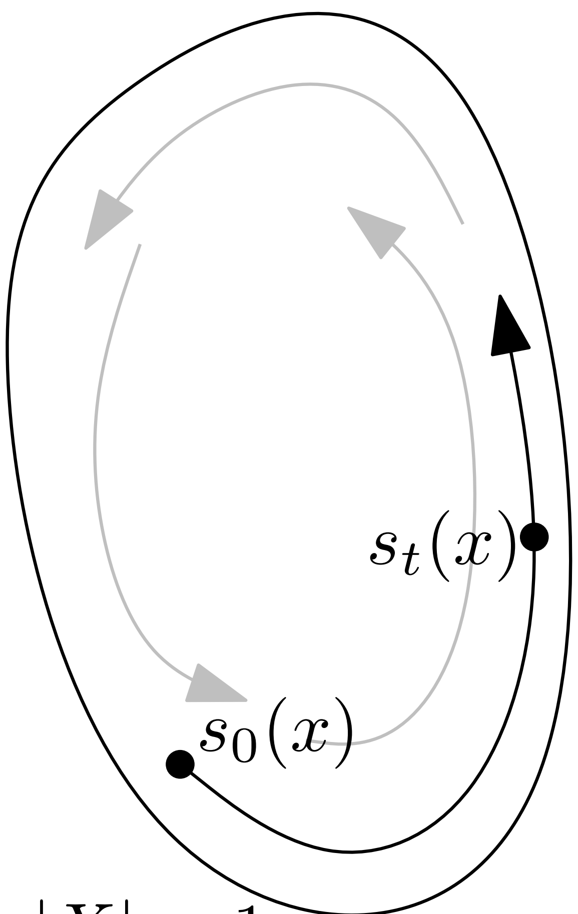


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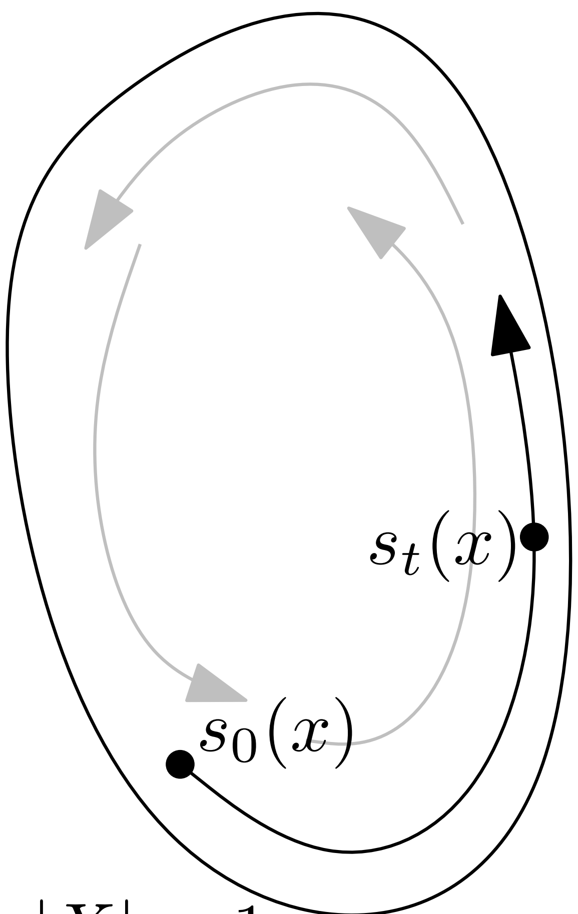


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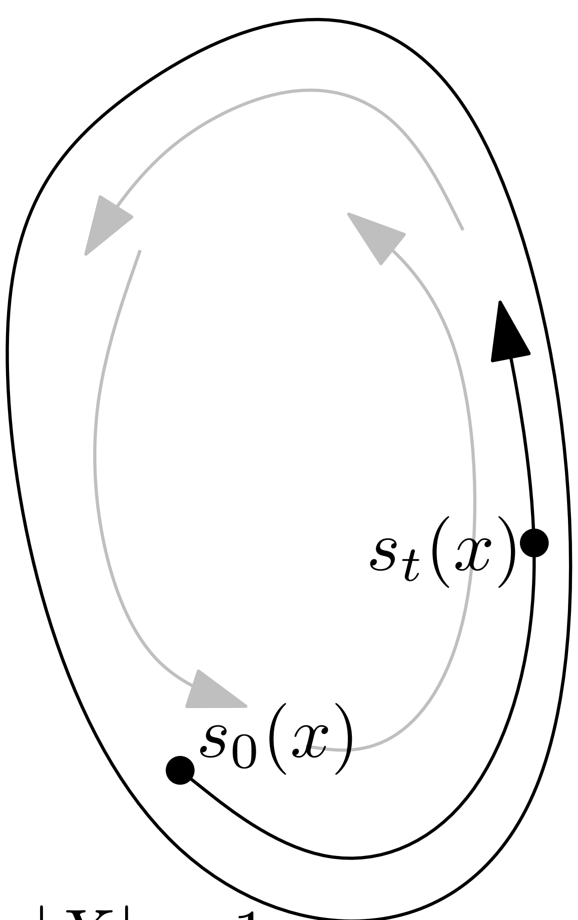


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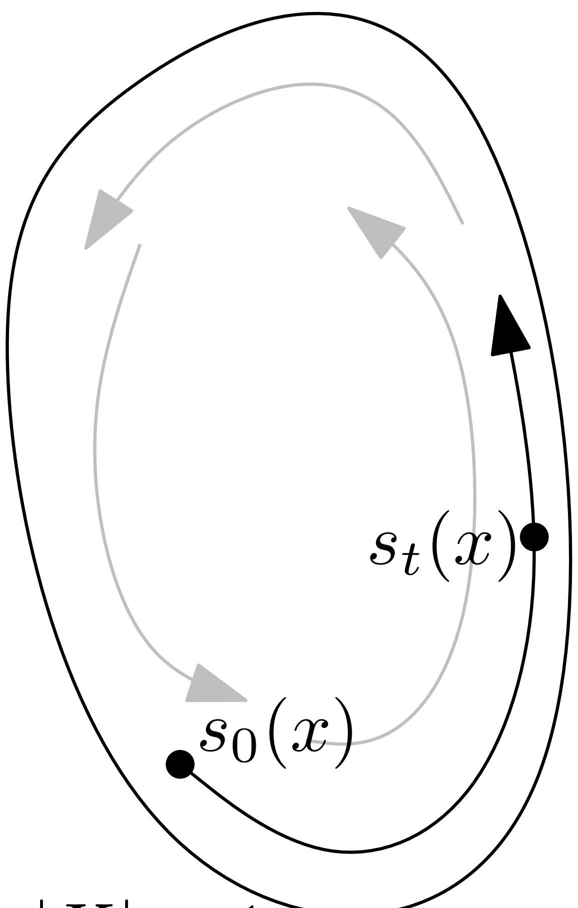


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$$\begin{cases} \partial_t u + (u \cdot \nabla) u = -\nabla p & \text{in } X \\ \mathrm{div} u = 0 & \text{in } X \\ u \cdot n = 0 & \text{on } \partial X \end{cases}$$

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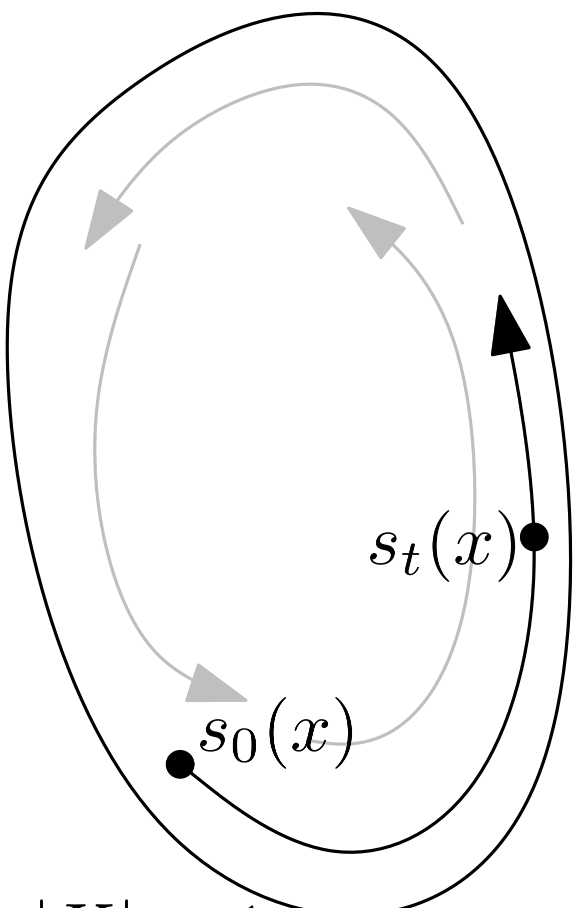


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This talk: Using this formulation for numerical computations (following Brenier):

$\longrightarrow$ Minimizing geodesics (with Jean-Marie Mirebeau, 2015)

$\longrightarrow$ Cauchy problem (with Thomas Gallouet, 2016).

# 1. Discretization of the Cauchy problem

Joint work with Thomas Gallouët

# Brenier's approximation of geodesics I

$$\begin{cases} \ddot{s}(t) \perp \mathrm{T}_{s(t)} S \\ s(t) \in S \\ (s(0), \dot{s}(0)) = (s_0, v_0) \end{cases}$$

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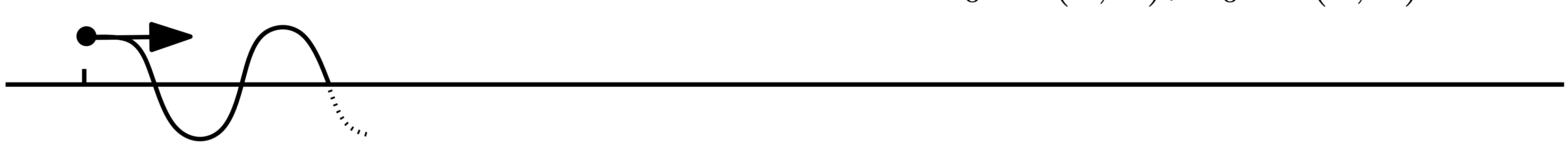

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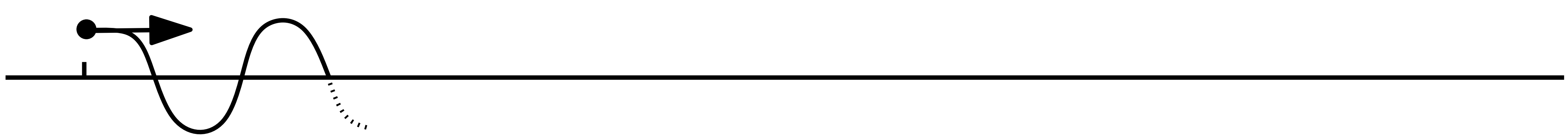

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$\longrightarrow$ $\mathcal{C}^1$ convergence towards the geodesic requires $\boxed{\frac{h}{\varepsilon} \longrightarrow 0.}$

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$\mathrm{Leb}$ = restriction of Lebesgue measure to a compact domain $X$

$\mathbb{S} = \{s : X \to X \mid s_{\#}\mathrm{Leb} = \mathrm{Leb}\} \longrightarrow$ "measure-preserving maps"

$\mathbb{M} = \mathrm{L}^2(X, \mathbb{R}^d)$

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  (NB: in $\mathbb{R}^d$ ($d \geq 2$), each iterations costs $N^3$... $\longrightarrow$ different approach needed)

# Distance to $\mathbb{S}$ and polar factorization of maps

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**Polar Factorization Theorem (Brenier):** For every map $m$ in $\mathbb{M} = \mathrm{L}^2(X, \mathbb{R}^d)$,

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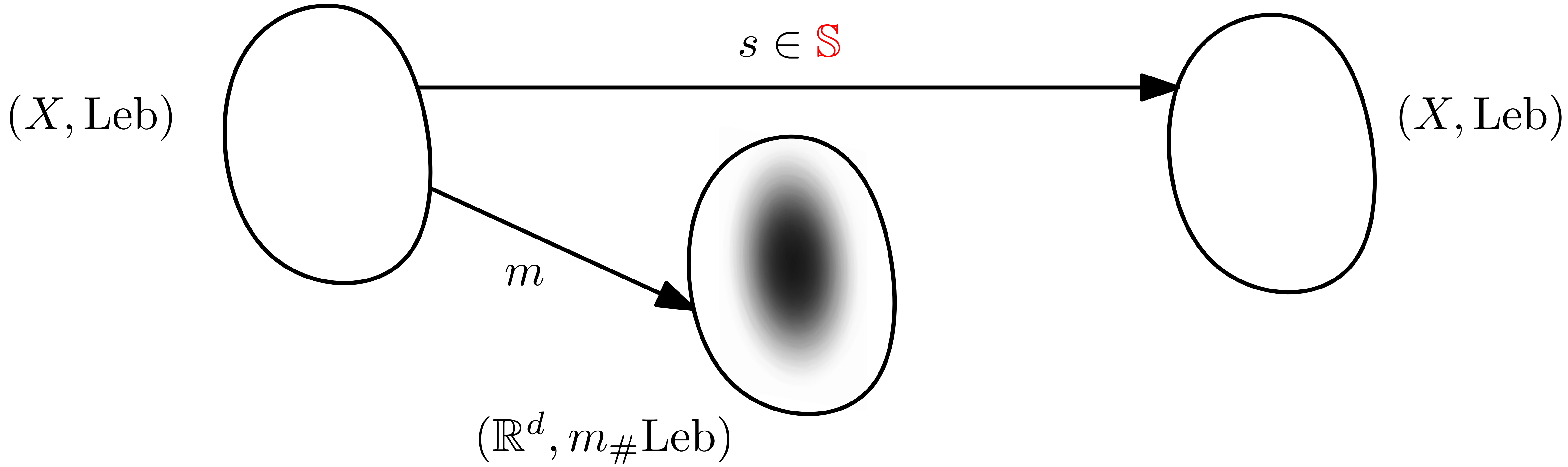

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Let $\bar{T}$ be the quadratic optimal transport map between $\mathrm{Leb}$ and $m_\# \mathrm{Leb}$. Then,

$$\Pi_{\mathbb{S}}(m) = \{\bar{s} \in \mathbb{S} \mid \bar{T} \circ \bar{s} = m\}$$

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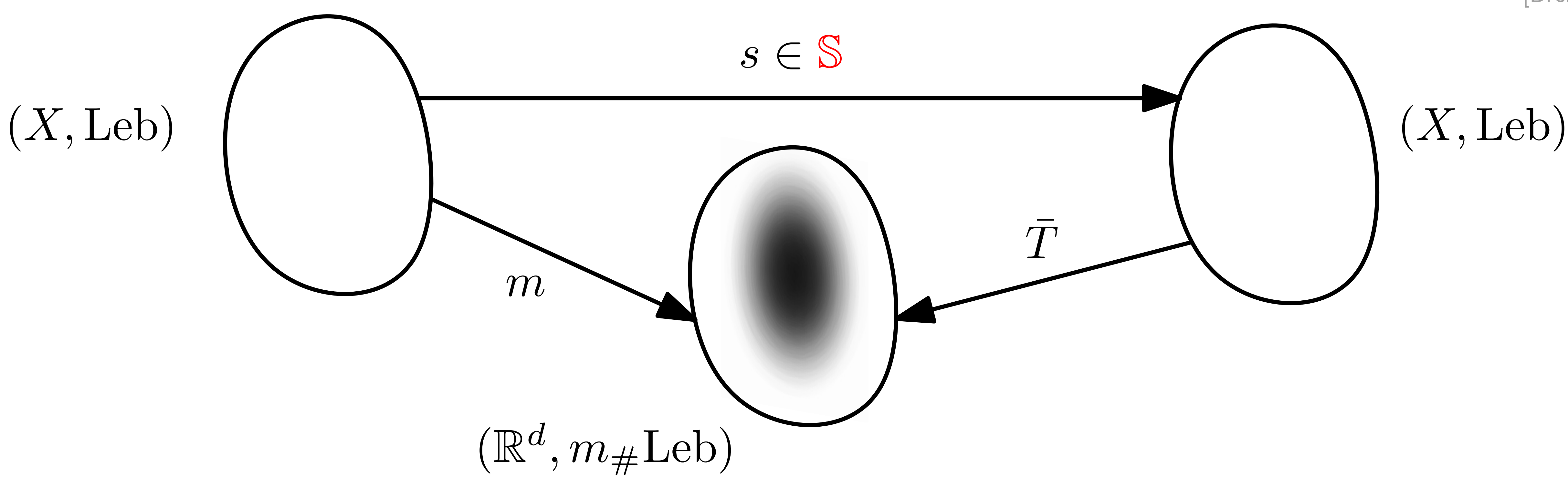

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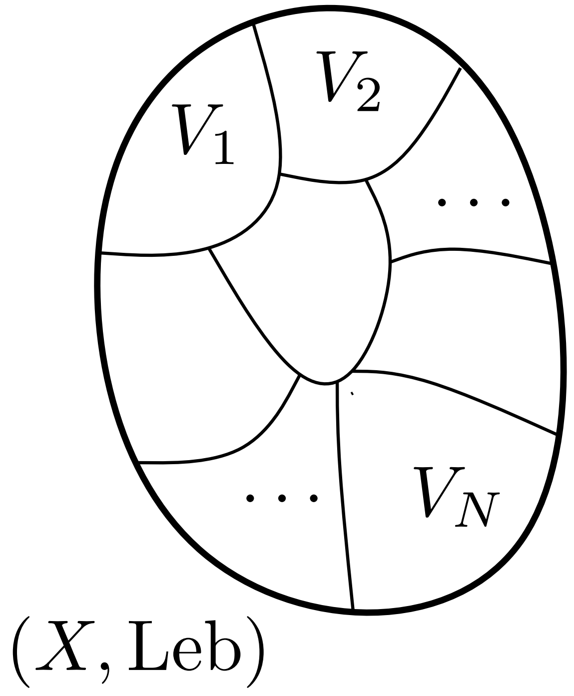
**Objective:** Constructing a finite-dimensional subspace of $\mathbb{M}$ and computing $\Pi_{\mathbb{S}}$

$\longrightarrow$ $X$ is partitioned into $(V_k)_{1\leq k\leq N}$ with $\mathrm{Leb}(V_k) = \frac{1}{N}$ and $\mathrm{diam}(V_k) \simeq N^{-\frac{1}{d}}$



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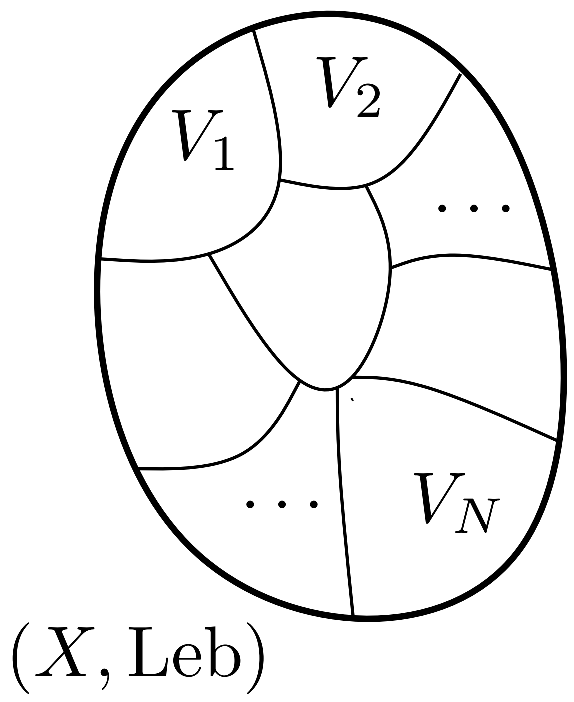

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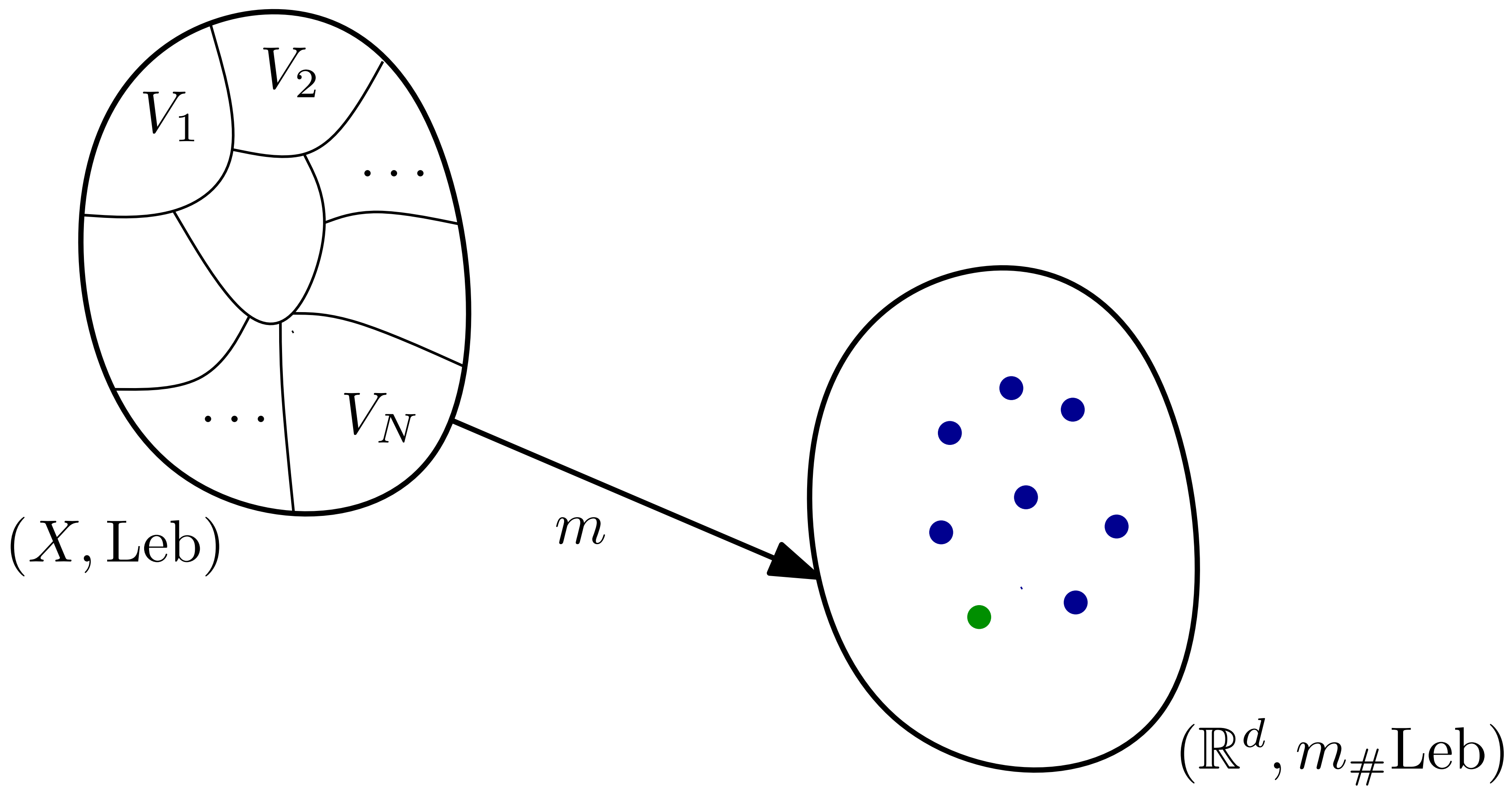

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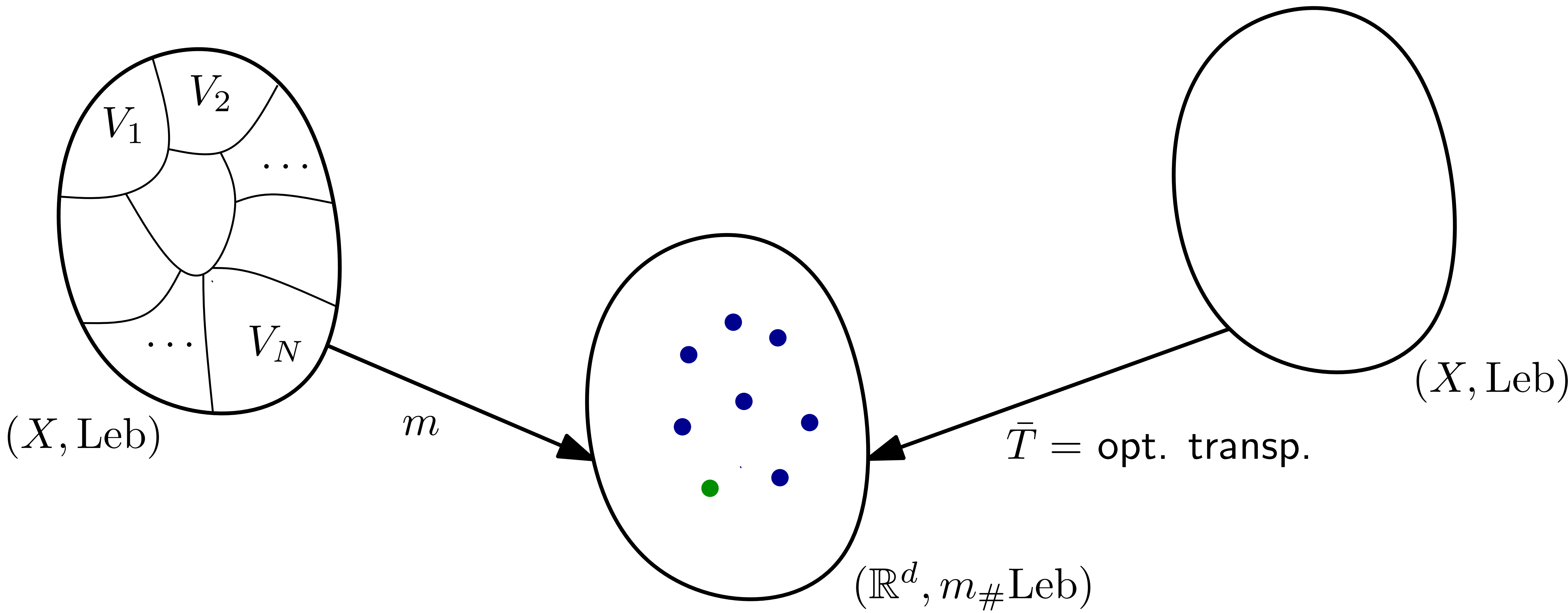


Fast computations of $\mathrm{d}^2_{\mathbb{S}}$ and $\nabla\,\mathrm{d}^2_{\mathbb{S}}$ are possible in 2D [M. '11] and 3D [Lévy '15]

