Applications of optimal transport to weather and climate

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For Yann Brenier's 60th birthday

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Background

Background

The governing equation used in weather and climate models have solutions that are far too complicated to compute.

Reduced models are needed to describe how the governing equations behave in particular circumstances, and thus to validate operational models.

This talk is mostly about the semi-geostrophic model, which describes the behaviour on scales dominated by the Earth's rotation. These are horizontal scales greater than 1000km.

The semi-geostrophic model with constant rotation

Euler equations

The incompressible Euler equations for a rotating, stratified, Boussinesq fluid are

$$D_t u_1 + \frac{\partial p}{\partial x_1} - f u_2 = 0$$

$$D_t u_2 + \frac{\partial p}{\partial x_2} + f u_1 = 0$$

$$D_t u_3 + \frac{\partial p}{\partial x_3} + \rho g = 0$$

$$D_t \rho = 0$$

$$\nabla \cdot u = 0$$

These are to be solved in a closed region Ω with no flow across the boundary.

Rotation dominated flows

Dimensionless equations, with $\varepsilon = U/fL = (H/L)^2$

$$\varepsilon D_t u_1 + \frac{\partial p}{\partial x_1} - u_2 = 0$$

$$\varepsilon D_t u_2 + \frac{\partial p}{\partial x_2} + u_1 = 0$$

$$\varepsilon^2 D_t u_3 + \frac{\partial p}{\partial x_3} + \rho = 0$$

$$D_t \rho = 0$$

$$\nabla \cdot u = 0$$

Semi-geostrophic limit

$$\nabla p = \left(u_{g2}, -u_{g1}, -\rho\right)$$

$$u = u_g + \mathrm{O}(\varepsilon)$$

$$\varepsilon D_t u_{g1} + \frac{\partial p}{\partial x_1} - u_2 = \mathrm{O}\left(\varepsilon^2\right)$$

$$\varepsilon D_t u_{g2} + \frac{\partial p}{\partial x_1} + u_1 = \mathrm{O}\left(\varepsilon^2\right)$$

$$\frac{\partial p}{\partial x_3} + \rho = \mathrm{O}\left(\varepsilon^2\right)$$

$$D_t \rho = 0$$

$$\nabla \cdot u = 0$$

Semi-geostrophic limit in dual variables

$$X_1 = x_1 + \varepsilon u_2$$

$$X_2 = x_2 - \varepsilon u_1$$

$$X_3 = -\rho$$

$$P = \varepsilon p + \frac{1}{2}\left(x_1^2 + x_2^2\right)$$

$$\left(X_1, X_2, X_3\right) = \nabla P$$

$$D_t X = \left(x_2 - X_2, X_1 - x_1, 0\right)$$

$$\nabla \cdot u = 0$$

Solution procedure

Discrete procedure-Cullen and Purser (1984,1989)

Represent data by finite volumes σ_i with X constant on each element.

Arrange in Ω so that P is convex. Requires monotonicity of X_i as function of x_i for each i.

Theorem of Alexandrov shows that a unique convex polyhedron P can be constructed such that the volume of each hyperface is σ_i.

Then solve evolution equations for X_i.

Does this converge to a 'continuous' solution?

Continuous procedure- Brenier (1991)

Start with data

$$X(t) = \nabla P(t)$$

Take a timestep by first updating *X*.

$$X^*(t+\delta t) = X(t) + \left(x_2 - X_2, X_1 - x_1, 0\right)\delta t$$

Brenier's polar factorisation theorem states

$$X^*(t+\delta t) = \nabla P(t+\delta t) \circ \mu$$

where μ is a measure-preserving mapping. So can set

$$X(t+\delta t) = \nabla P(t+\delta t)$$

Solution using optimal transport (Benamou/Brenier)

Define mass in dual variables.

$$\sigma = \nabla P \# \mathcal{L}^3$$

Define velocity in dual variables

$$D_t\left(X_1, X_2, X_3\right) \equiv \mathbf{U} = \left(X_2 - x_2, x_1 - X_1, 0\right)$$

Solve the mass conservation equation in dual variables.

$$\frac{\partial \sigma}{\partial t} + \nabla \cdot \left(\sigma \mathbf{U}\right) = 0$$

Map solution back to physical space at each time using optimal transport to calculate physical space velocity.

Optimal transport problem

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Find map *T* minimising energy *E* where

$$\left(x_1, x_2, x_3\right) = T\left(X_1, X_2, X_3\right)$$

$$E = \left\{ \int \frac{1}{2}\left(\left(x_1 - X_1\right)^2 + \left(x_2 - X_2\right)^2 \right) - x_3 X_3 \right\} \mathrm{d}L^3$$

$$T_{\#}\sigma = L^3$$

Solution satisfies

$$\left(X_1, X_2, X_3\right) = \nabla P$$

as required

Numerical solution

Represent σ as a set of Dirac masses σ_i.