# Space-discretization

**Objective:** Constructing a finite-dimensional subspace of $\mathbb{M}$ and computing $\Pi_{\mathbb{S}}$

$\longrightarrow$ $X$ is partitioned into $(V_k)_{1\leq k\leq N}$ with $\mathrm{Leb}(V_k) = \frac{1}{N}$ and $\mathrm{diam}(V_k) \simeq N^{-\frac{1}{d}}$

$\longrightarrow$ $\mathbb{M}_N :=$ { piecewise constant functions on $(V_k)$} $\subseteq \mathbb{M}$

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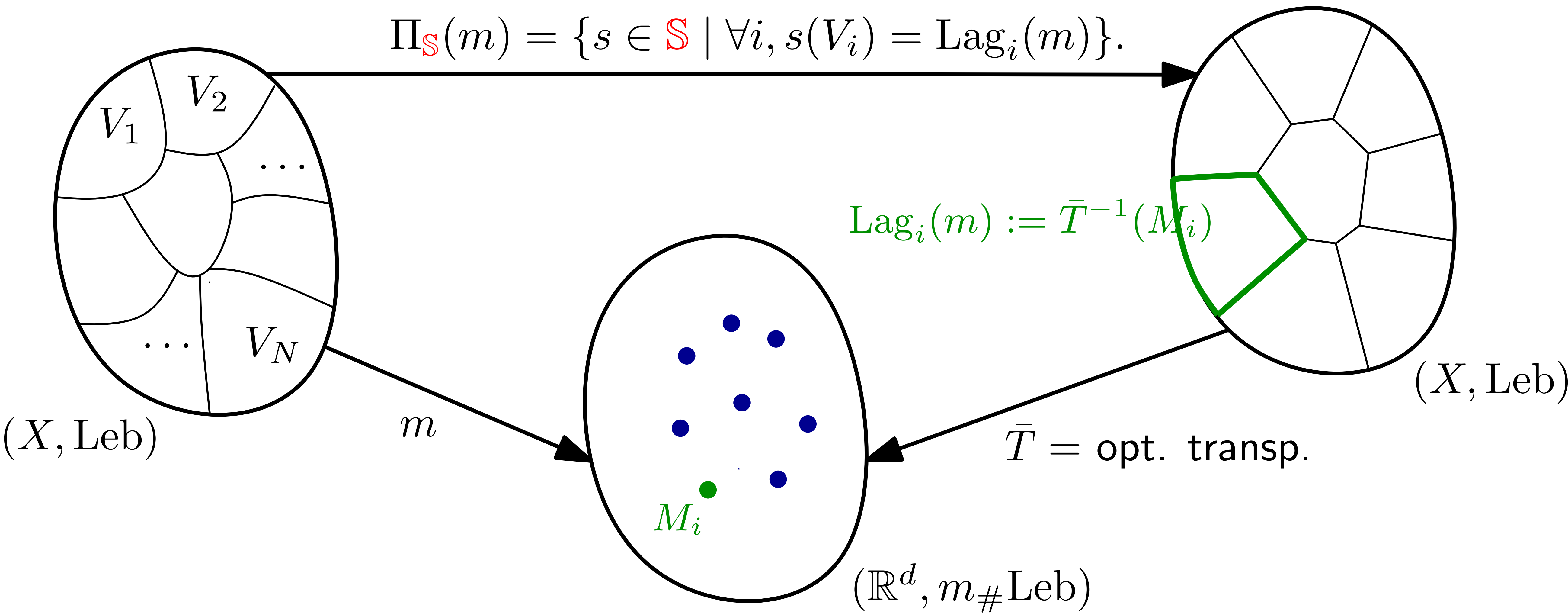


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# Convergence of the space-discretization

space-discretization: (1) $\begin{cases} \ddot{m}(t) + \frac{1}{2\varepsilon^2}(m - \Pi_{\mathbb{M}_N} \circ \Pi_{\mathbb{S}}(m(t))) = 0 \\ m(t) \in \mathbb{M}_N \\ (m(0), \dot{m}(0)) = (\Pi_{\mathbb{M}_N}(\mathrm{id}), \Pi_{\mathbb{M}_N}(u_0)) \end{cases}$

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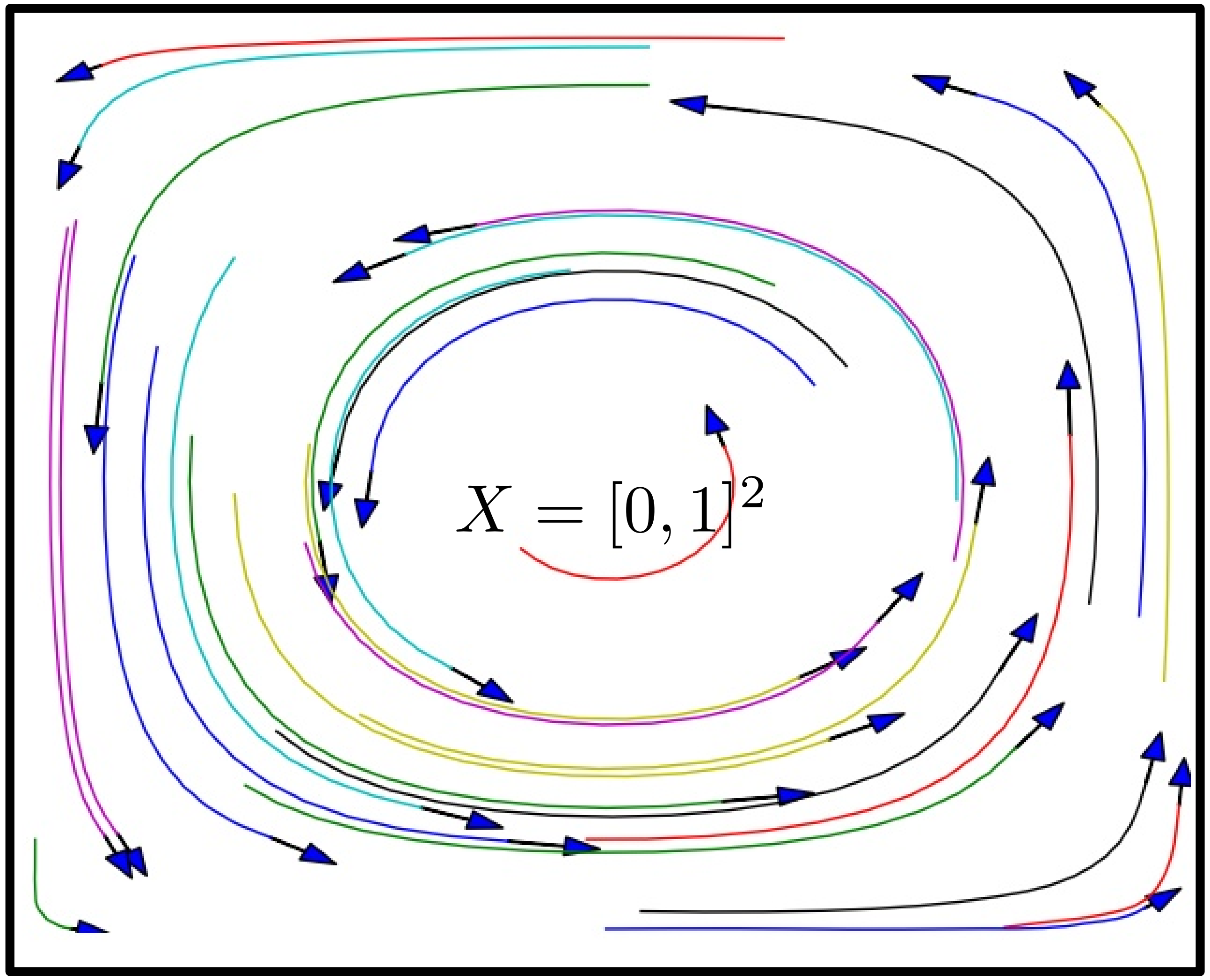
$\longrightarrow$ Convergence of a time-discretization using the symplectic Euler scheme.

# Numerical result: Stationary flow on $[0,1]^2$

**Stationary flow on $[0,1]^2$:** speed: $u(\mathbf{x}) = (\cos(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\cos(\pi x_2))$

pressure: $p(\mathbf{x}) = \frac{1}{4}(\sin^2(\pi x_1) + \sin^2(\pi x_2))$



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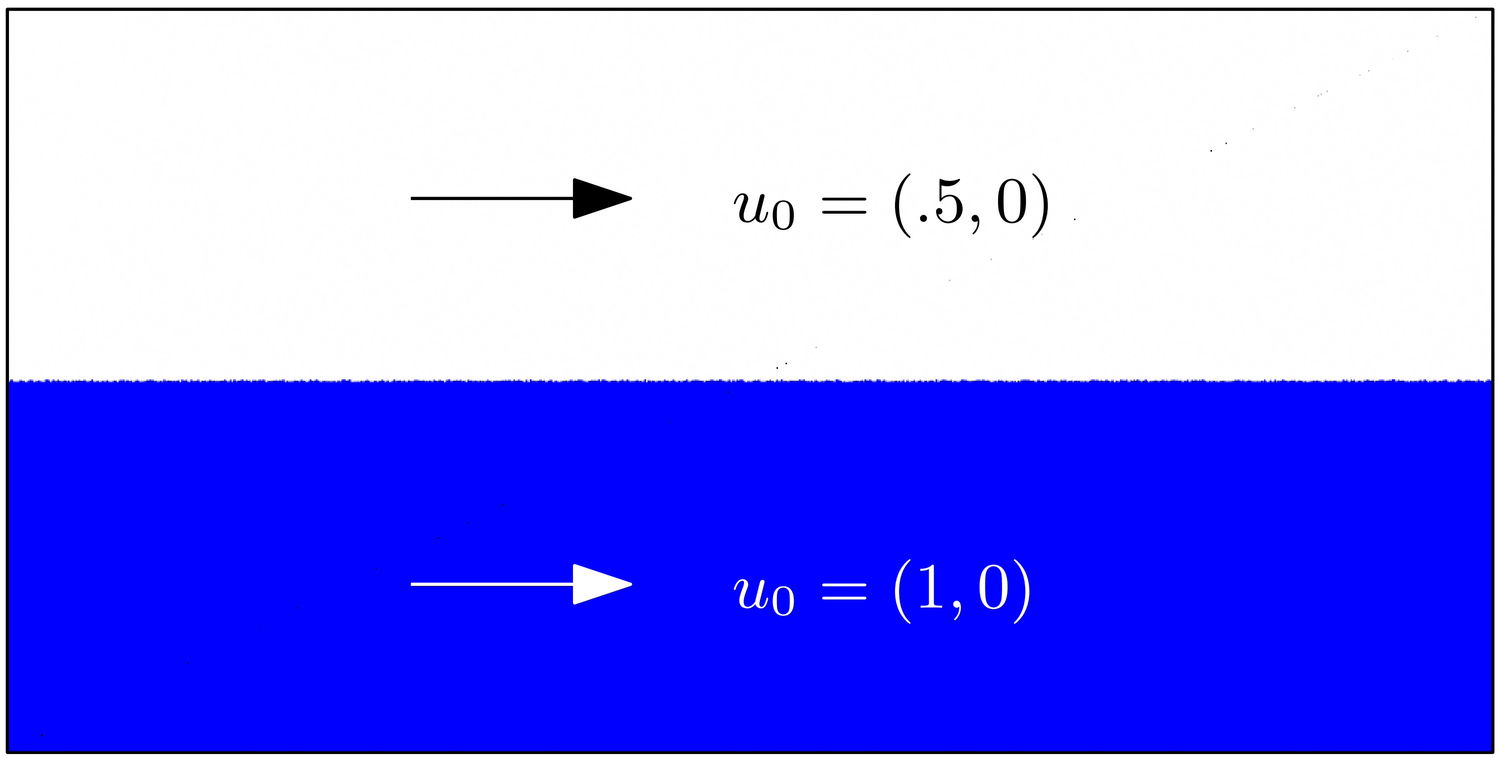


$X = [0,2] \times [-.5,.5]/(x = 0 \sim x = 2)$

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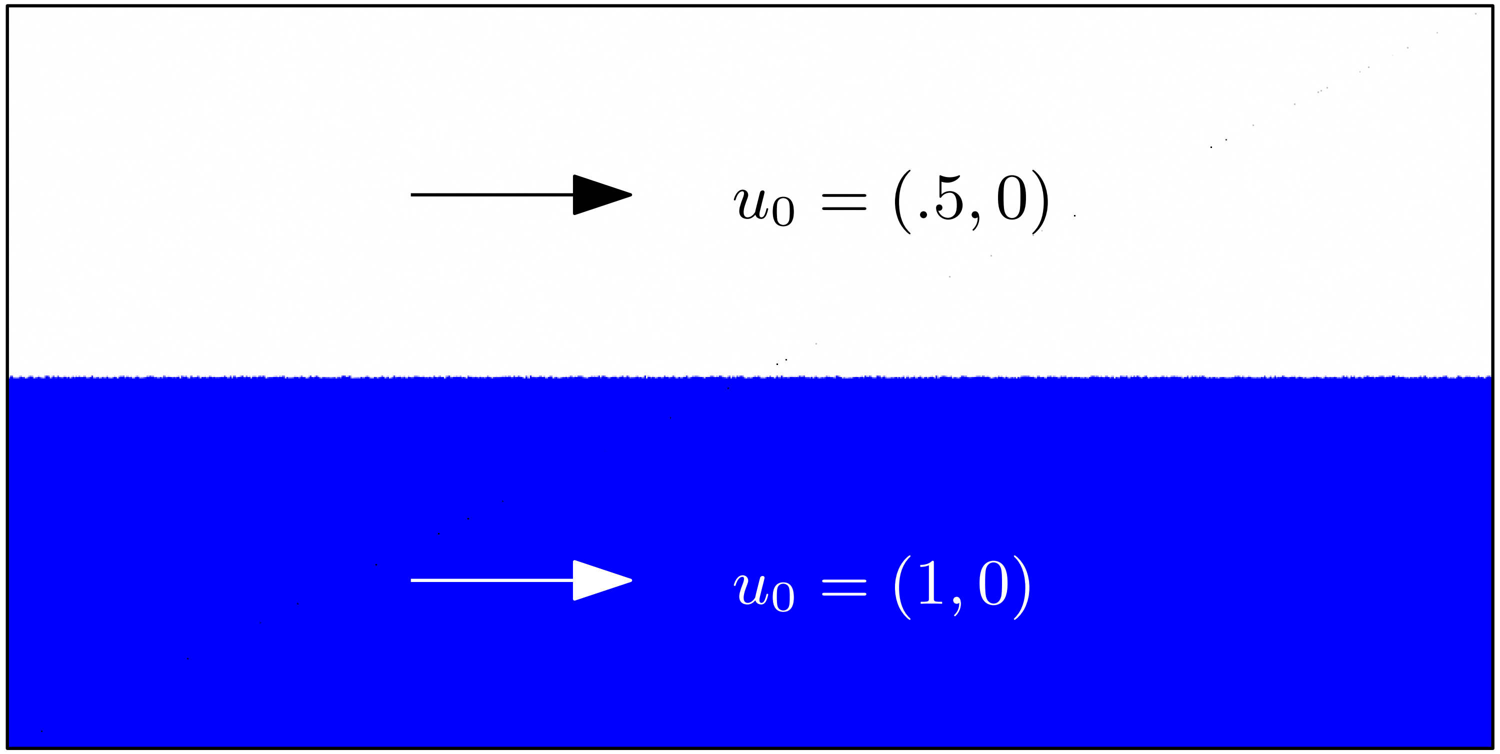


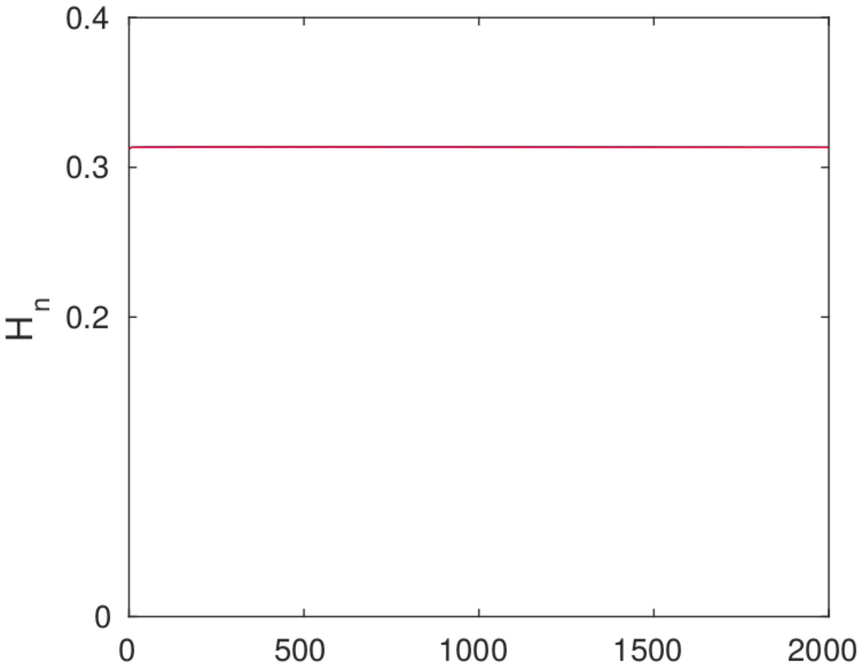


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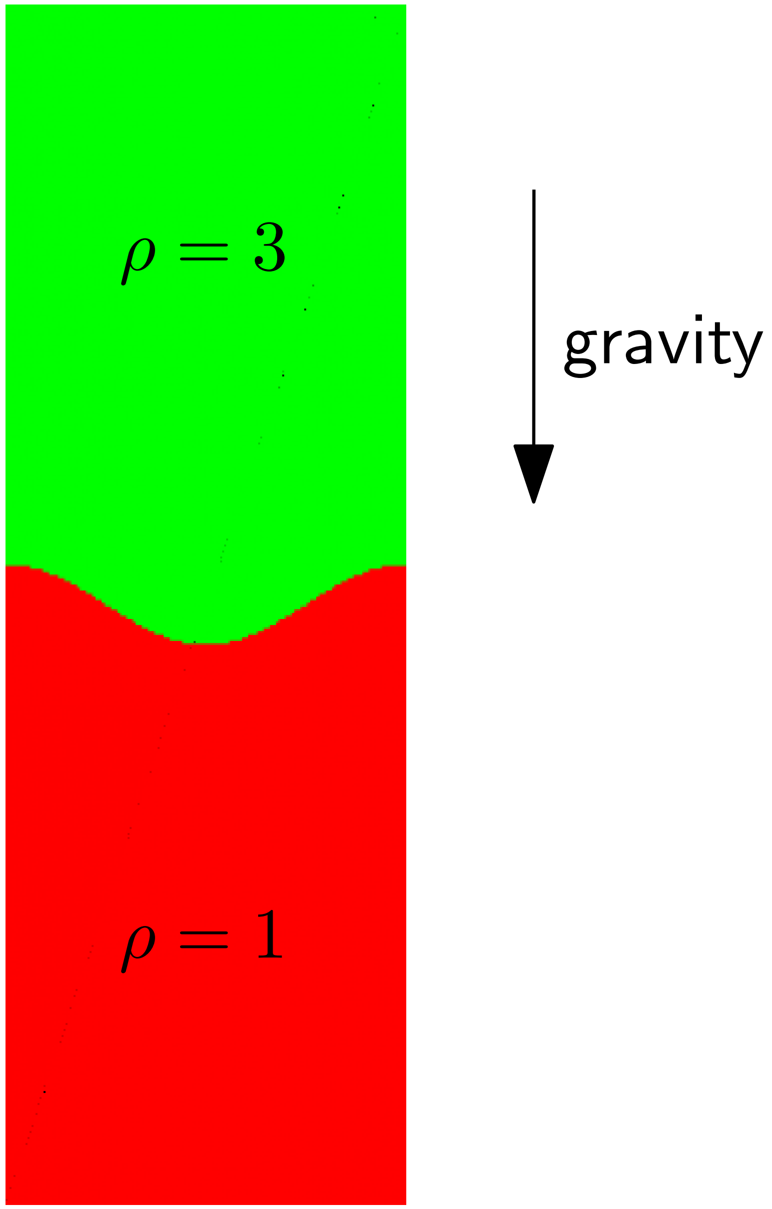
**B.** Rayleigh-Taylor instability (Inhomogeneous fluid)




$X = [-1, 1] \times [-3, 3]$

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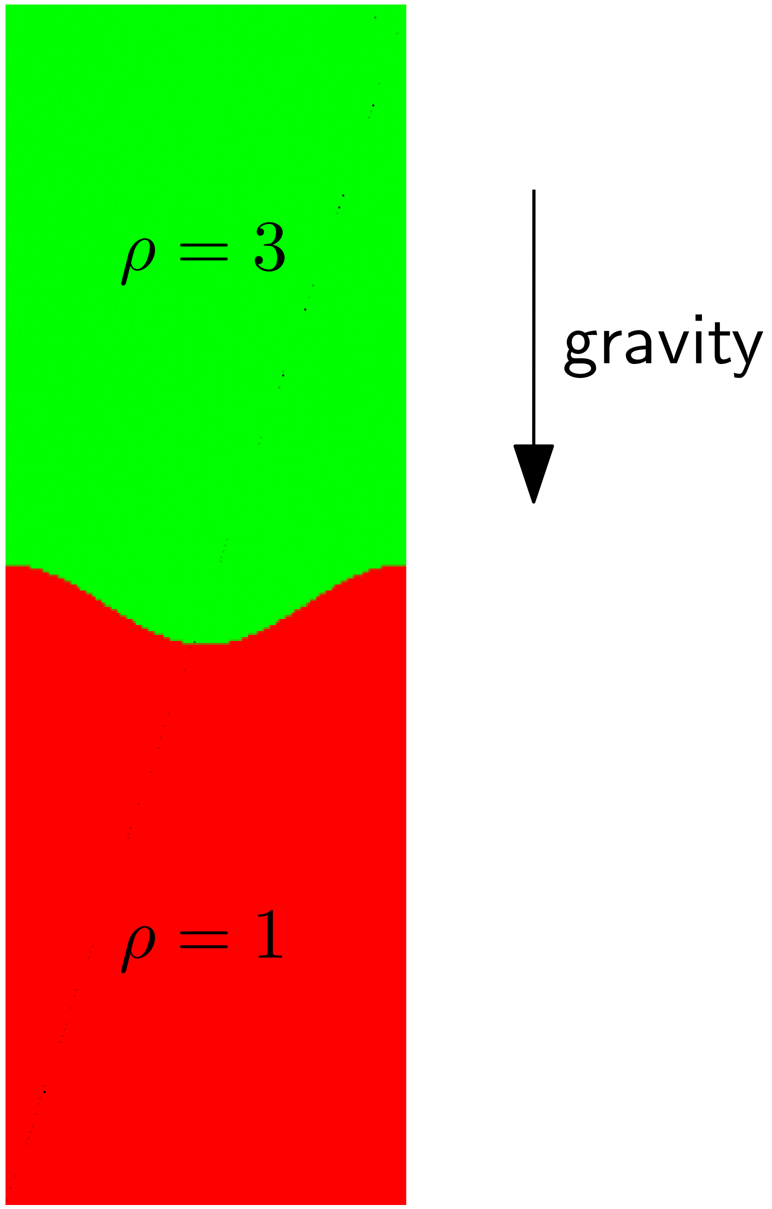


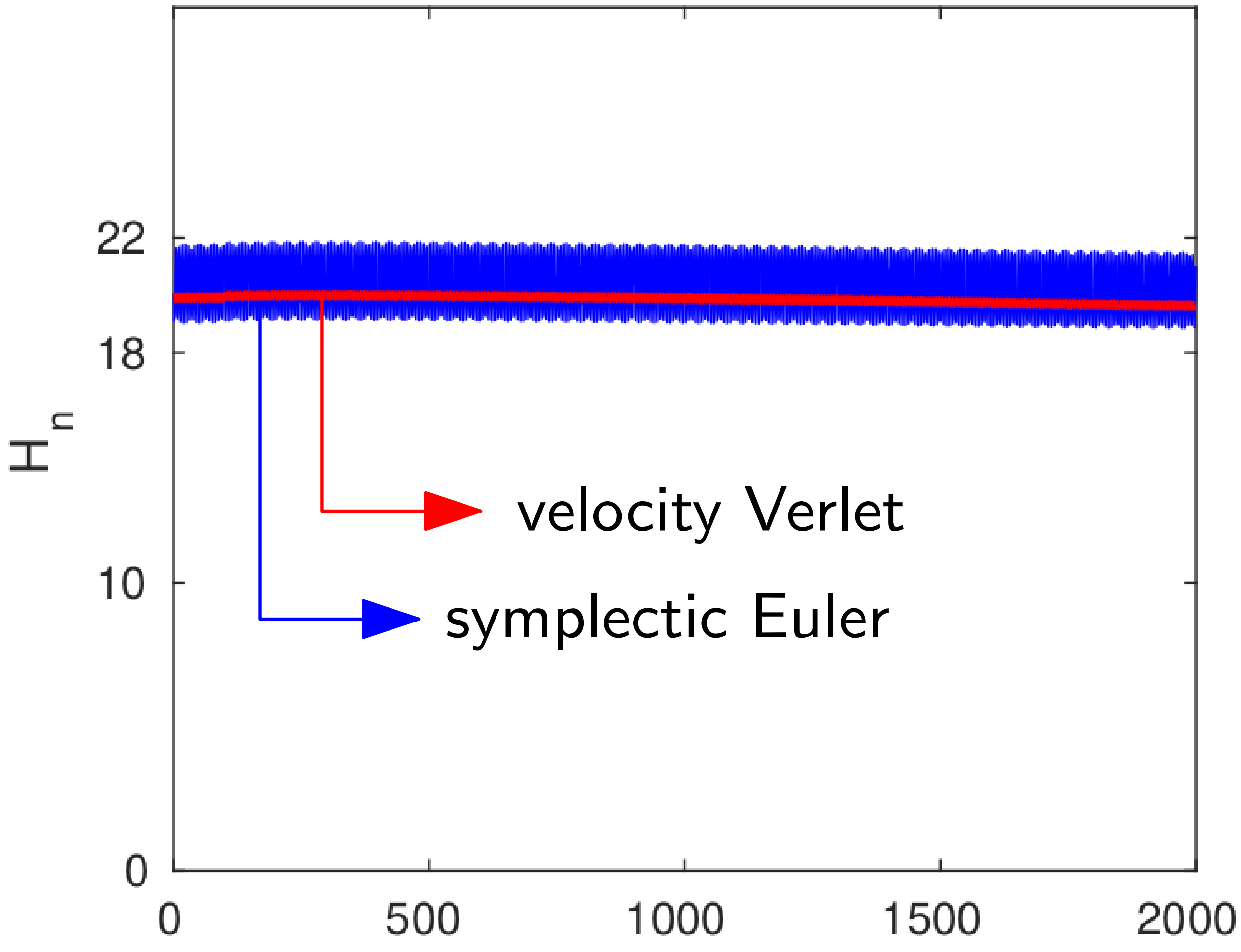


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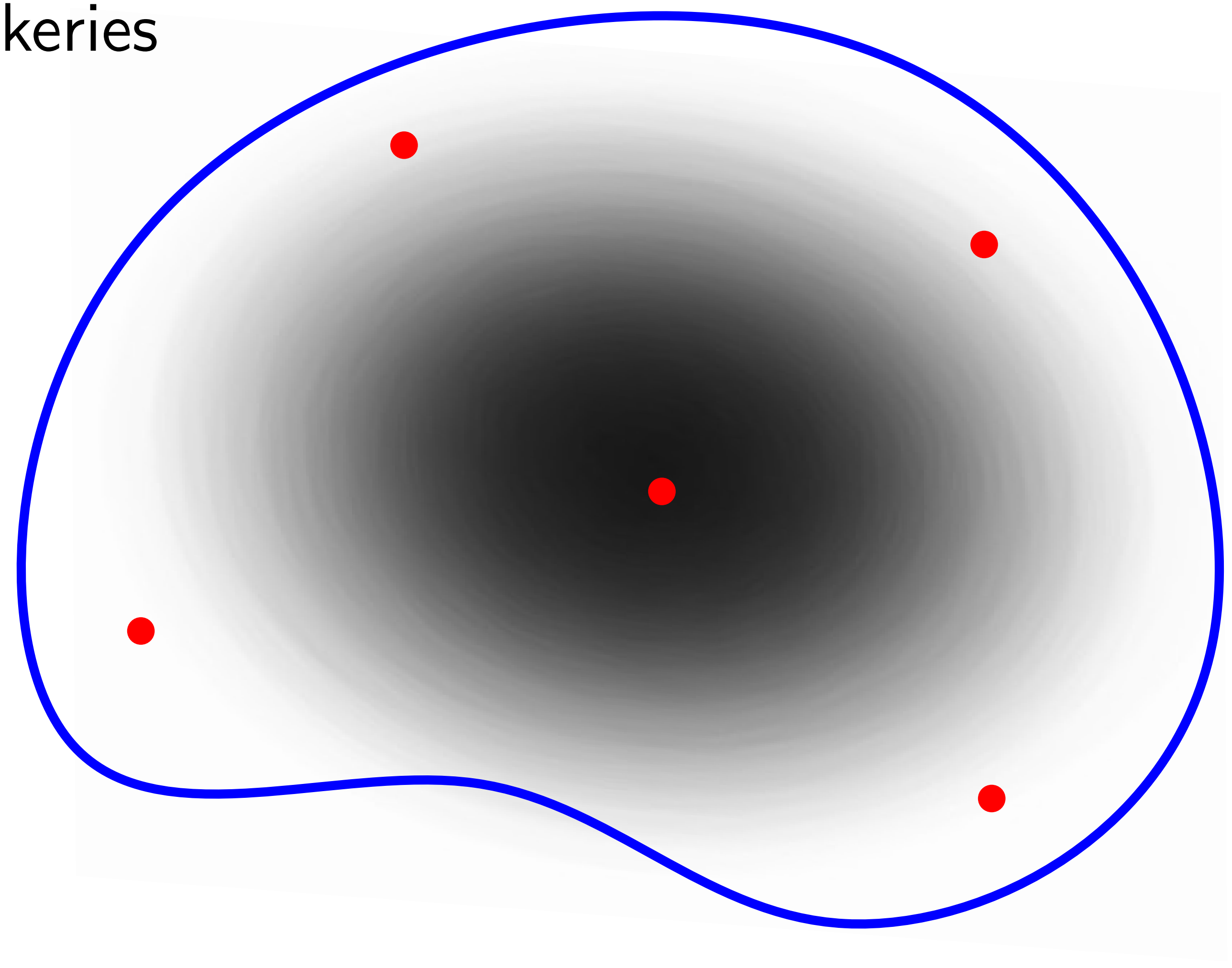
# 2. Semi-discrete optimal transport

# An economic metaphor

$\rho : X \to \mathbb{R}$ density of population

$c(x, y) := \|x - y\|^2$ cost of walking from $x$ to $y$

$Y$ = location of bakeries



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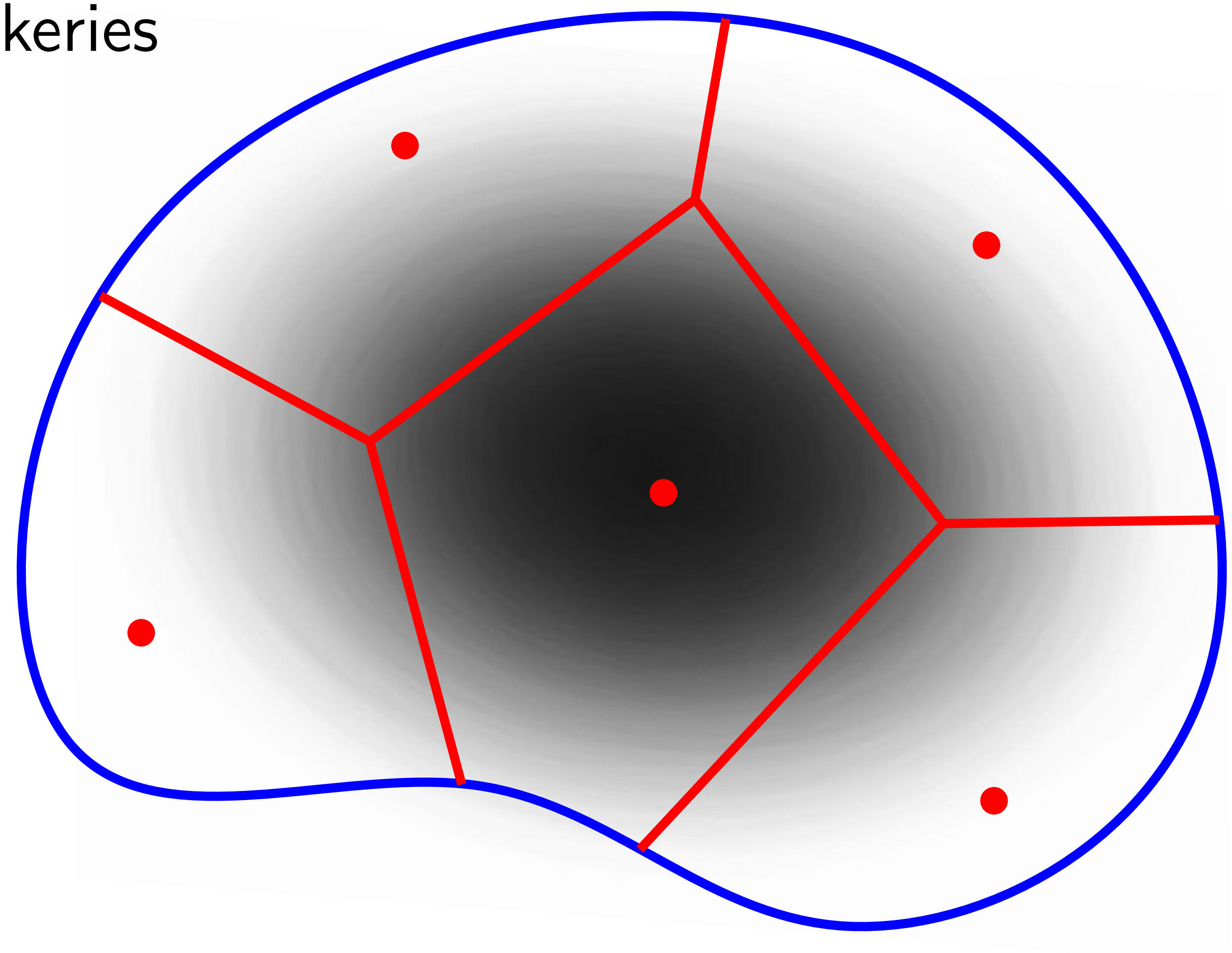

- If the price of bread is uniform, people go the closest bakery:

$$\mathrm{Vor}(y) = \{x \in X; \forall z \in Y,\ c(x,y) \leq c(x,z)\}$$

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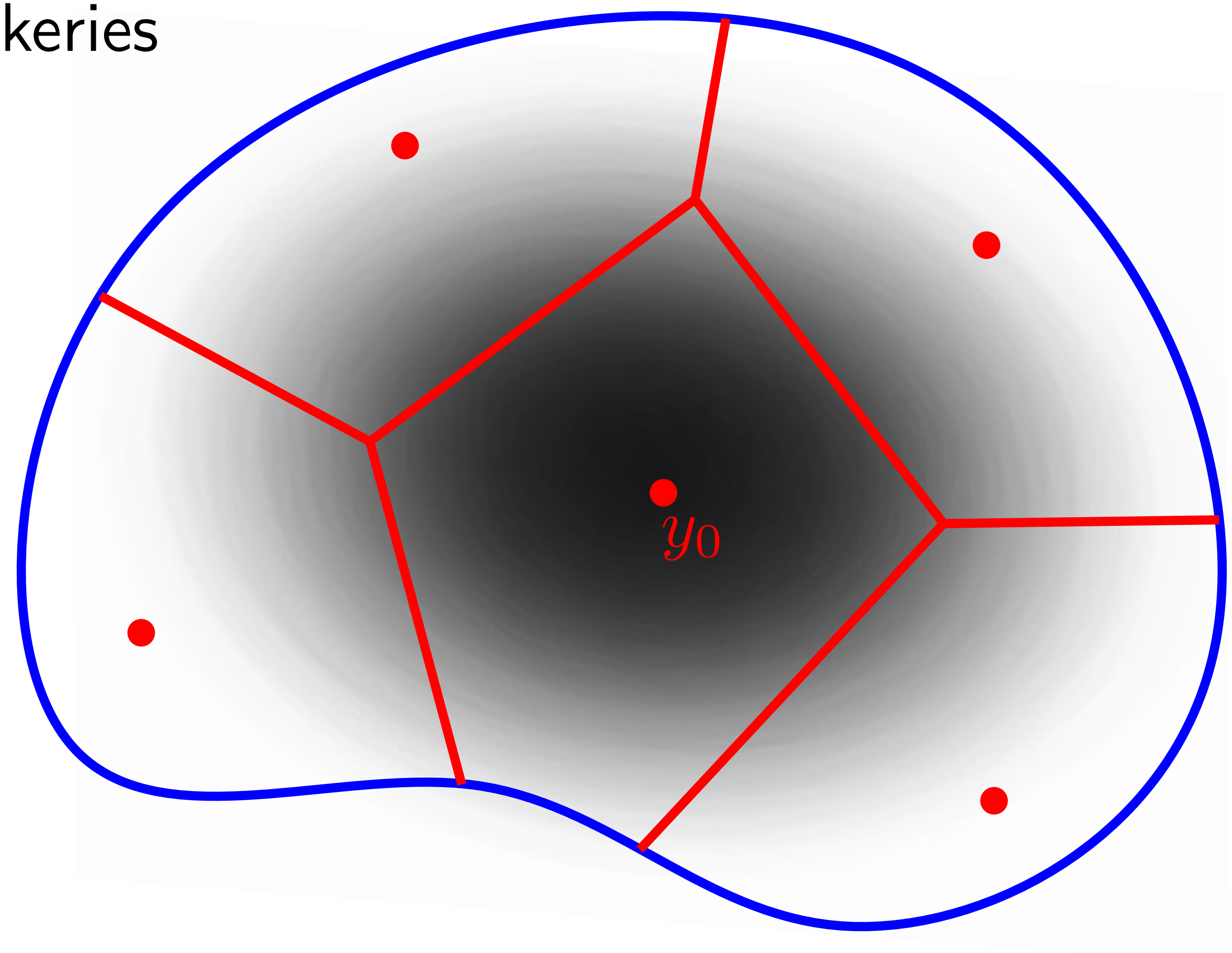


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Minimizes total distance walked ... **but** might exceed the capacity of bakery $y_0$!

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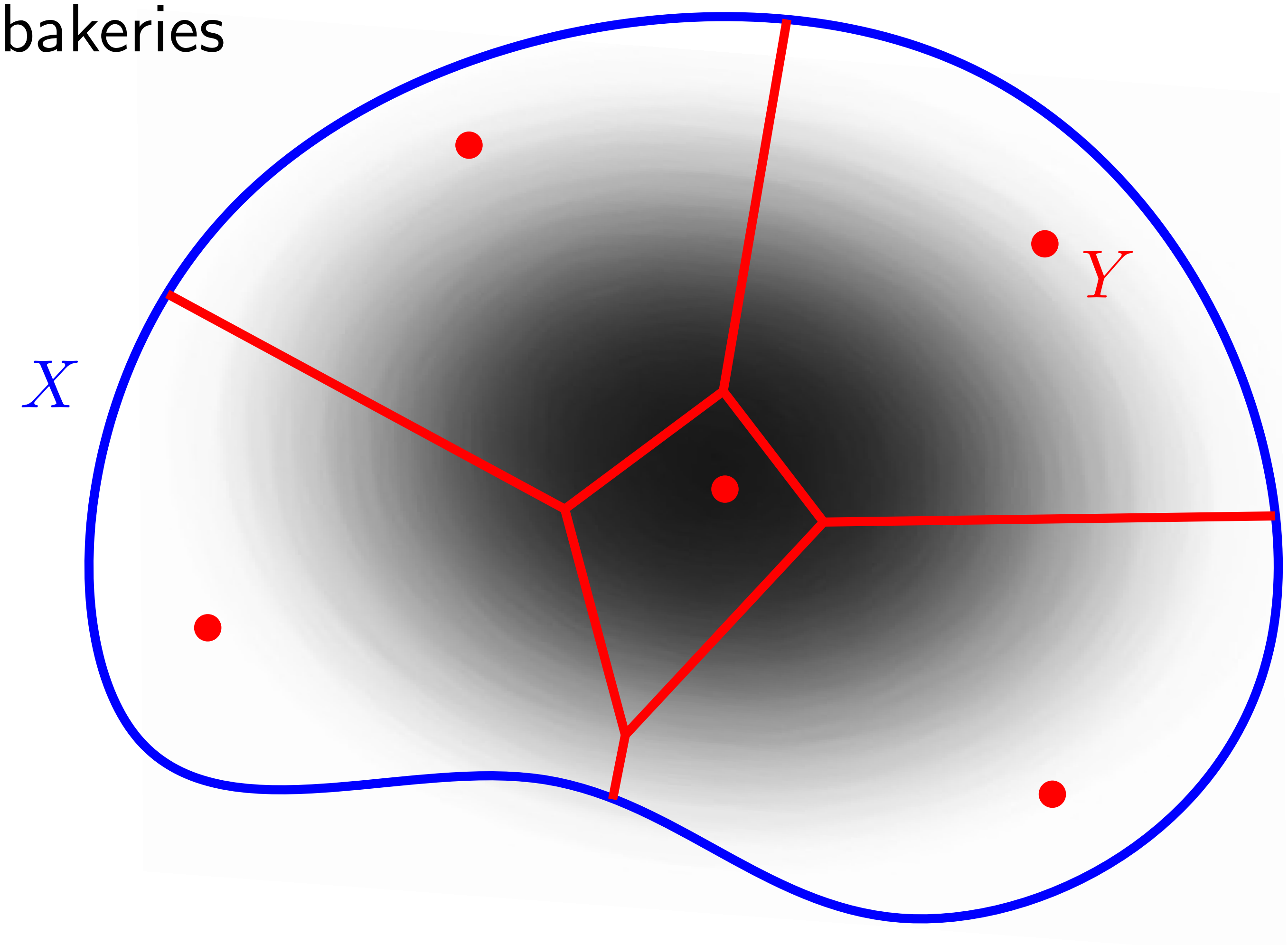


- If prices are given by $\psi : Y \to \mathbb{R}$, people make a compromise:

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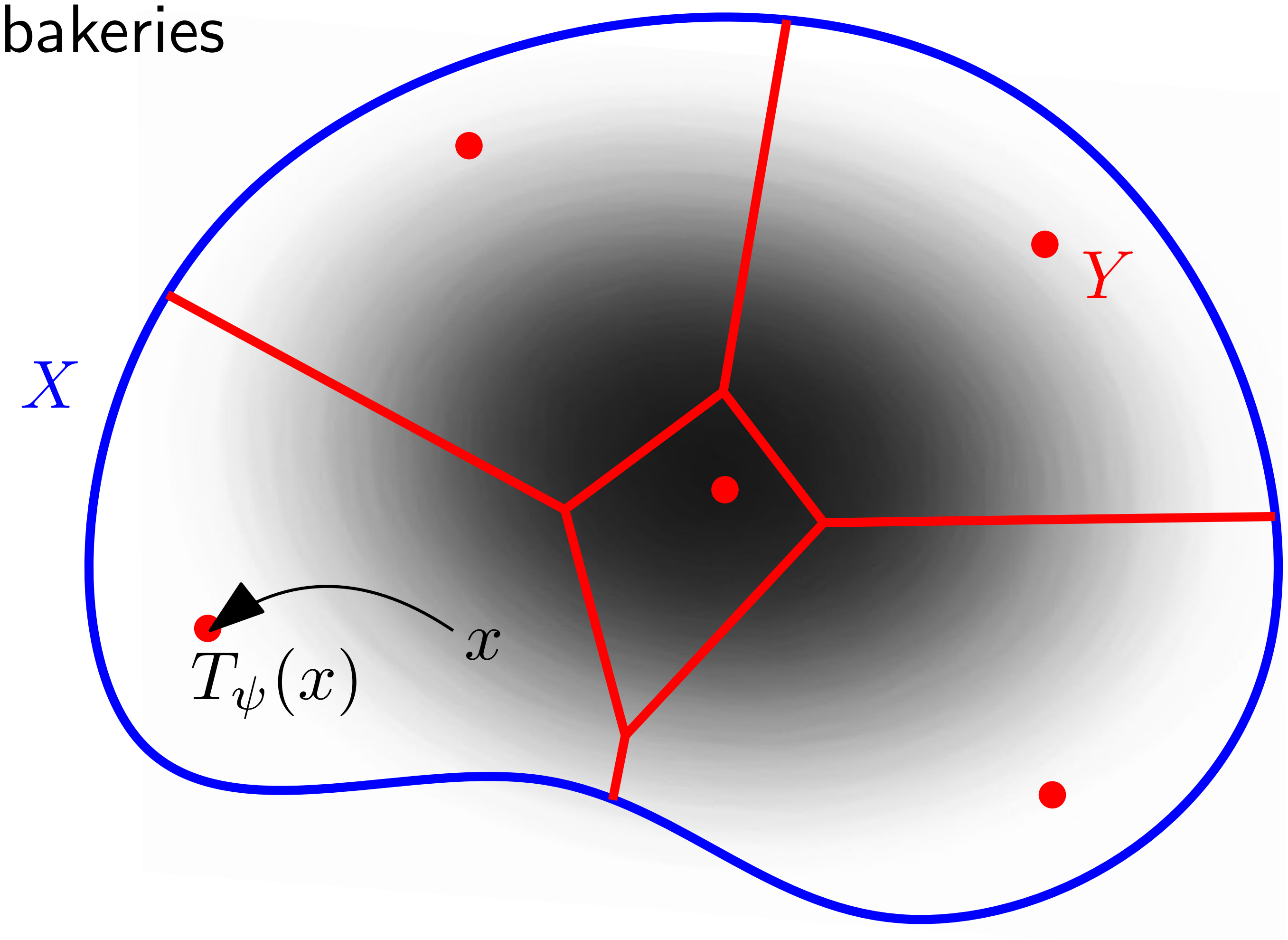


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**Lemma:** The map $T_\psi$ induced by this decomposition is a $c$-**optimal transport** between $\rho$ and $\nu_\psi := T_{\psi\#}\nu = \sum_{y \in Y} \rho(\mathrm{Lag}_y(\psi))\delta_y$.

# SD-OT as Concave Maximization

**Theorem:** Finding an **optimal transport** between $\rho$ and $\nu = \sum_Y \nu_y \delta_y$

$\Longleftrightarrow$ finding **prices** $\psi$ on $Y$ such that $\nu_\psi = \nu$ [Gangbo McCann '96]

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In the simulations, we use a (damped) Newton's algorithm, solving a sequence of linearized **discrete** Monge-Ampère equations.

# Numerical example

- Simple damped Newton's algorithm, with global linear convergence, [Mirebeau 15]
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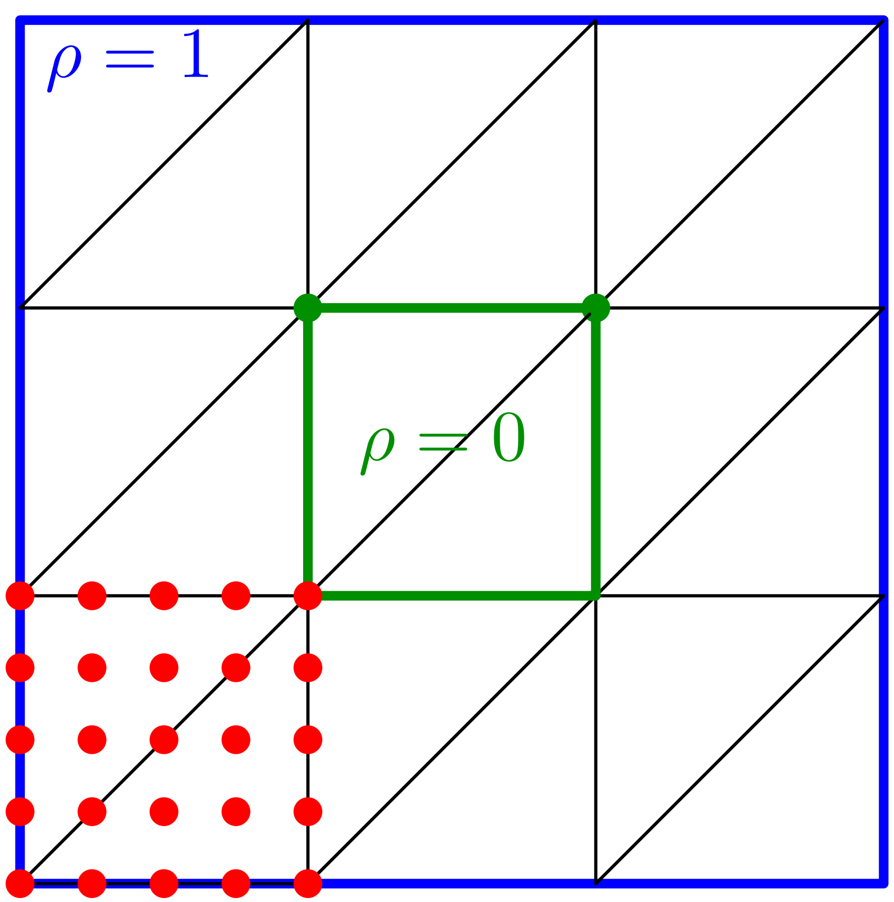


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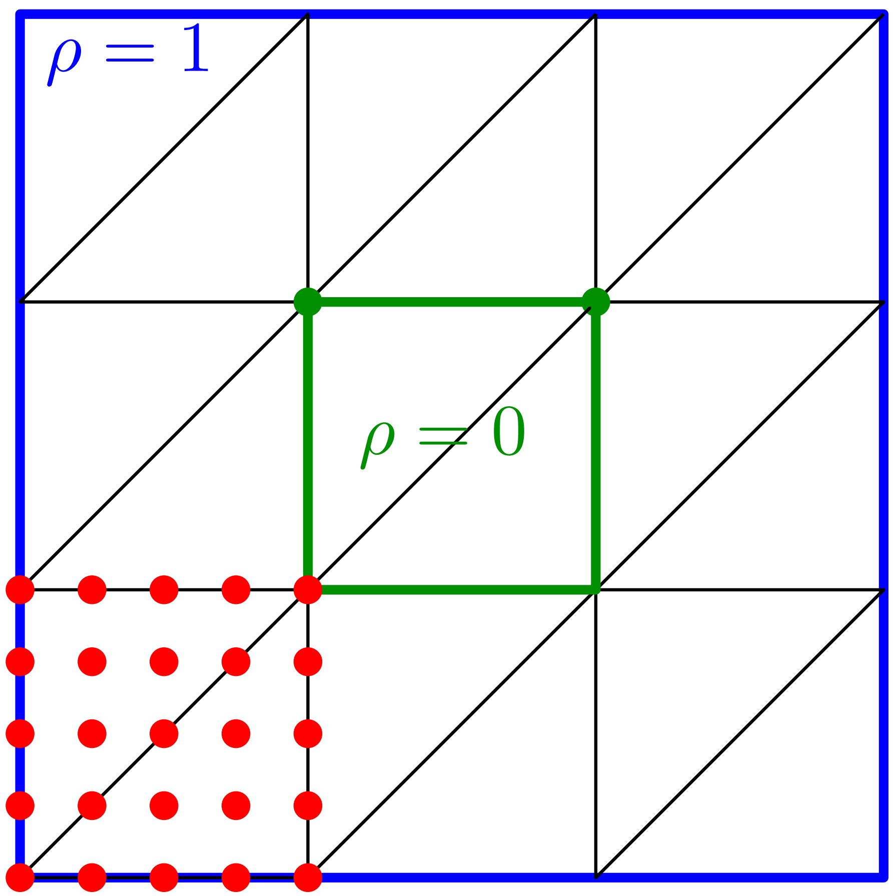


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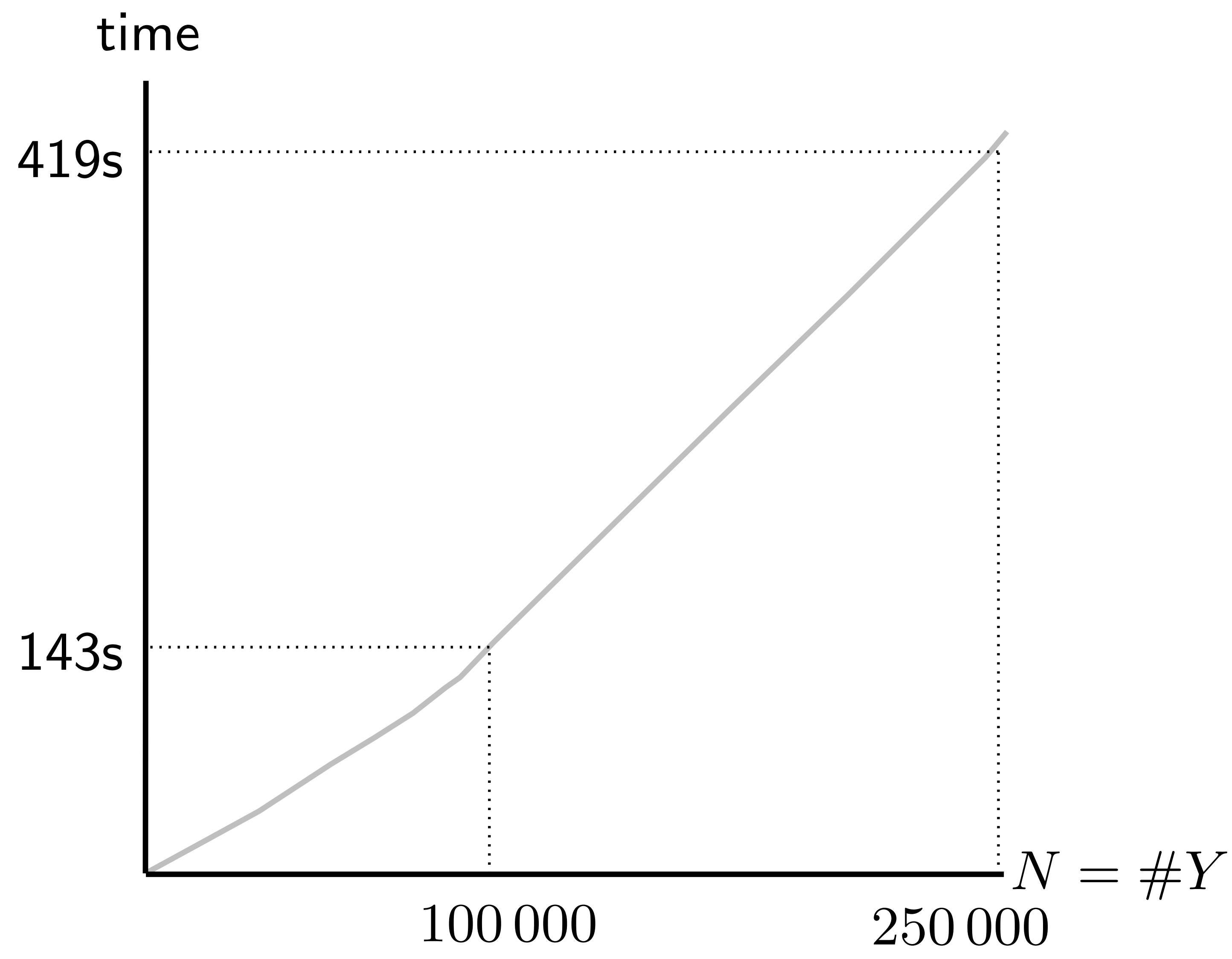

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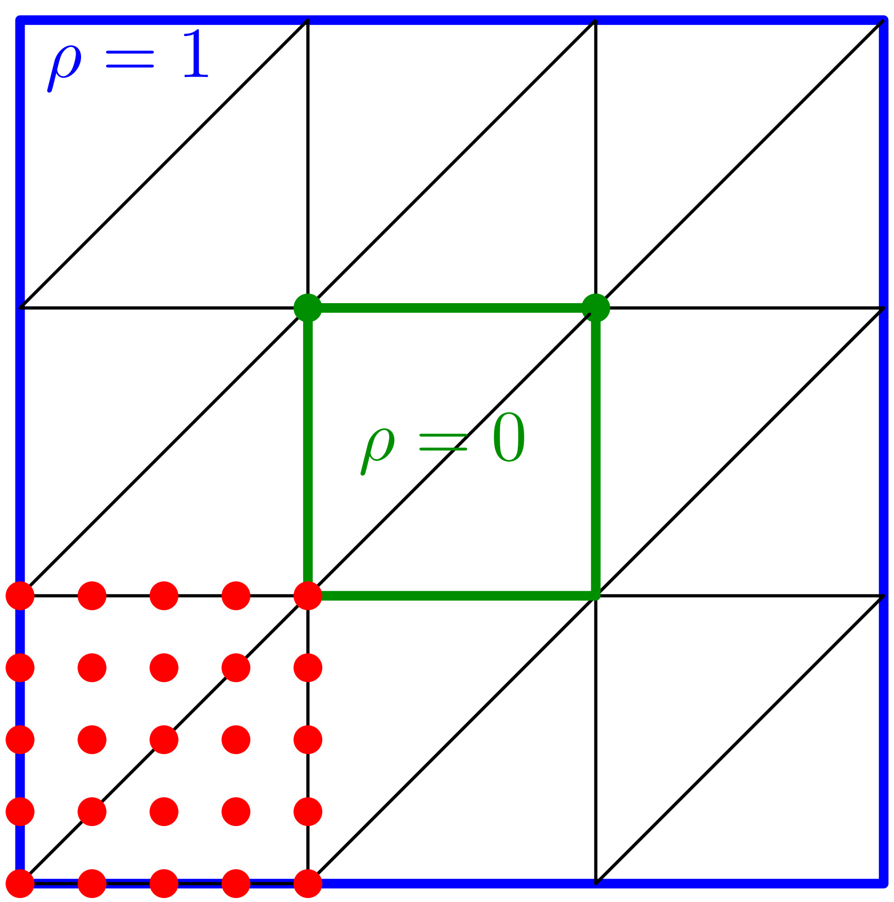


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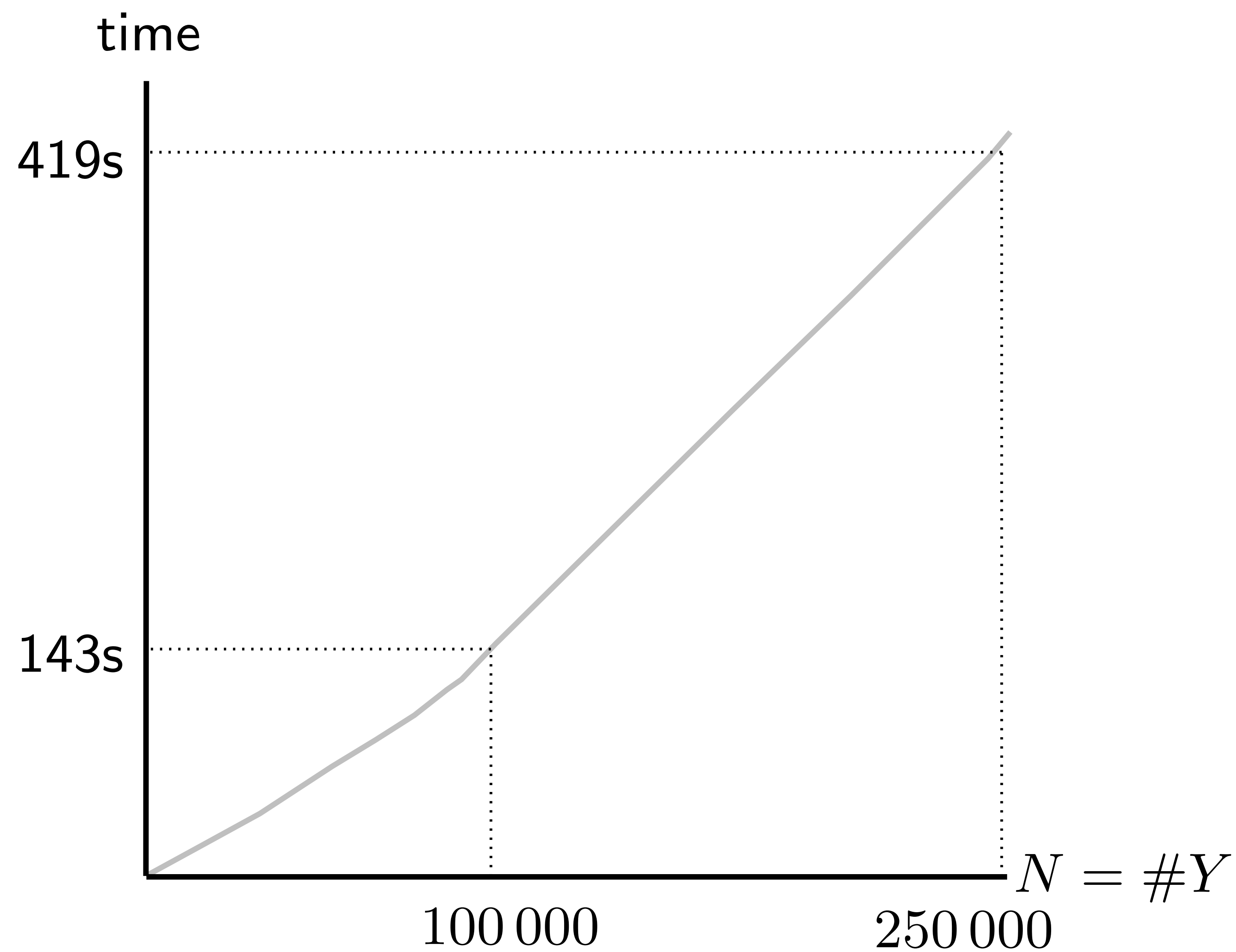


**Near-$\mathcal{O}(N)$ vs $\mathcal{O}(N^3)$ complexity for fully discrete (combinatorial) OT.**

# 3. Minimizing geodesics in $\mathbb{S}\mathrm{Diff}$

Joint work with Jean-Marie Mirebeau

# Finite-dimensional example

Let $S$ be a submanifold in $\mathbb{R}^d$, whose minimizing geodesics need to be approximated.

- **Minimizing geodesics:** $\min_{s:[0,1]\to\mathbb{R}^d} \frac{1}{2}\int_0^1 \|\dot{s}_t\|^2 \,\mathrm{d}\,t$ where $\begin{cases} \forall t \in [0,1],\ s_t \in S \\ s_0 = s_*, s_1 = s^* \end{cases}$

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- **Relaxation:** Given a penalization parameter $\alpha > 0$, consider

$$\min_{m:[0,1]\to\mathbb{R}^d} \frac{1}{2}\int_0^1 \|\dot{m}_t\|^2 \,\mathrm{d}\,t + \alpha\left(\int_{[0,1]} \mathrm{d}_S^2(m_t)\,\mathrm{d}\,t + \|m_0 - s_*\|^2 + \|m_1 - s^*\|^2\right).$$

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# Finite-dimensional example

Let $S$ be a submanifold in $\mathbb{R}^d$, whose minimizing geodesics need to be approximated.

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Imagine now that only a finite sample $S_K \subseteq S$ is known, with $\mathrm{Card}(S_K) = K$.
$\longrightarrow$ How should $\lambda = \lambda(T,K)$ be chosen ?

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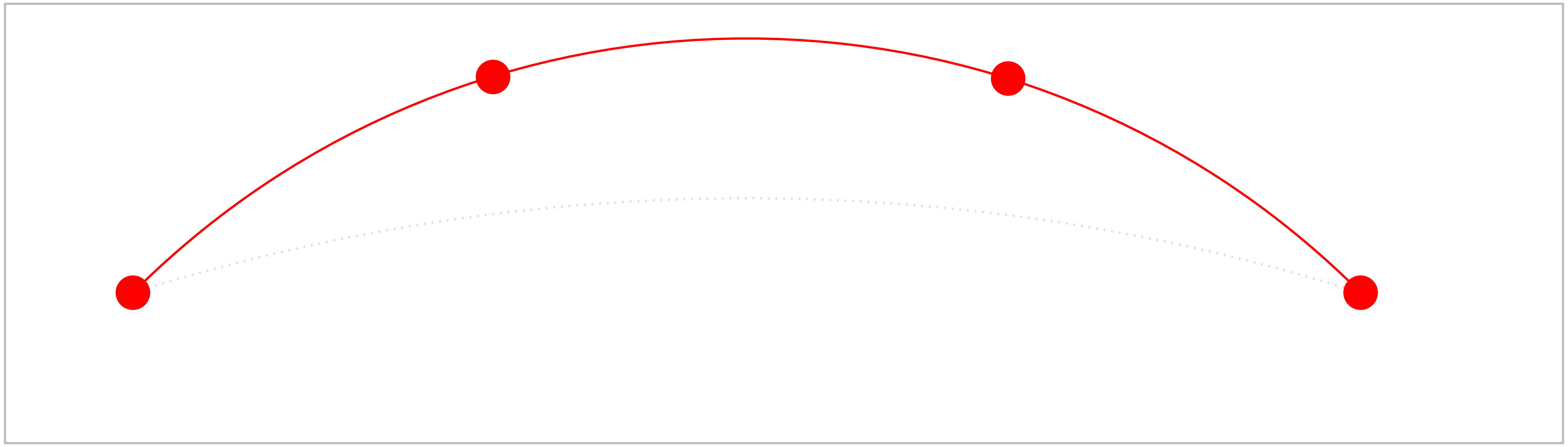

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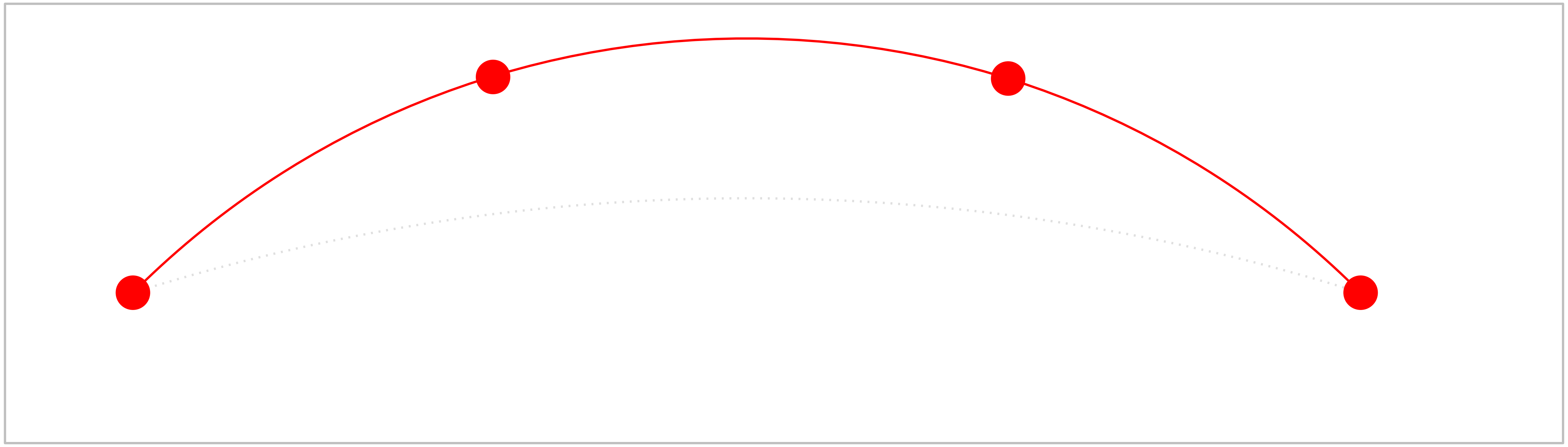

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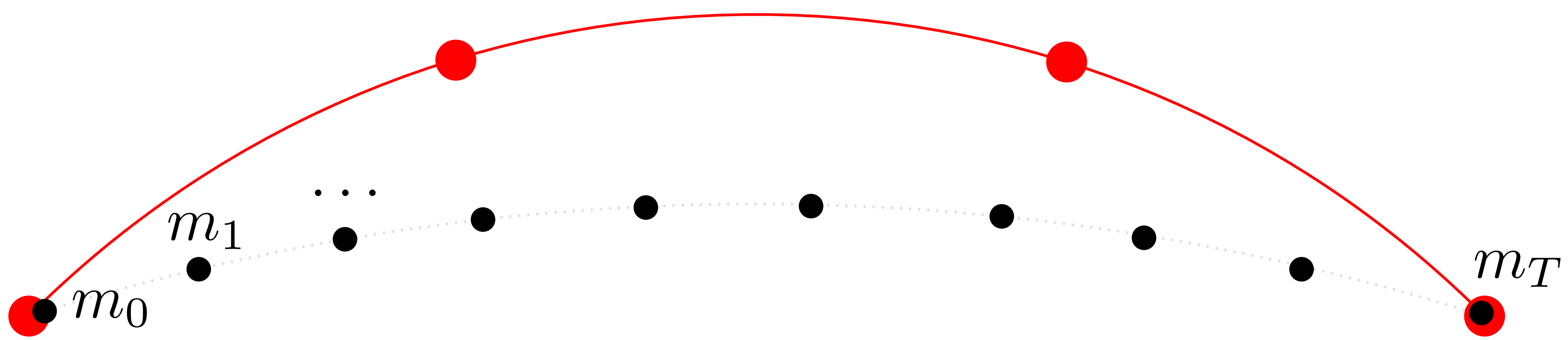


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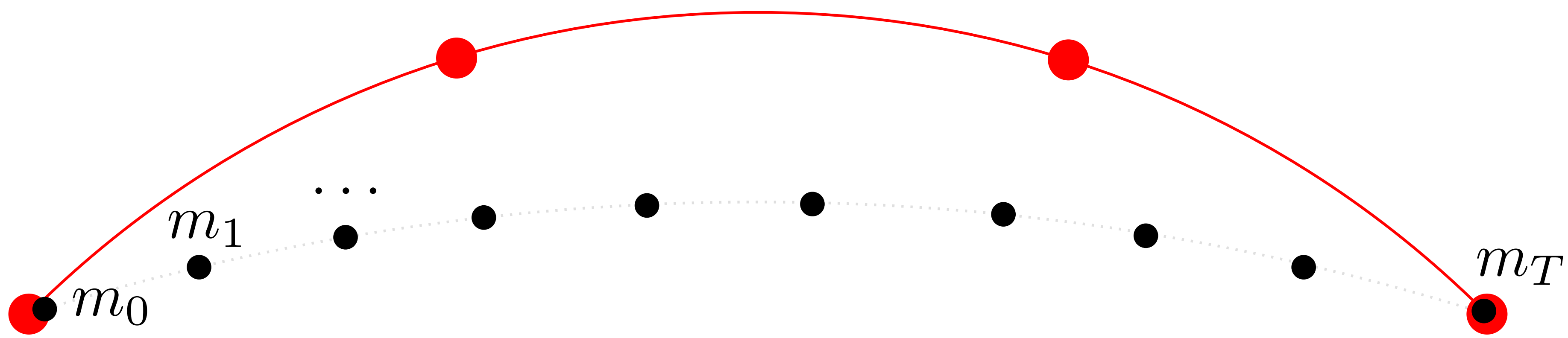


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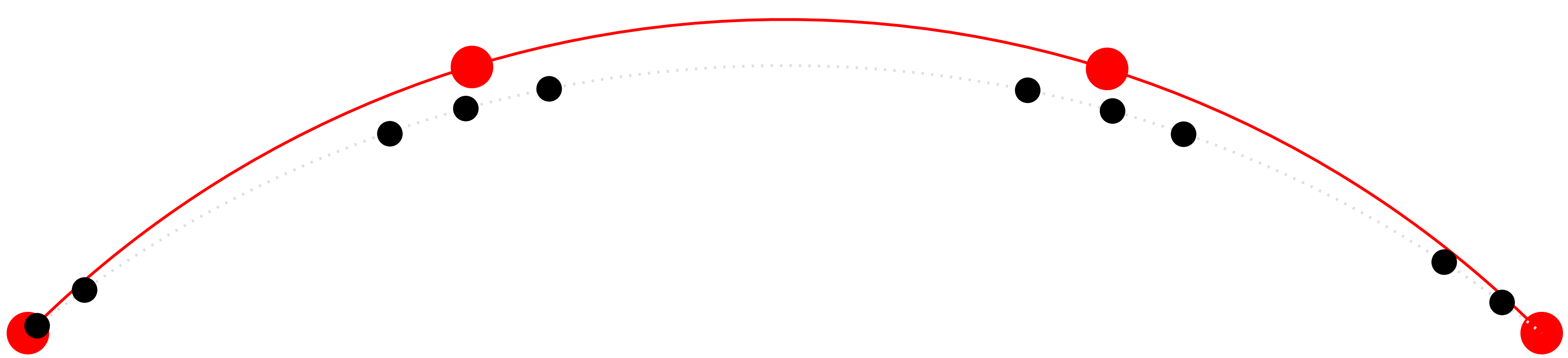

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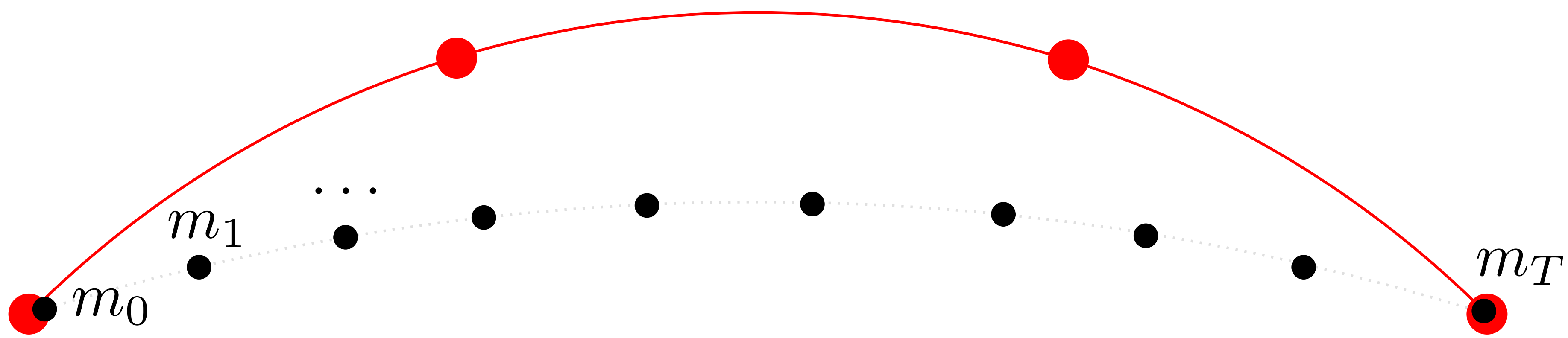


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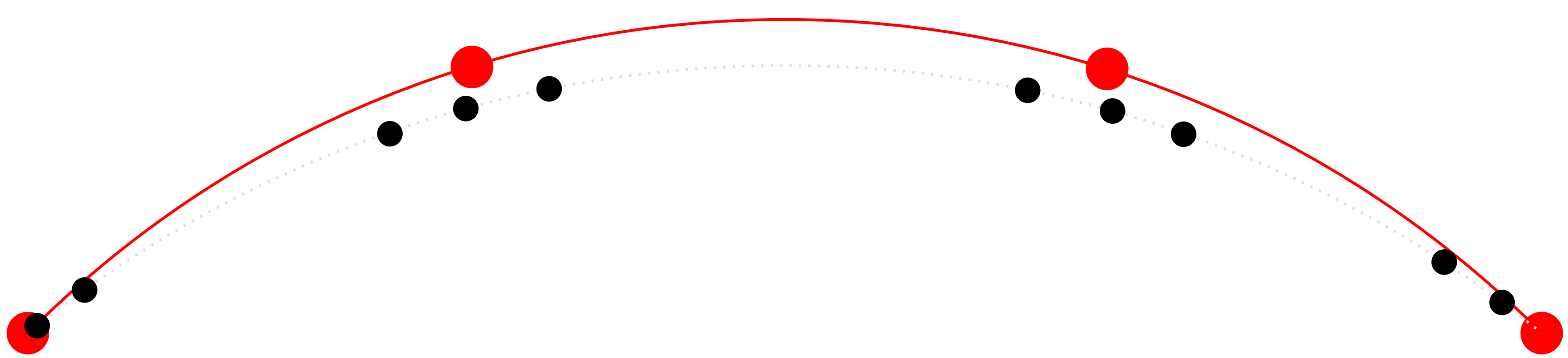

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$\longrightarrow$ combinatorial optimization pb (when $\lambda = +\infty$)

# Minimal geodesics in $\mathbb{S}\mathrm{Diff}$ and relaxations

$\mathrm{Leb}$ = restriction of Lebesgue measure to a compact domain $X$

$\mathbb{S}\mathrm{Diff} = \{s : X \to X$ diffeomorphism $\mid s_{\#}\mathrm{Leb} = \mathrm{Leb}\} \subseteq \mathbb{M} = \mathrm{L}^2(X, \mathbb{R}^d)$

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**C.** relaxation involving measures over the set $\Gamma$ of $\mathcal{C}^0$ paths in $X$.

[Brenier '89]
[Schinerelman '94]

# Brenier's generalized geodesics

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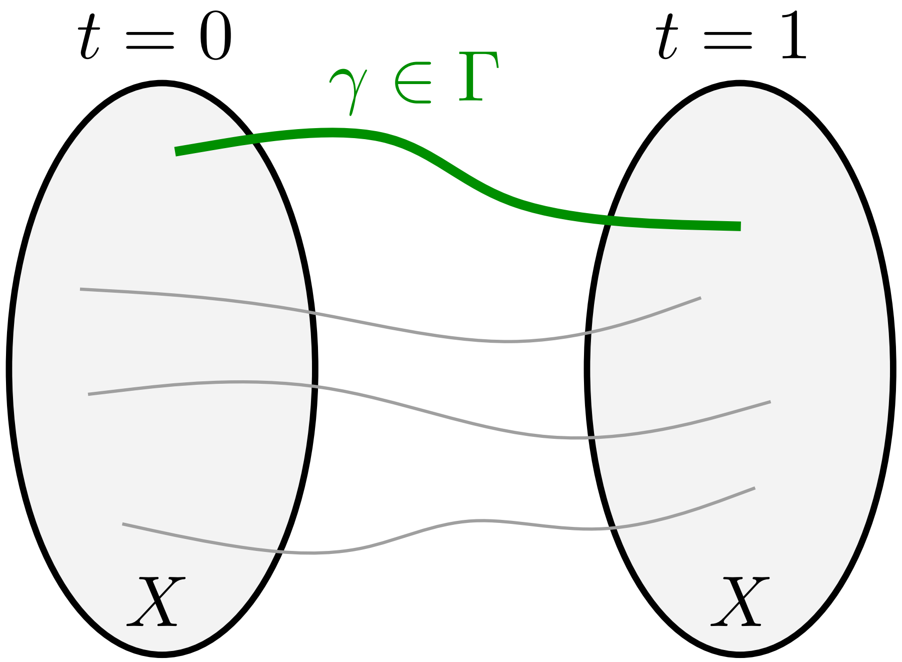

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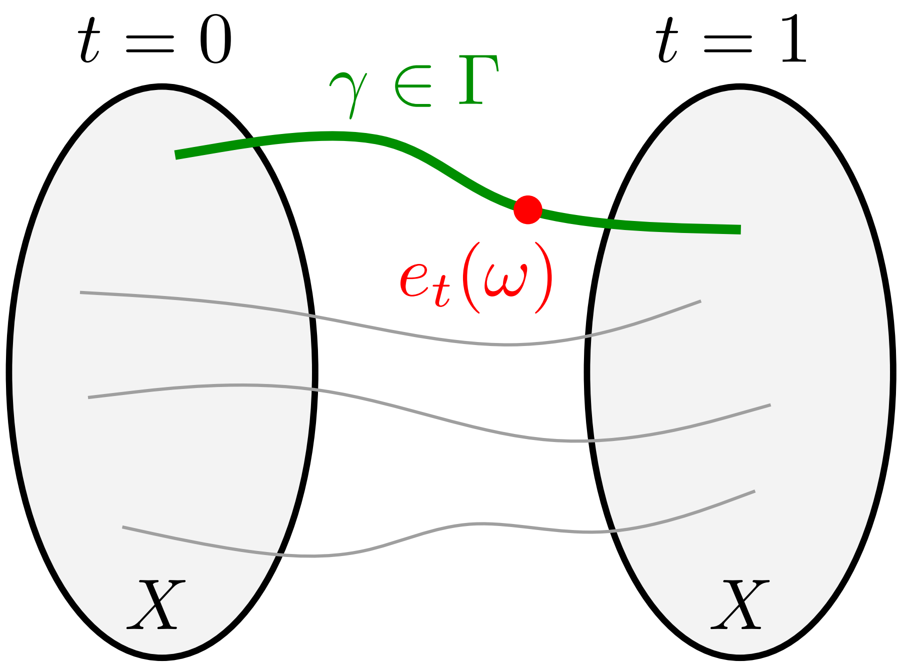


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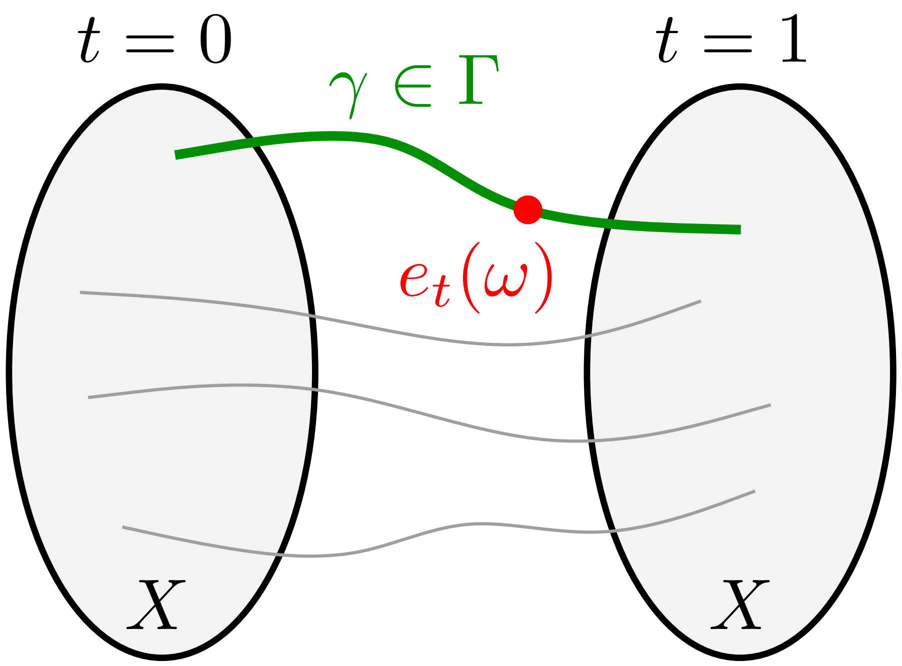


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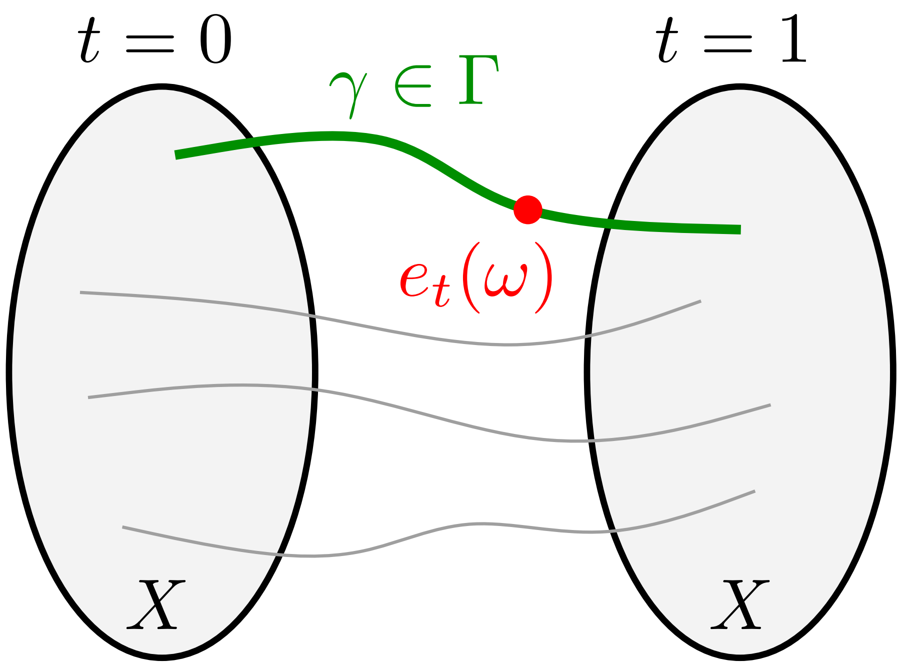


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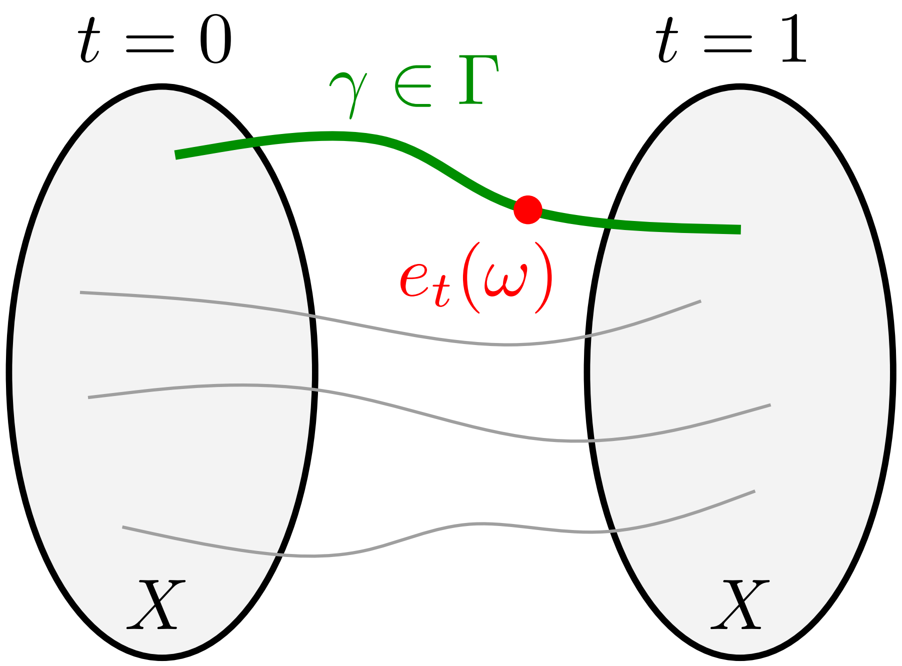


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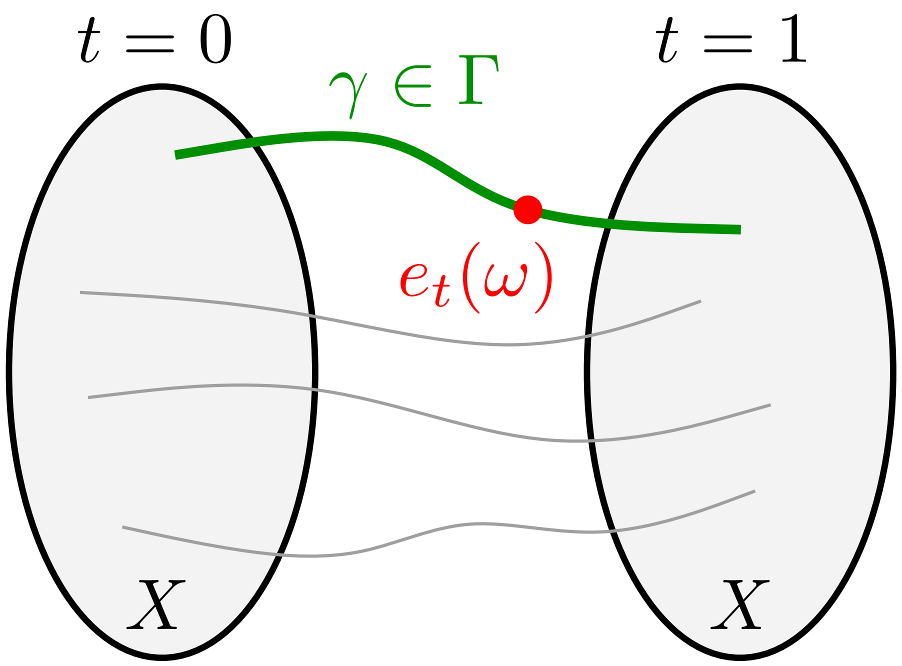


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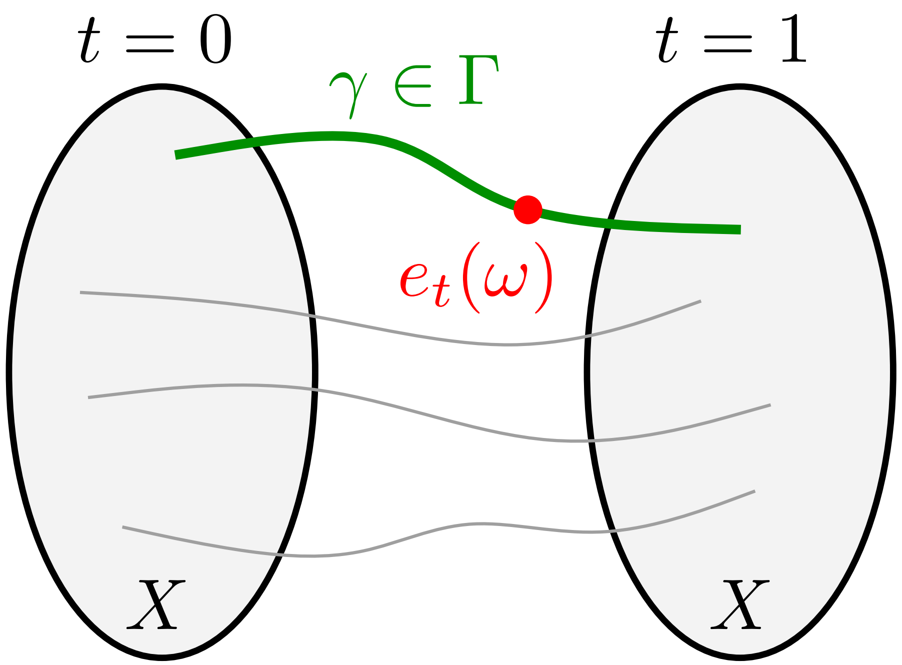


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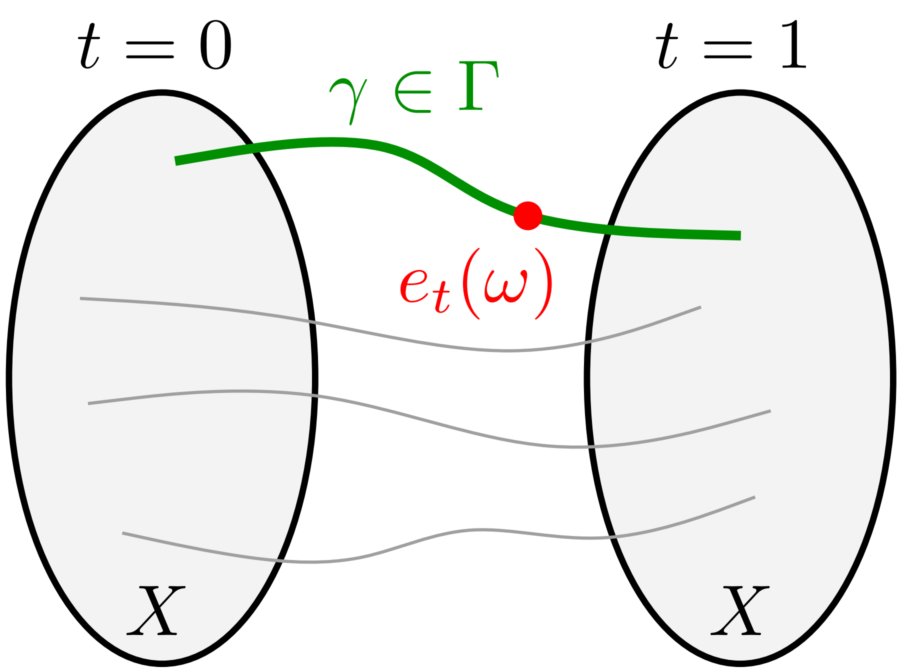


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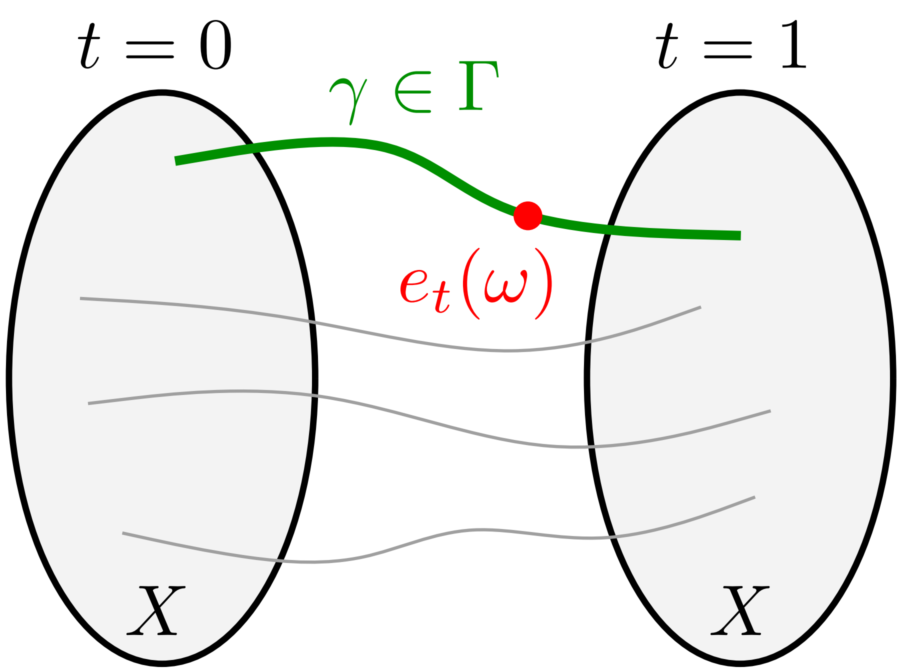


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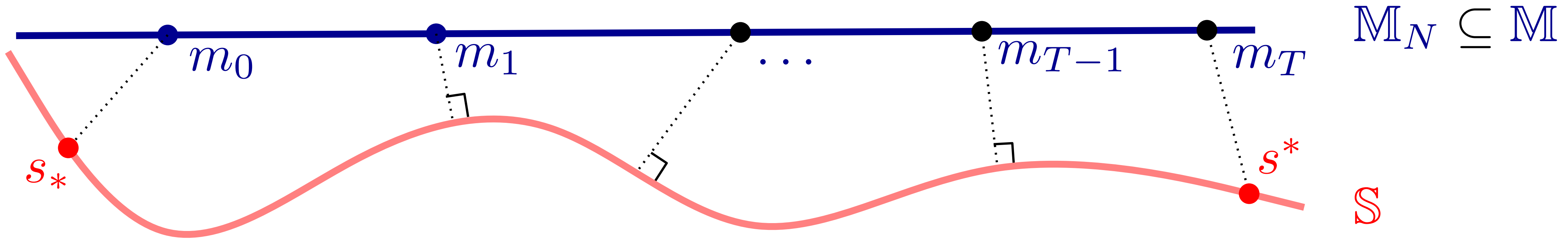
- **Numerics:** mostly in 1D using permutations [Brenier '87, Brenier–Roesch '98]

# Time-discretization

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action $\qquad$ boundary conditions $\qquad$ incompressibility

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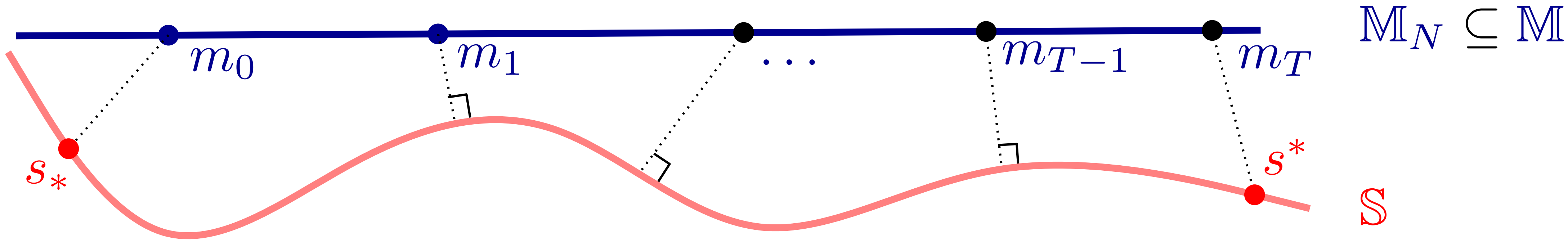


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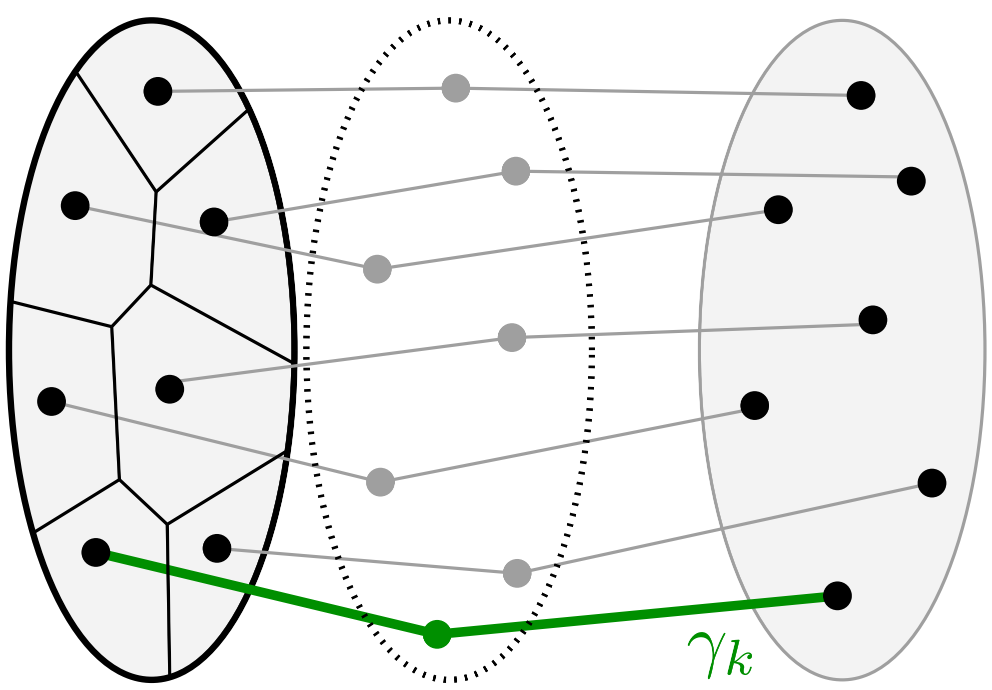

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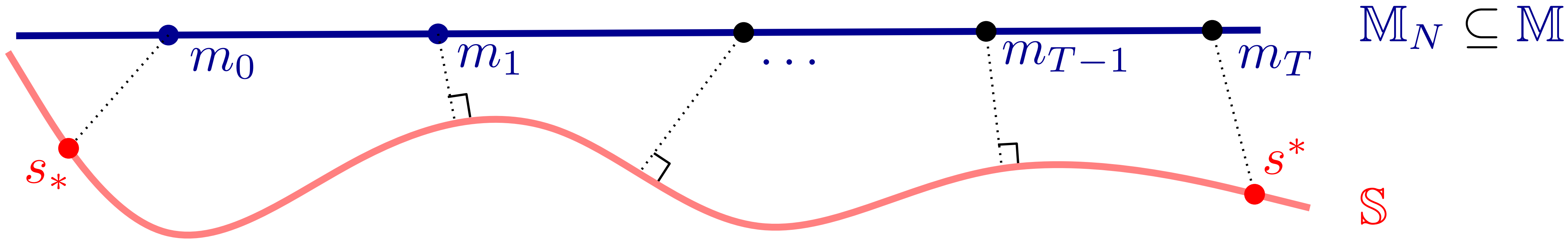


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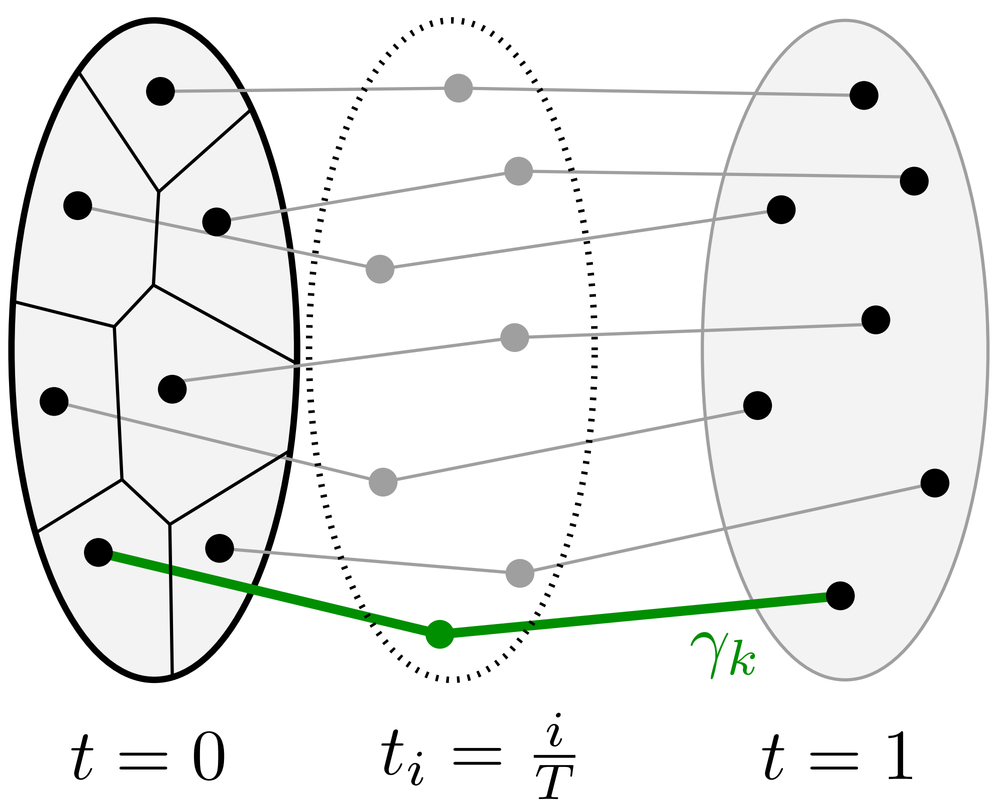
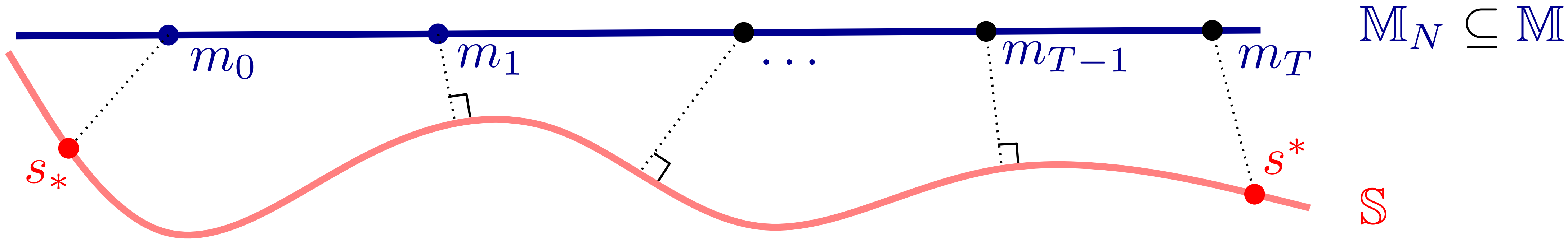



Then, with $\mu_m = \frac{1}{N} \sum_{k=1}^N \delta_{\gamma_k} \in \mathrm{Prob}(\Gamma)$,

$$\mathcal{E}_{N,T,\lambda}(m) = \mathcal{E}(\mu_m) + \lambda \sum_{i=1}^T \mathrm{W}_2^2(e_{t_i \#}\mu_m, \mathrm{Leb}) + \text{boundary cond.}$$

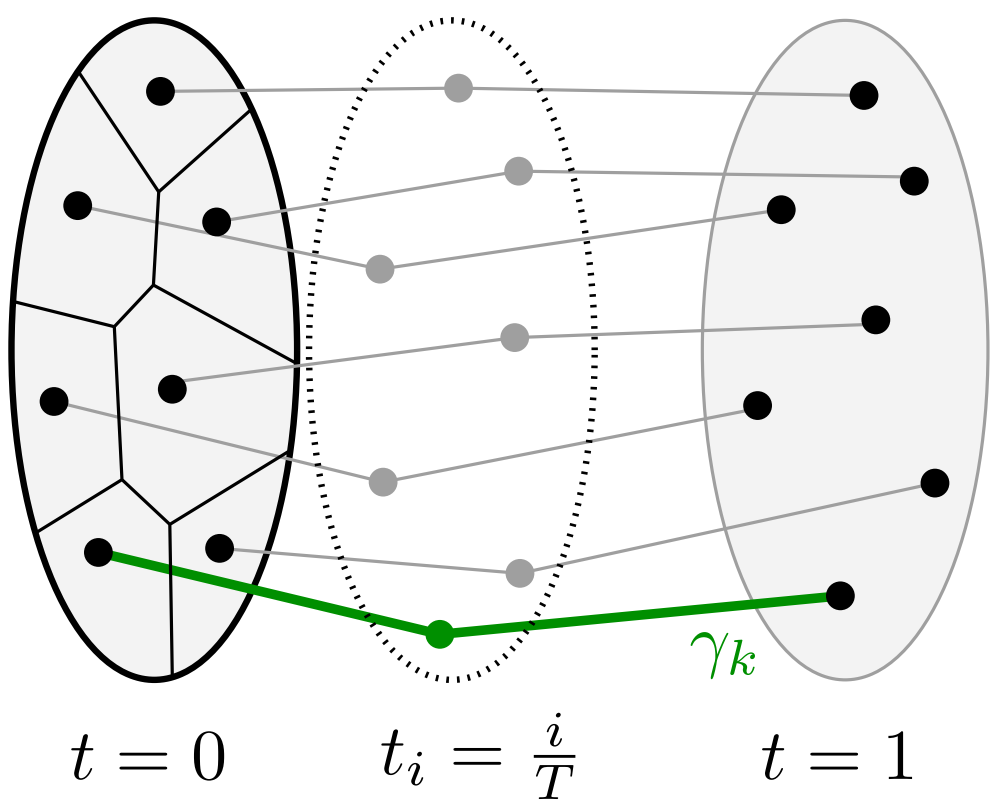
# Time-discretization




- Time-discretization of geodesic with endpoints $s_*, s^* \in \mathbb{S}$ $\qquad \mathcal{E}_{N,T,\lambda} : (\mathbb{M}_N)^T \to \mathbb{R}$,

$$\mathcal{E}_{N,T,\lambda}(m) := \frac{T}{2} \sum_{i=0}^{T-1} \|m_{i+1} - m_i\|_2^2 + \lambda \left( \|m_0 - s_*\|_2^2 + \|m_T - s^*\|_2^2 + \sum_{i=1}^{T-1} \mathrm{d}_{\mathbb{S}}^2(m_i) \right)$$

- Given $m = (m_1, \dots, m_T) \in \mathbb{M}_N^T$, let $\gamma_k \in \mathcal{C}^0([0,1], \mathbb{R}^d)$ be PL with $\gamma_k(t_i) = m_i(V_k)$




Then, with $\mu_m = \frac{1}{N} \sum_{k=1}^N \delta_{\gamma_k} \in \mathrm{Prob}(\Gamma)$,

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$\longrightarrow \simeq$ Common discretization for both relaxations!

$\longrightarrow$ Choice of penalization parameter?

# Convergence theorem

**Regular generalized geodesic**: a probability measure $\mu \in \mathrm{Prob}(\Gamma)$ s.t.

*(Regularity)* $\exists p$ with Lipschitz gradient s.t. $\forall \gamma \in \mathrm{spt}(\mu), \quad \ddot{\gamma} = -\nabla p \circ \gamma,$

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Then, up to subsequences, $\mu_{m_N} \in \mathrm{Prob}(\Gamma)$ converges weakly to a minimizing generalized geodesic between $s_*$ and $s^*$.

[Mirebeau–M., 2015]

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  for $h_N := N^{-\frac{1}{D}}$, with $D \in \mathbb{N}$ to be determined.

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- It turns out that one can take $D := \dim(\mathrm{spt}(\mu^{\mathrm{opt}}))$

  $\longrightarrow$ For a classical solution $s : [0,1] \to \mathbb{S}$, $\dim(\mathrm{spt}(\mu^{\mathrm{opt}})) = d$. $\qquad (\lambda_N = N^d)$

  $\longrightarrow$ For a regular generalized solution, $\dim(\mathrm{spt}(\mu^{\mathrm{opt}})) \leq 2d$. $\qquad (\lambda_N = N^{2d})$

# Energy estimate for classical solutions

**Proposition:** Assume that the minimizing geodesic $s$ between $s_*$ and $s^*$ is classical and that $s \in \mathrm{L}^\infty([0,1], H^1(X))$. Then, with $h_N = N^{-1/d}$,

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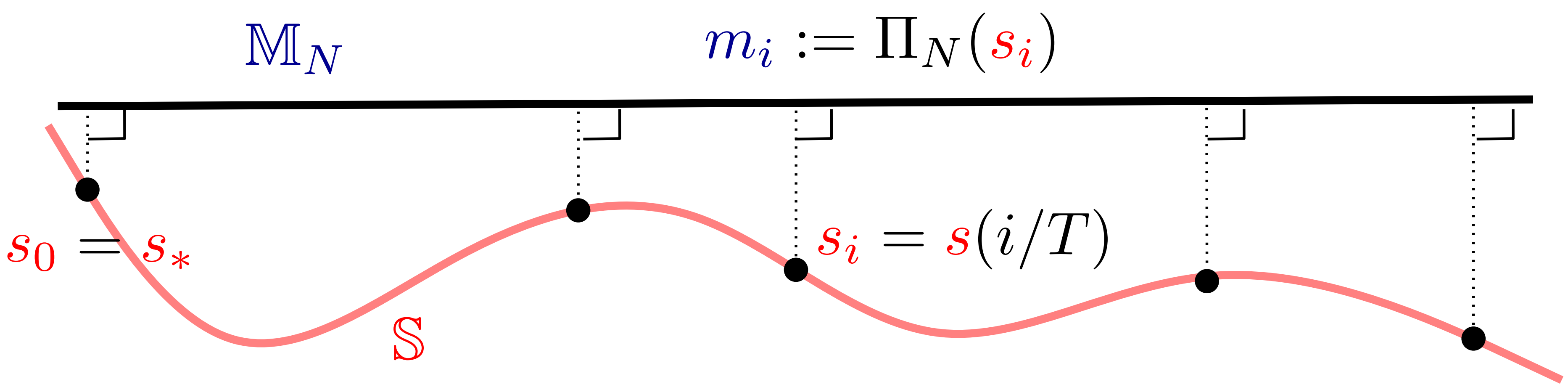

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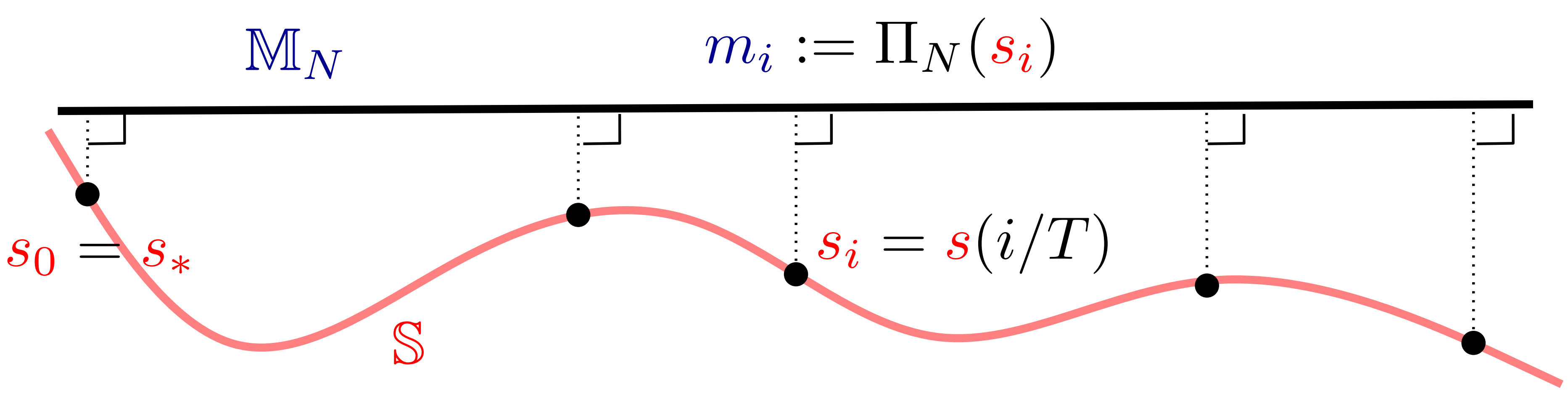
Then, $\mathcal{E}_{N,T,\lambda}(m)$ is upper bounded using the Poincaré-Wirtinger inequality.



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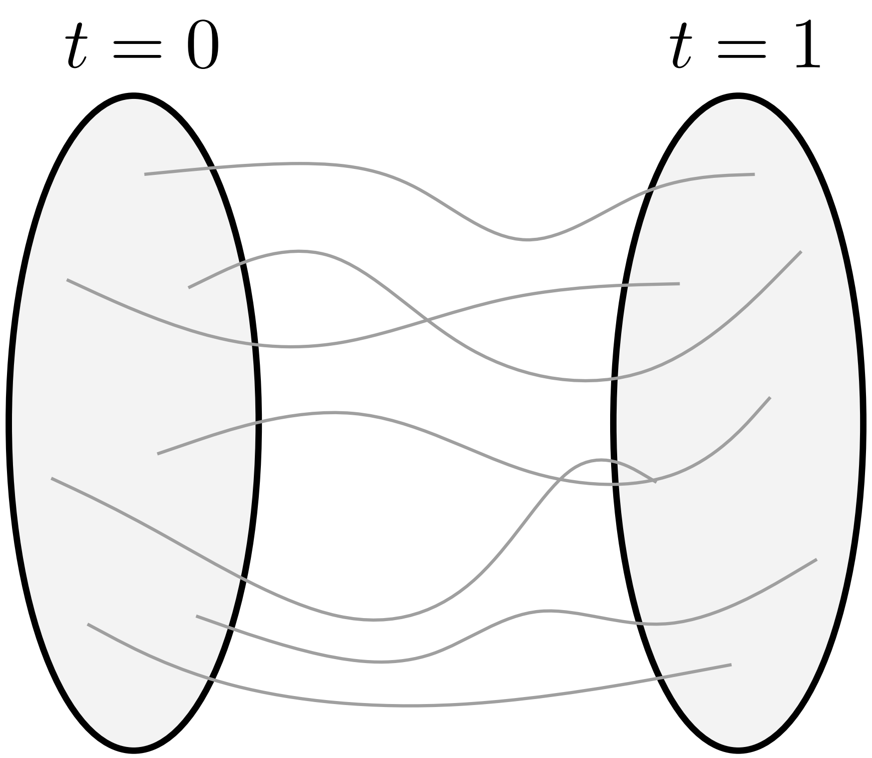


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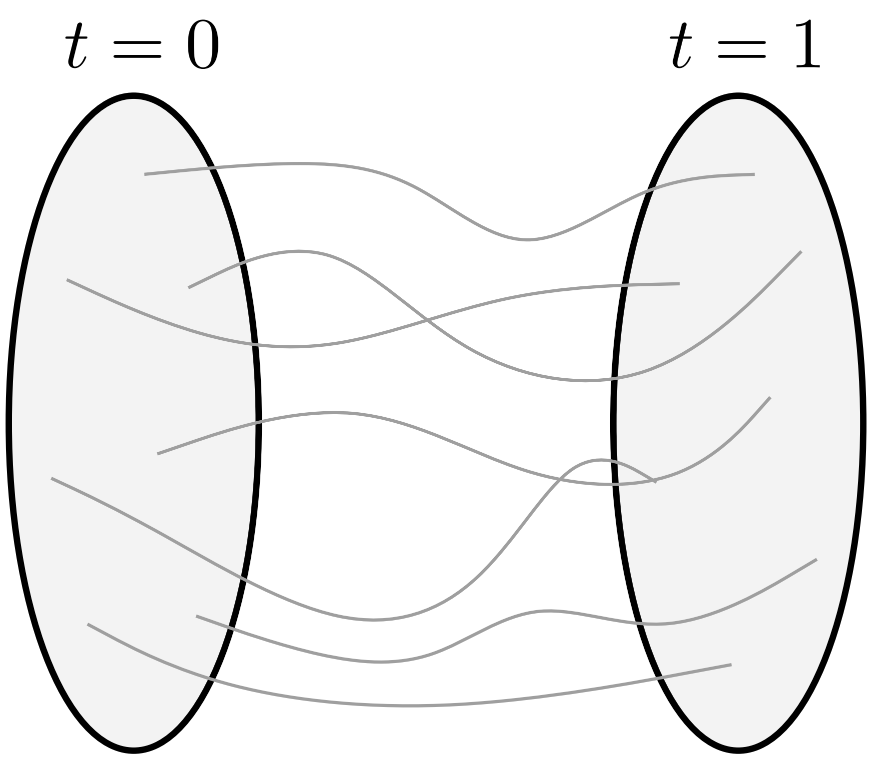


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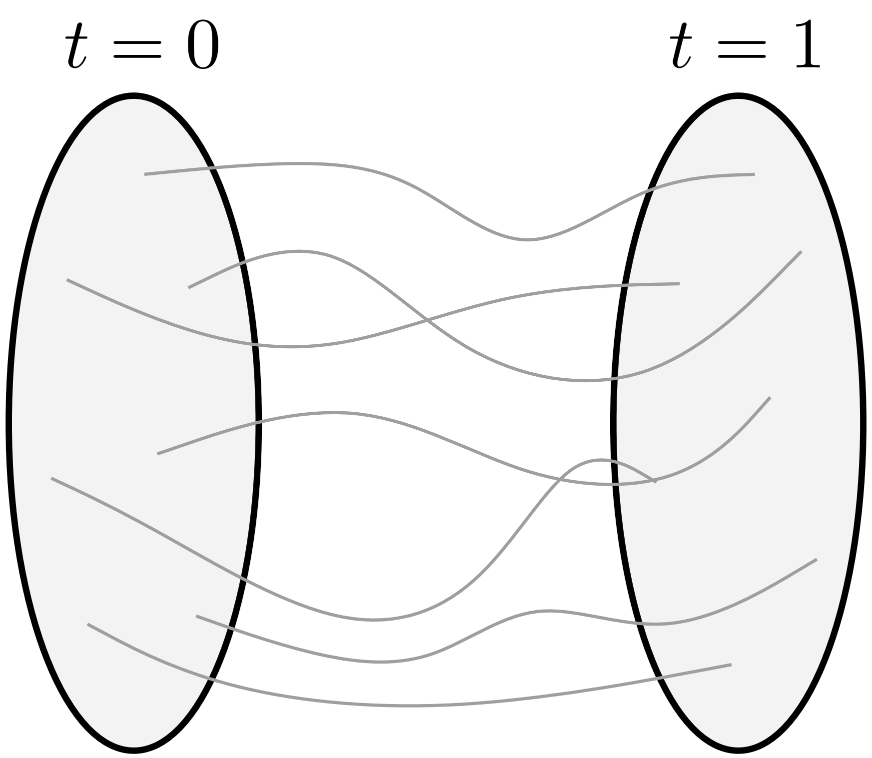


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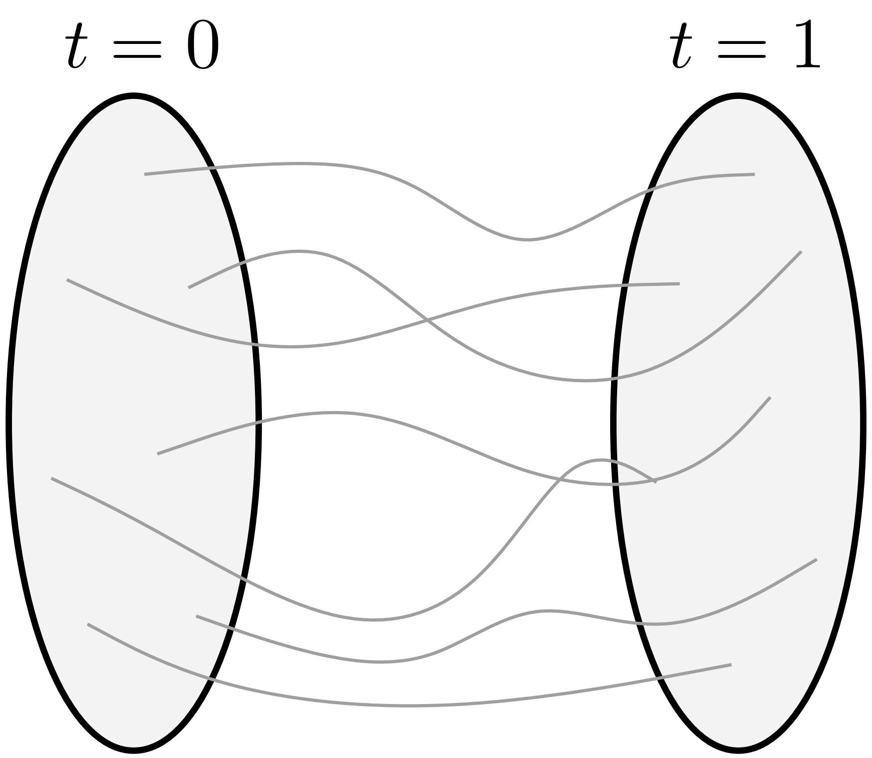


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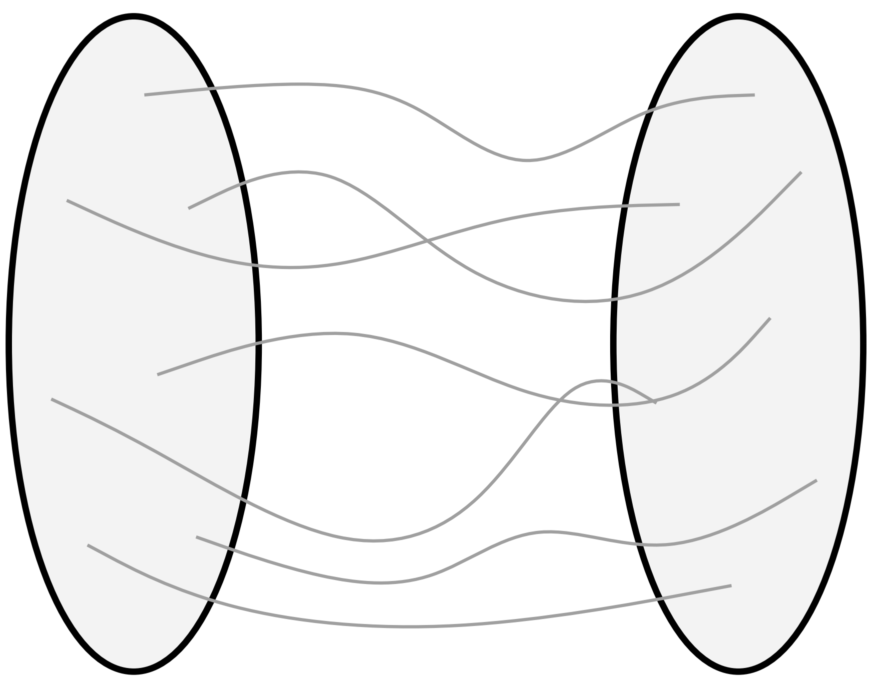


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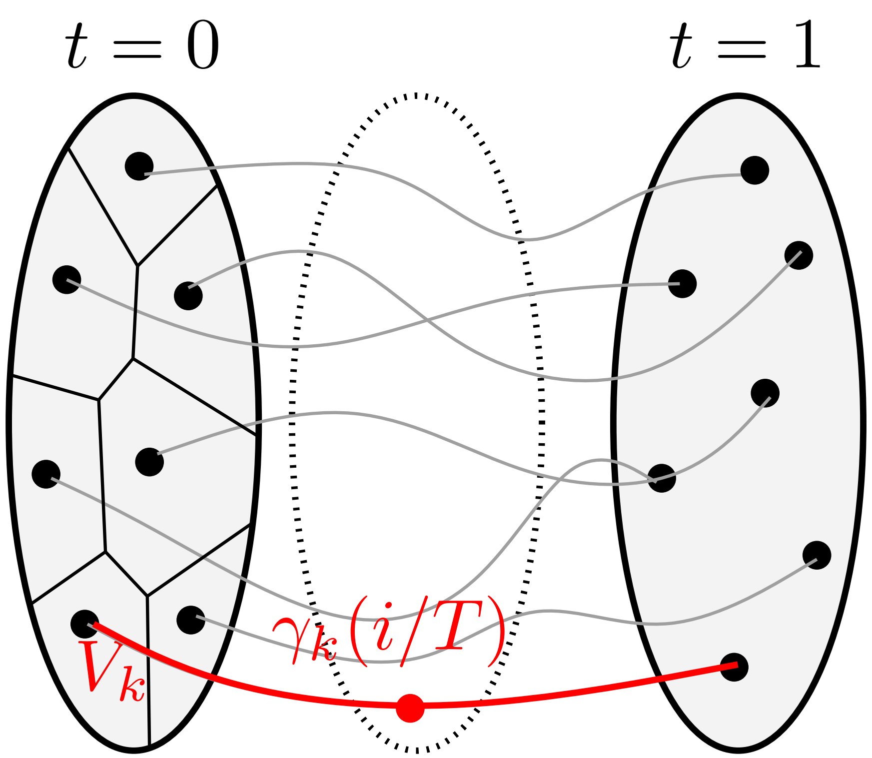


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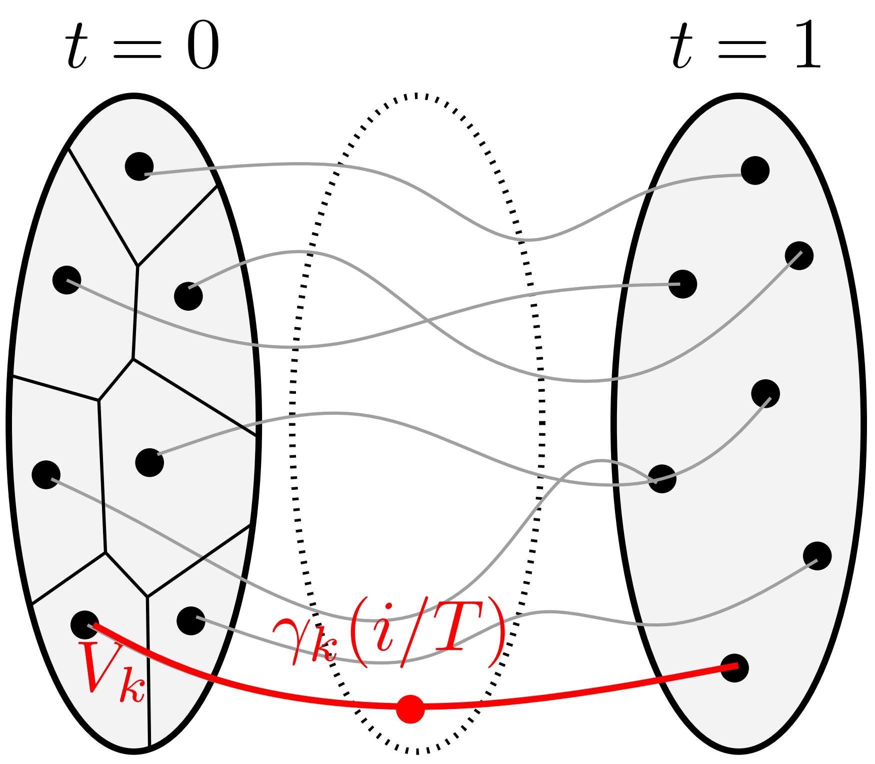


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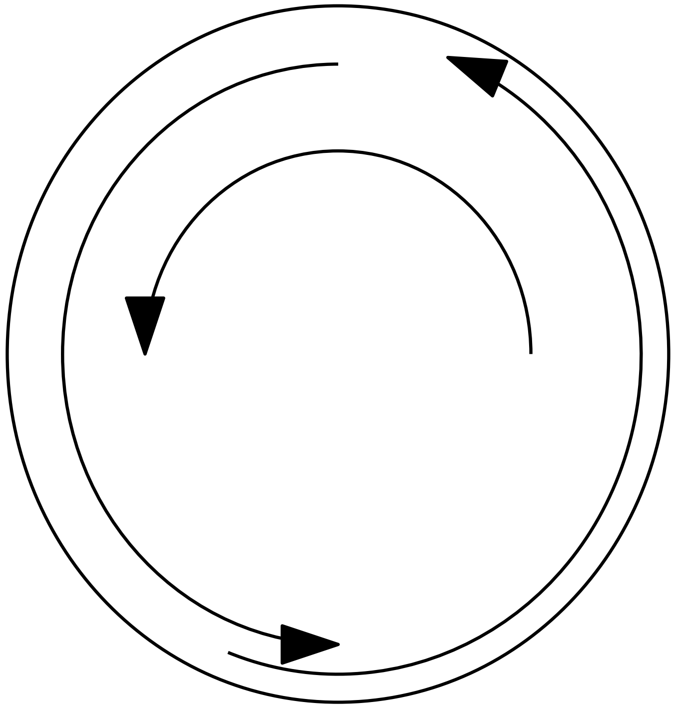
**E.** Upper bound $\mathcal{E}_{N,T,\lambda}(m)$ using the quantization estimate.

# Numerical result: Inversion of the Disk



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**Classical solutions:** clockwise/counterclockwise rotations $\mu_\pm$

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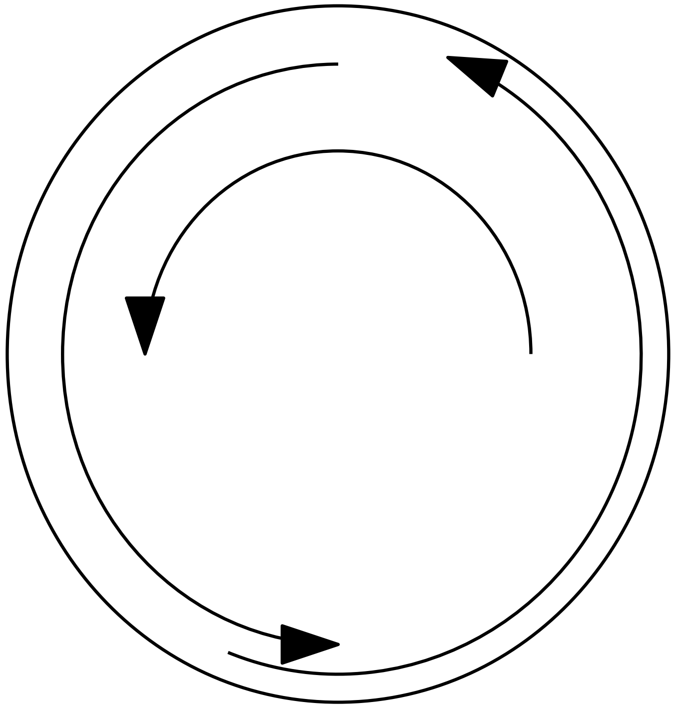

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**Examples of generalized solutions:**

linear combination $\mu_{\frac{1}{2}}$ of $\mu_\pm$ constructed from rotations

NB: $\dim(\mathrm{spt}(\mu_{\frac{1}{2}})) = 2$

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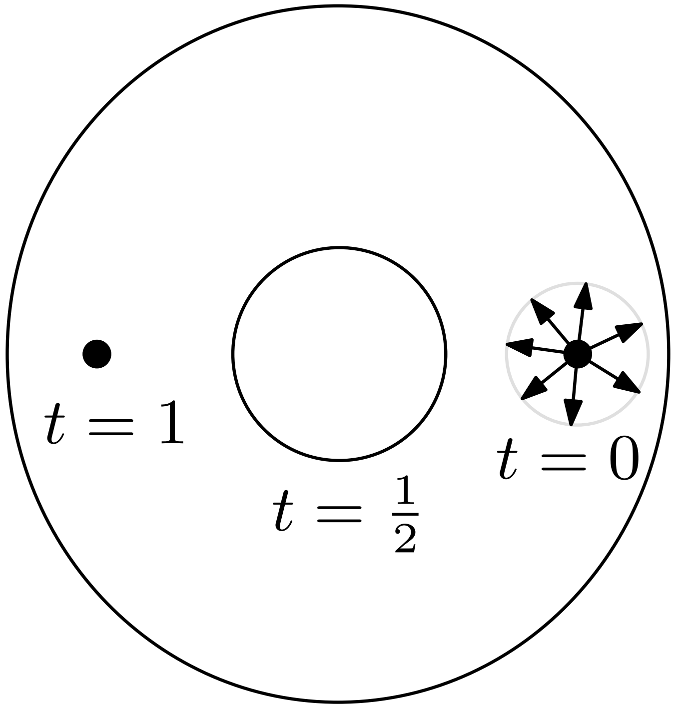


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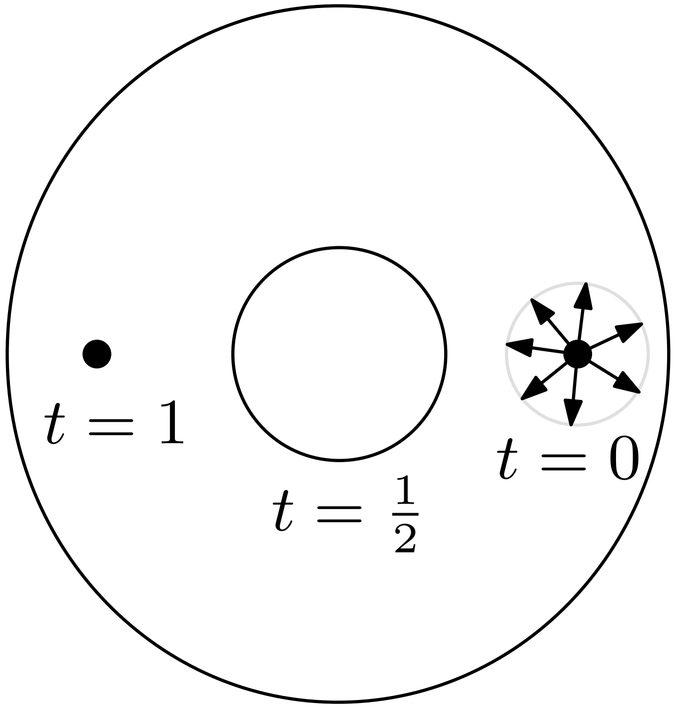


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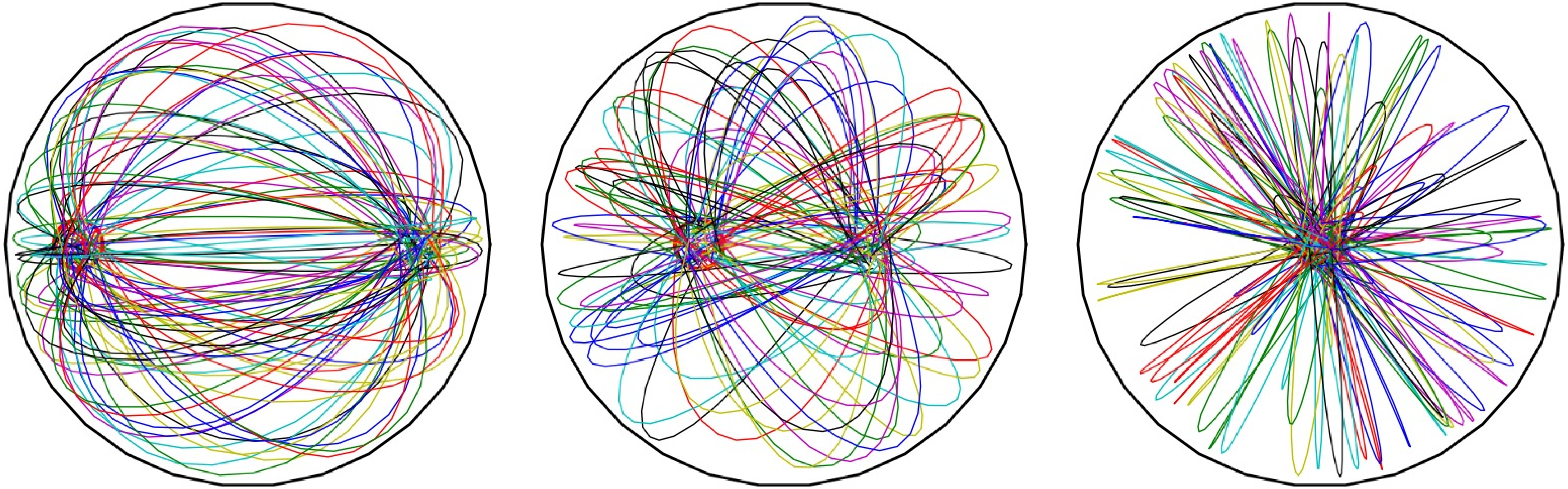
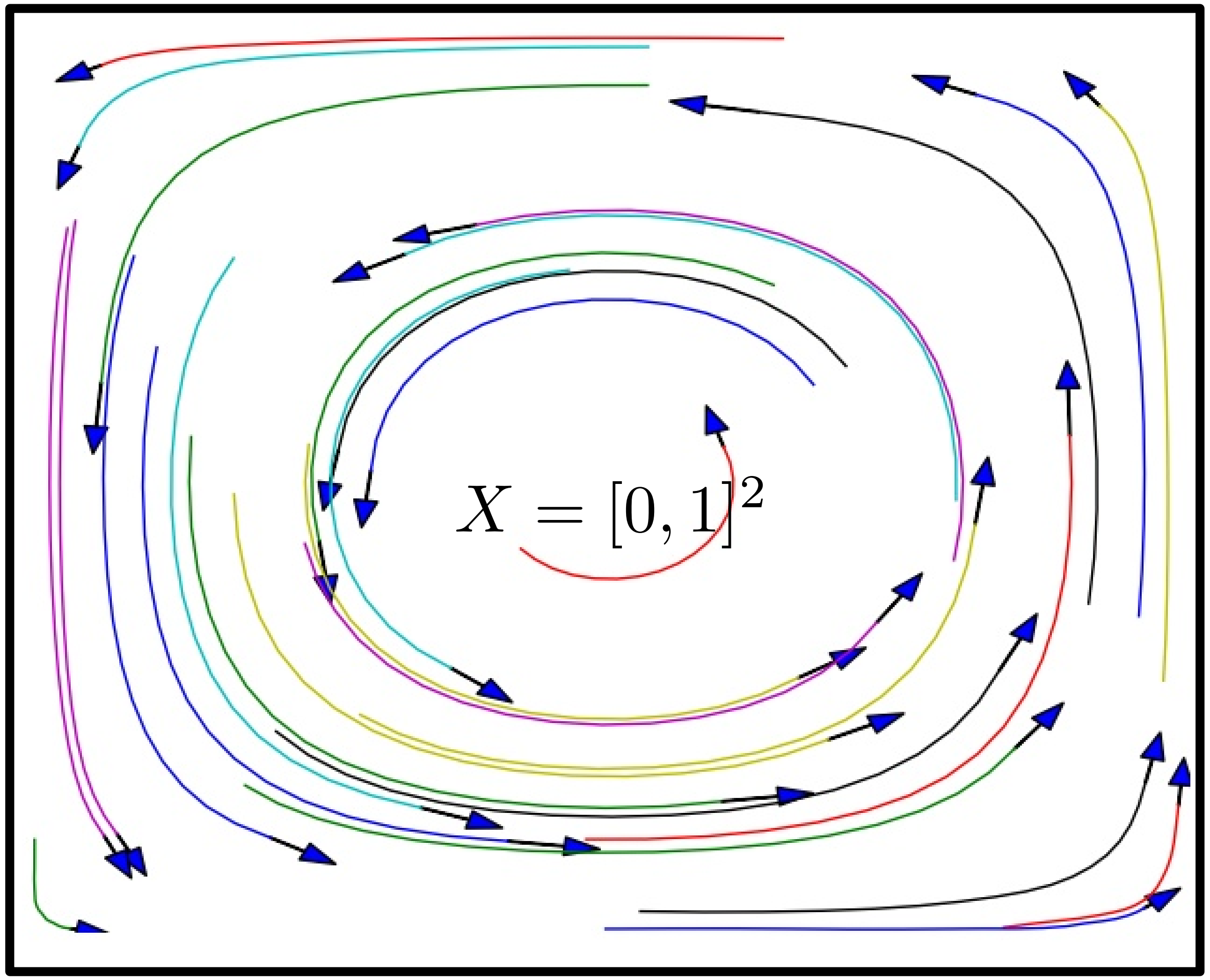
Computed trajectories for $N = 10^5,\ T = 17$



# Numerical result: Beltrami Flow in Square

**Stationary flow on $[0,1]^2$:** speed: $u(\mathbf{x}) = (\cos(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\cos(\pi x_2))$

[Brenier–Roesch] pressure: $p(\mathbf{x}) = \frac{1}{4}(\sin^2(\pi x_1) + \sin^2(\pi x_2))$



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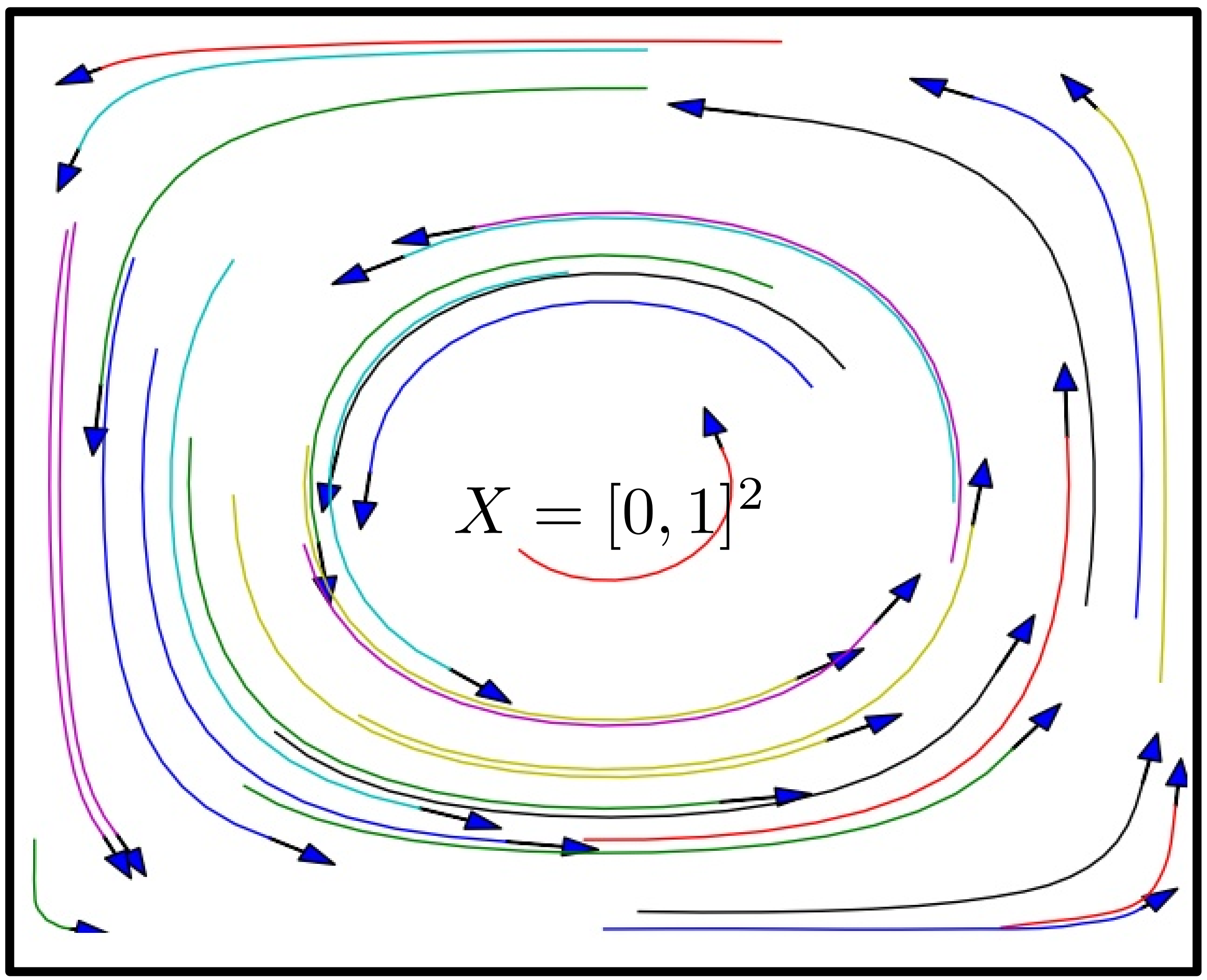


**Exact Lagrangian solution:**

$s_0^e = \mathrm{id} \qquad \dot{s}_t^e = u \circ s_t$

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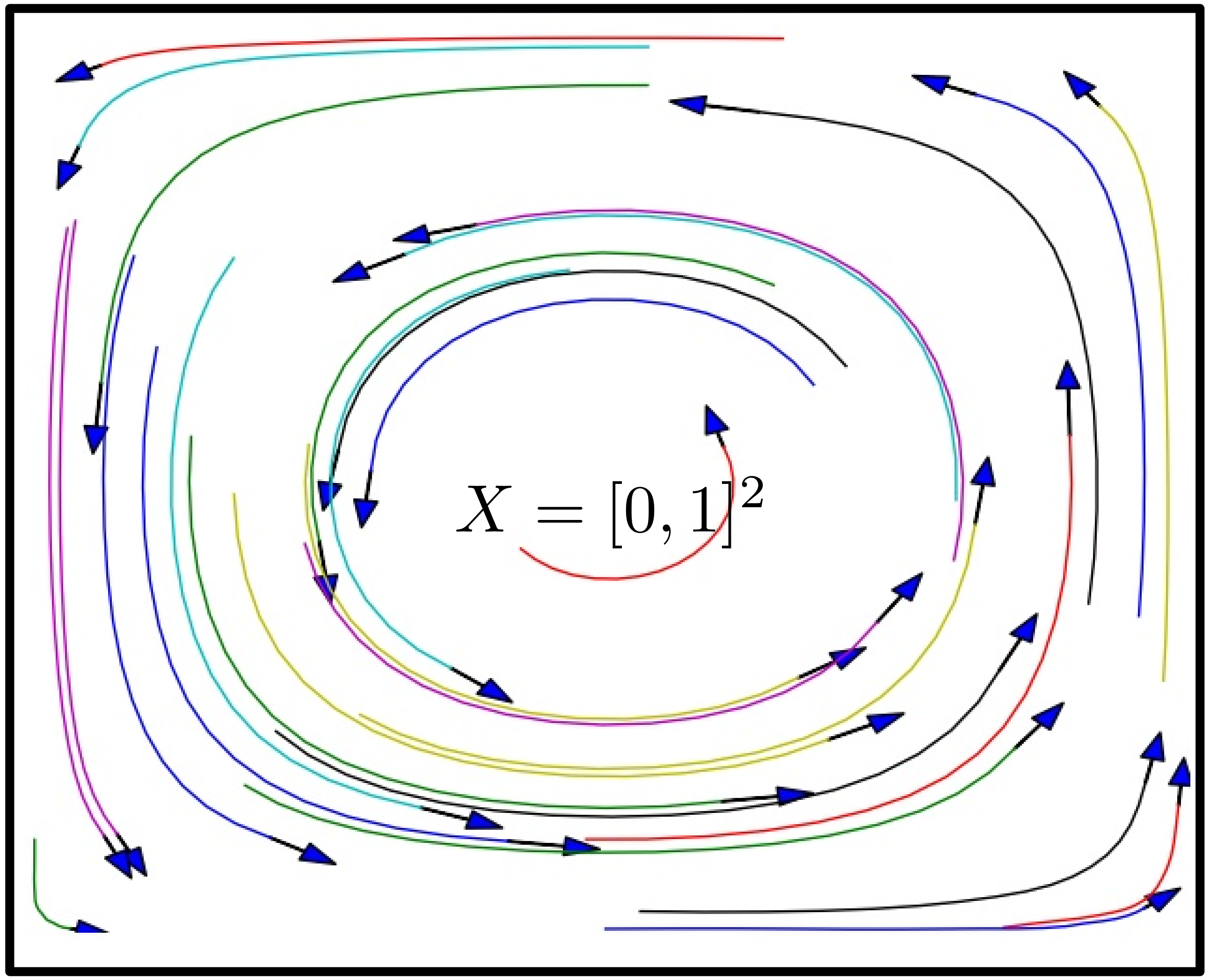


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NB: $s^e$ is minimizing on $[0,1]$

**Reconstruction problem:**

$\min \mathcal{E}_{N,T,\lambda}$

$s_* = s^e_0,\ s^* = s^e_{t_{\max}}$

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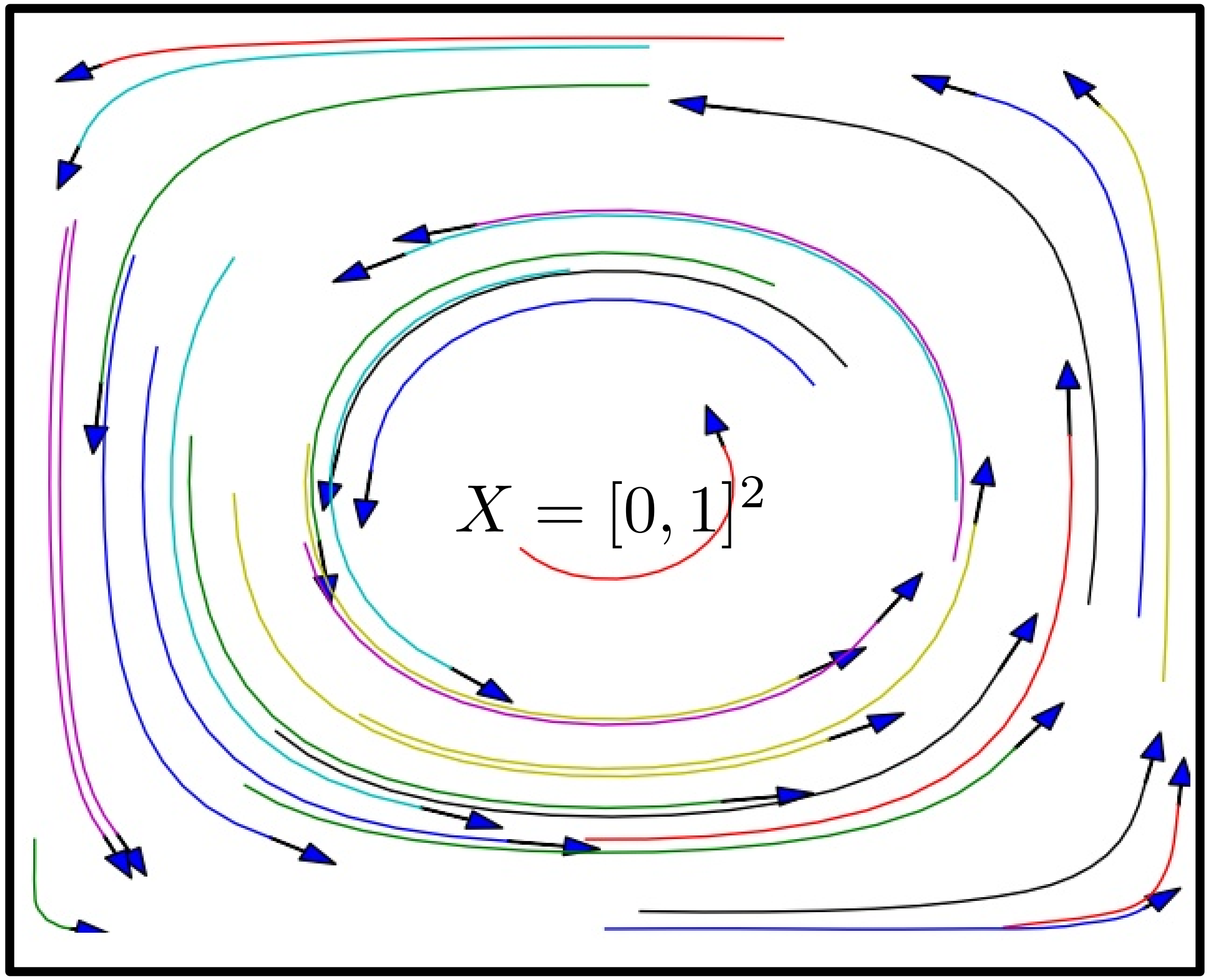
**Stationary flow on $[0,1]^2$:** speed: $u(\mathbf{x}) = (\cos(\pi x_1)\sin(\pi x_2), \sin(\pi x_1)\cos(\pi x_2))$

[Brenier–Roesch] pressure: $p(\mathbf{x}) = \frac{1}{4}(\sin^2(\pi x_1) + \sin^2(\pi x_2))$




**Exact Lagrangian solution:**

$s_0^e = \mathrm{id} \qquad \dot{s}_t^e = u \circ s_t$

NB: $s^e$ is minimizing on $[0,1]$

**Reconstruction problem:**

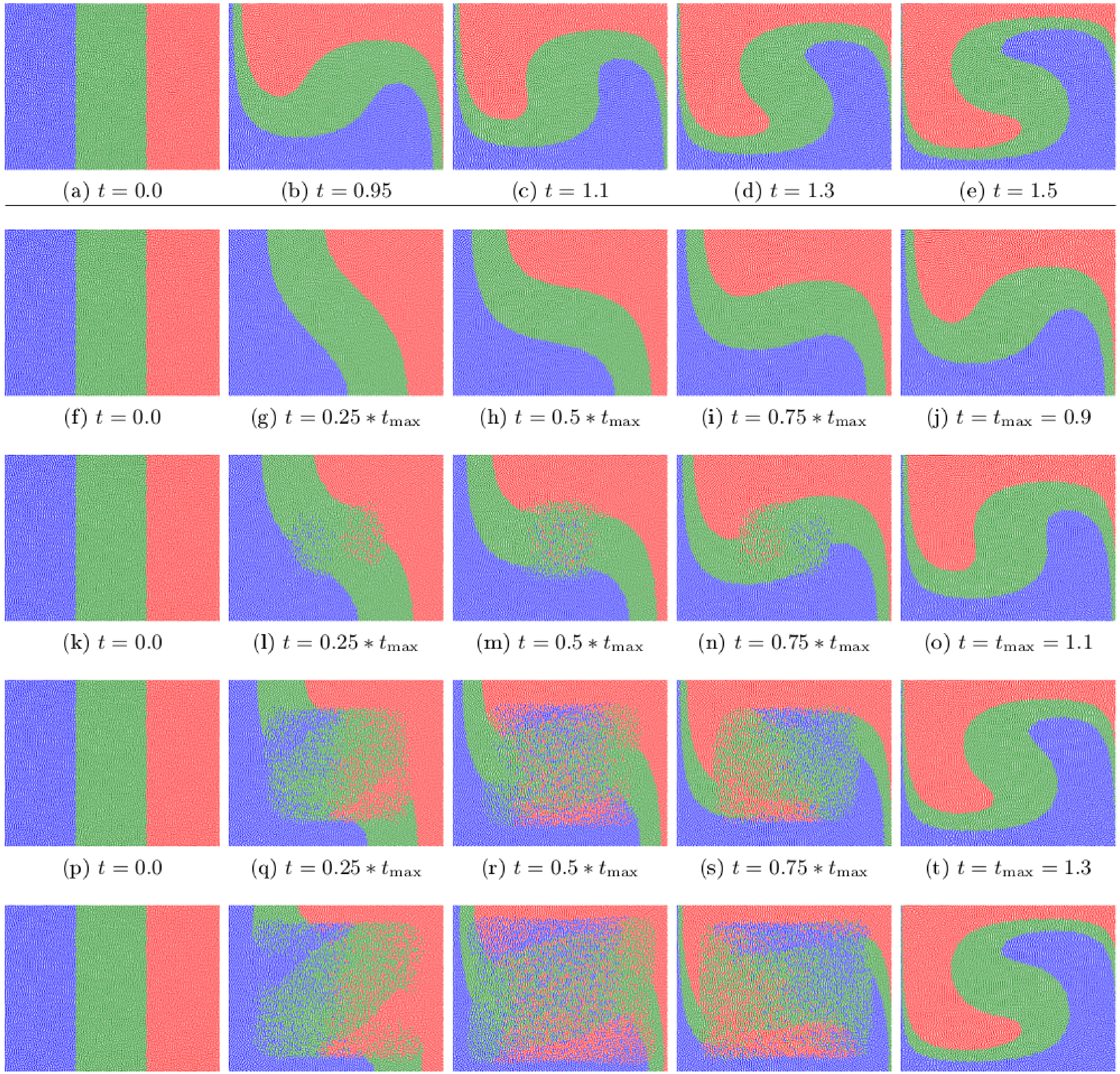
$\min \mathcal{E}_{N,T,\lambda}$

$s_* = s_0^e,\ s^* = s_{t_{\max}}^e$

**Parameters:**

$t_{\max} \in \{0.9, 1.1, 1.3, 1.5\}$

# Numerical result: Beltrami Flow in Square



(a) $t = 0.0$ (b) $t = 0.95$ (c) $t = 1.1$ (d) $t = 1.3$ (e) $t = 1.5$

(f) $t = 0.0$ (g) $t = 0.25 * t_{\max}$ (h) $t = 0.5 * t_{\max}$ (i) $t = 0.75 * t_{\max}$ (j) $t = t_{\max} = 0.9$

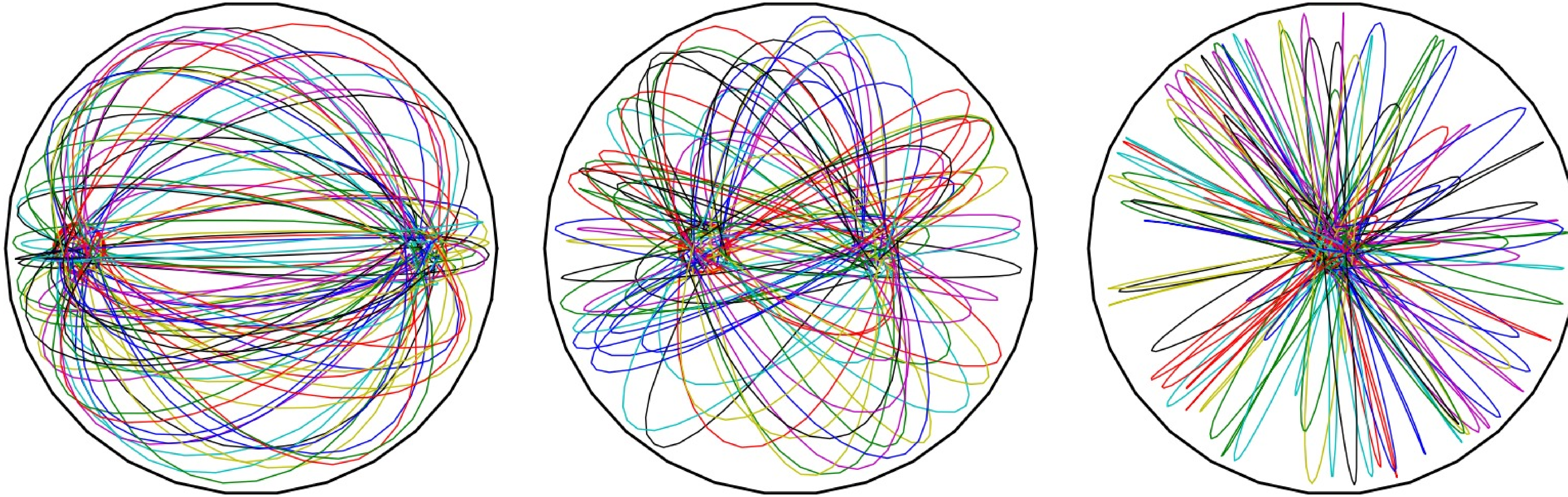
(k) $t = 0.0$ (l) $t = 0.25 * t_{\max}$ (m) $t = 0.5 * t_{\max}$ (n) $t = 0.75 * t_{\max}$ (o) $t = t_{\max} = 1.1$

(p) $t = 0.0$ (q) $t = 0.25 * t_{\max}$ (r) $t = 0.5 * t_{\max}$ (s) $t = 0.75 * t_{\max}$ (t) $t = t_{\max} = 1.3$

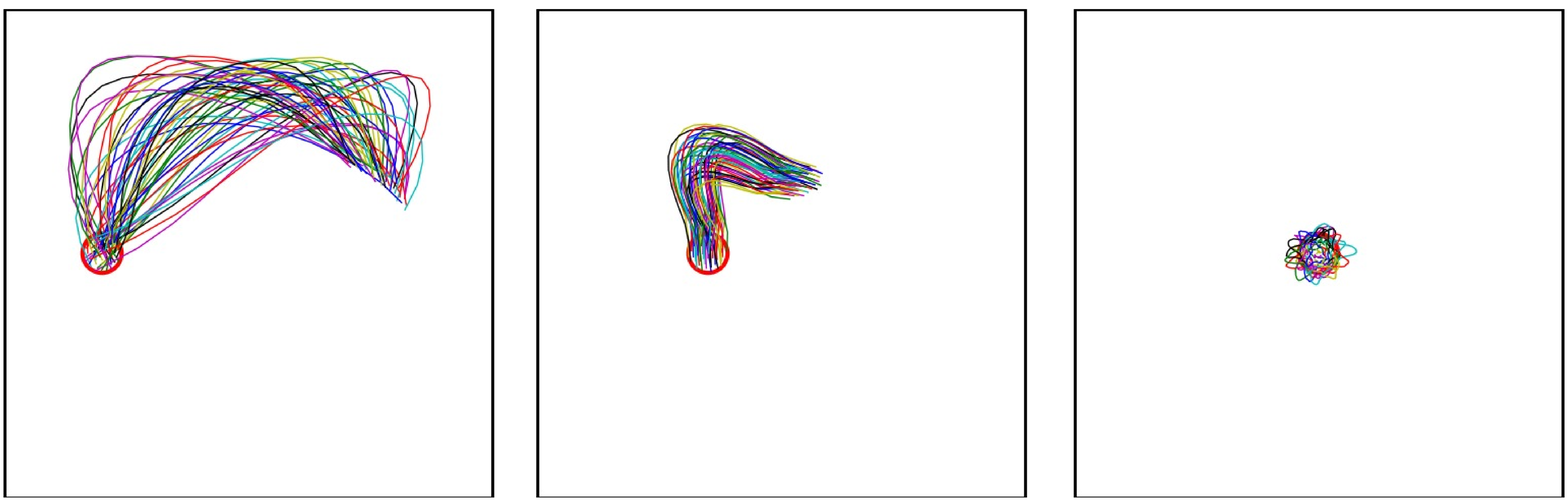
(u) $t = 0.0$ (v) $t = 0.25 * t_{\max}$ (w) $t = 0.5 * t_{\max}$ (x) $t = 0.75 * t_{\max}$ (y) $t = t_{\max} = 1.5$

NB: qualitatively similar results by Luca Nenna and J.D. Benamou

# Numerical result: Comparison of Trajectories



Disk inversion



Square, $t_{\max} = 1.5$

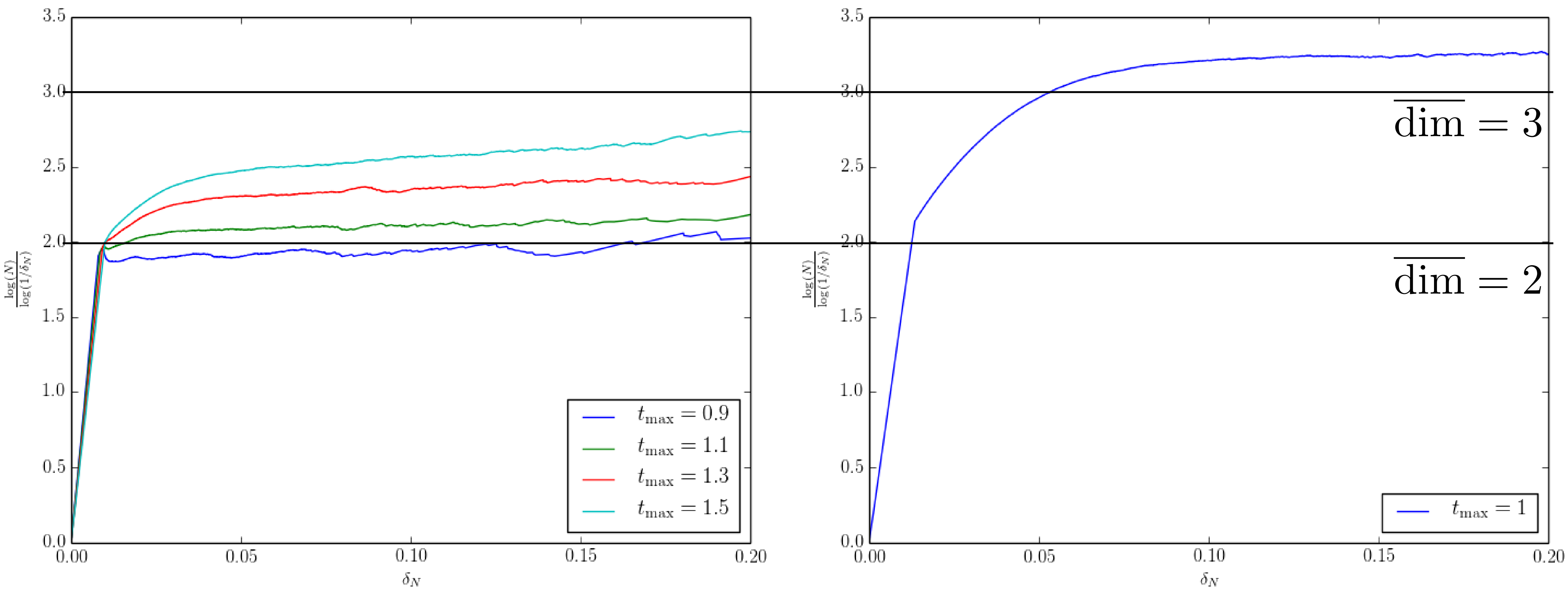
# Comparison of Minkowski dimensions

**Minkowski dimension** Let $S \subseteq \Gamma$ be a compact subset of a metric space.

$$\overline{\dim}(S) = \limsup_{N\to\infty} \log(N)/\log(1/\delta_N)$$

where $\delta_N$ = minimum radius required to cover $S$ with $N$ balls.

Estimation of $\dim(\mathrm{spt}(\mu))$ via $\log(N)/\log(1/\delta_N)$




Square rotation, $t_{\max} \in \{0.9, 1.1, 1.3, 1.5\}$

Disk inversion

# Perspectives

A) More realistic numerical schemes for the Cauchy problem (e.g. without $\varepsilon$) ?

B) Changing the polar factorization theorem $\longrightarrow$ other fluid models,

e.g. fluid-structure interactions / Camassa-Holm equation [Gallouet-Vialard 16], pressureless Euler equation with congestion [Maury-Preux '15]

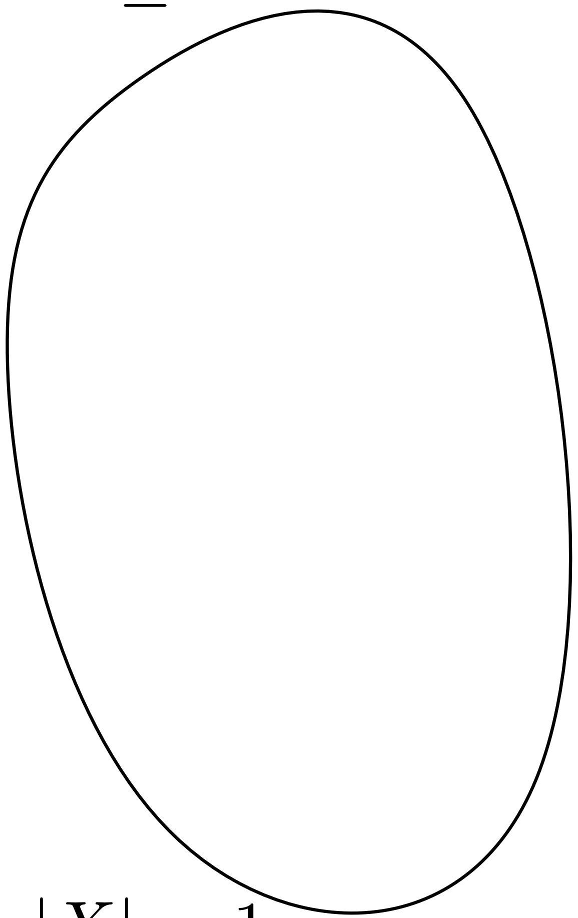
C) Viscosity?

# Solutions to Euler's equations as geodesics in $\mathbb{S}\mathrm{Diff}$

$\mathbb{S}\mathrm{Diff}$ = measure-preserving diffeomorphisms from $X$ to itself $\subseteq \mathrm{L}^2(X, \mathbb{R}^d)$

[Arnold 1966]

$X \subseteq \mathbb{R}^d$ bounded



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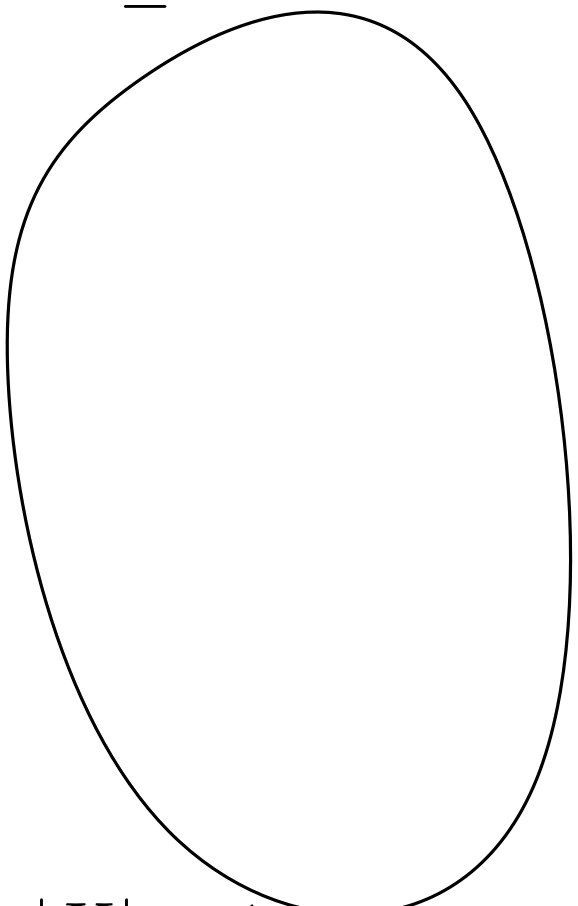

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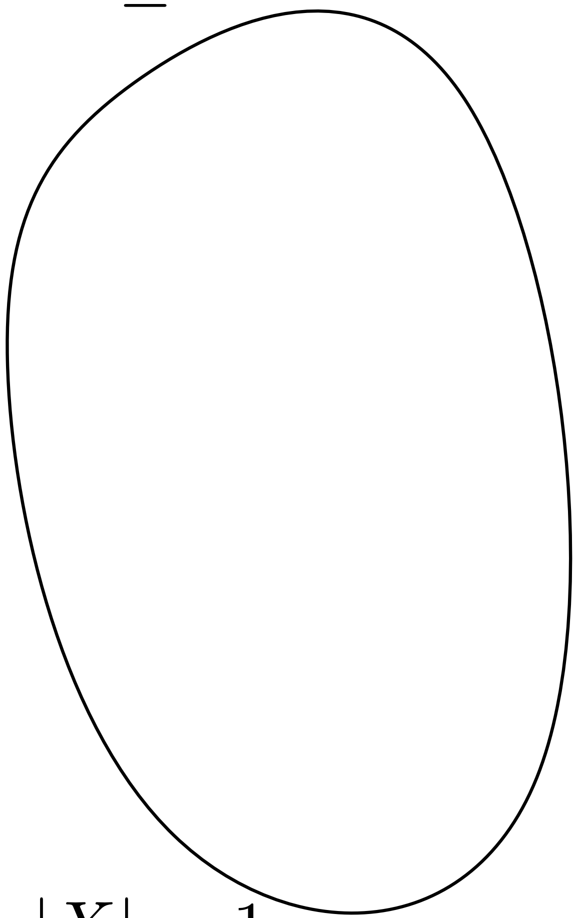

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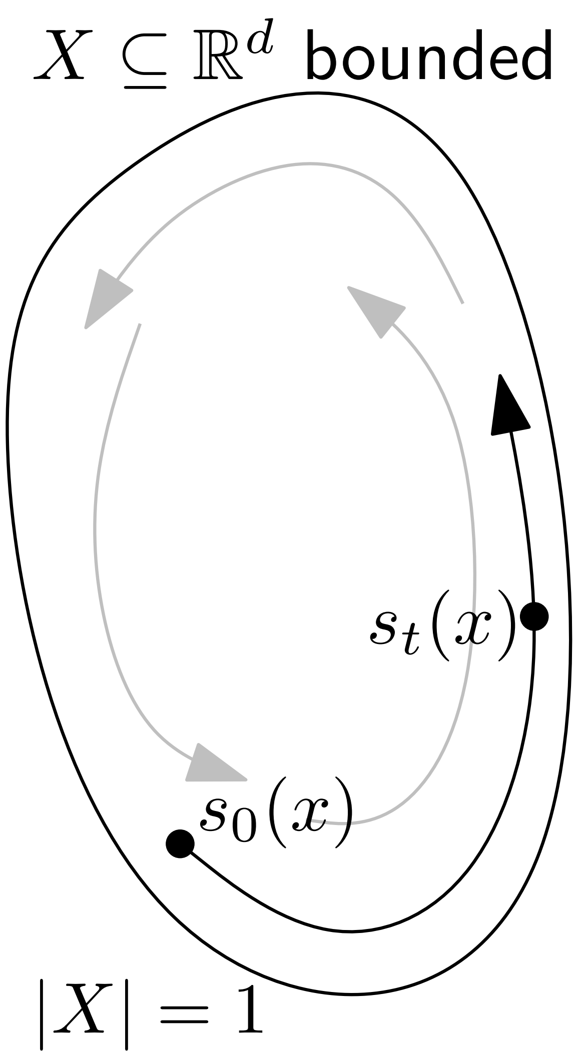


$\longrightarrow$ $\mathrm{T}_{\mathrm{id}}\mathbb{S}\mathrm{Diff}$ = divergence-free vector fields

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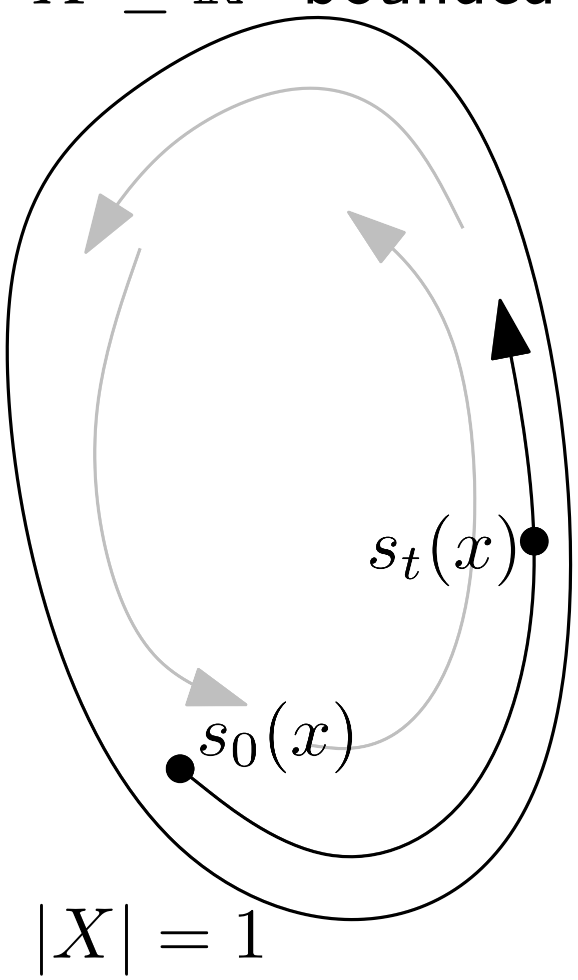


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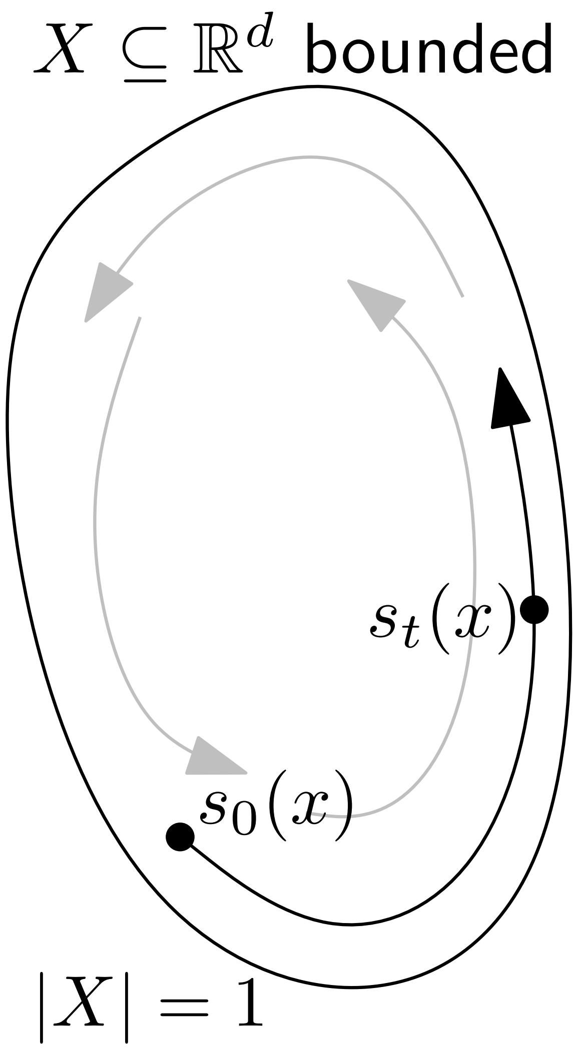


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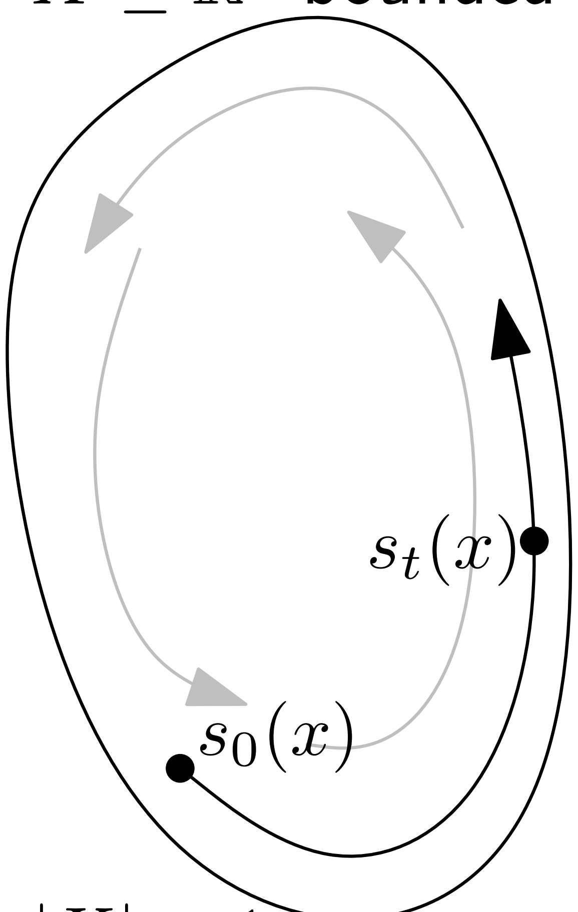


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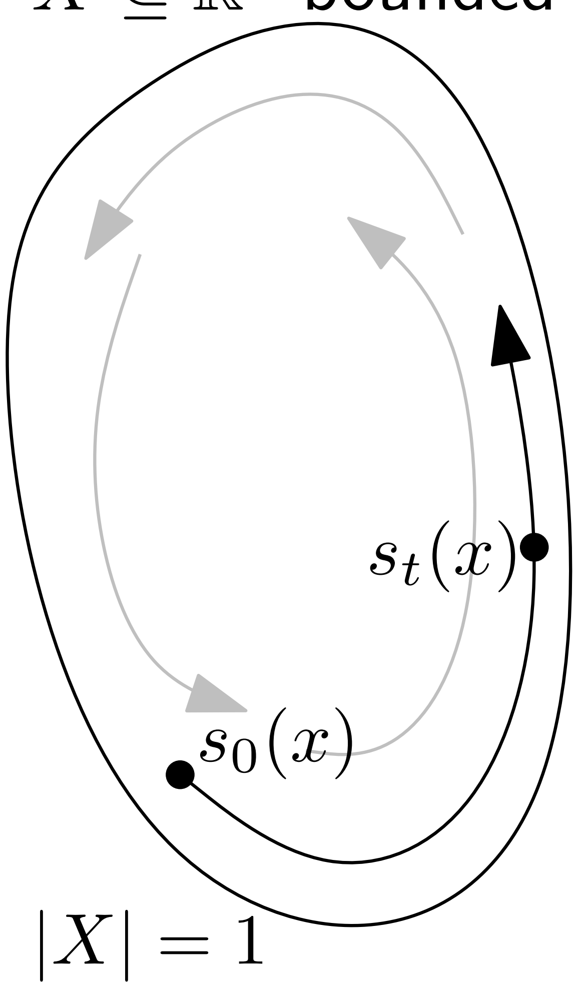


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$$\begin{cases} \partial_t u + (u \cdot \nabla)u = -\nabla p & \text{in } X \\ \mathrm{div} u = 0 & \text{in } X \\ u \cdot n = 0 & \text{on } \partial X \end{cases}$$

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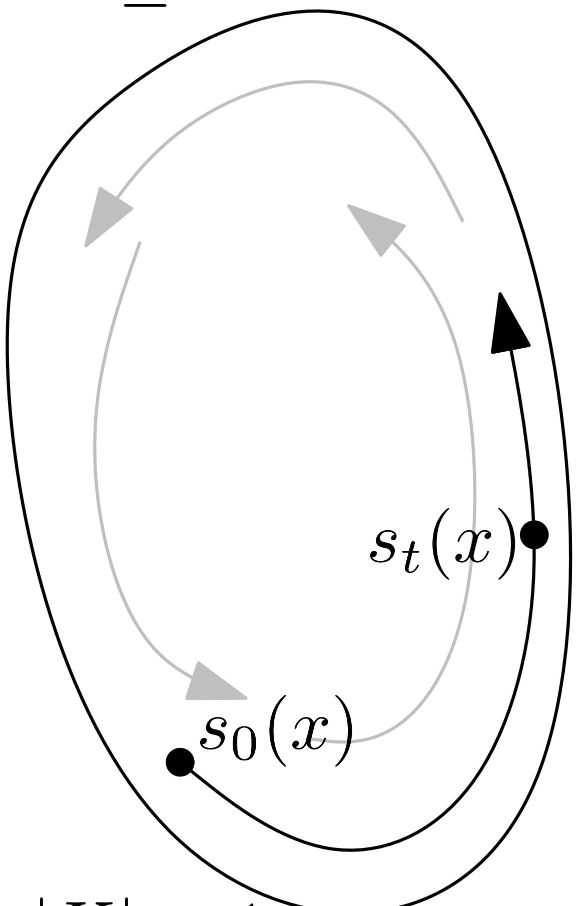


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Use this formulation for numerical computations (following Brenier):
$\longrightarrow$ Minimizing geodesics (with Jean-Marie Mirebeau, 2015)
$\longrightarrow$ Cauchy problem (with Thomas Gallouet, 2016).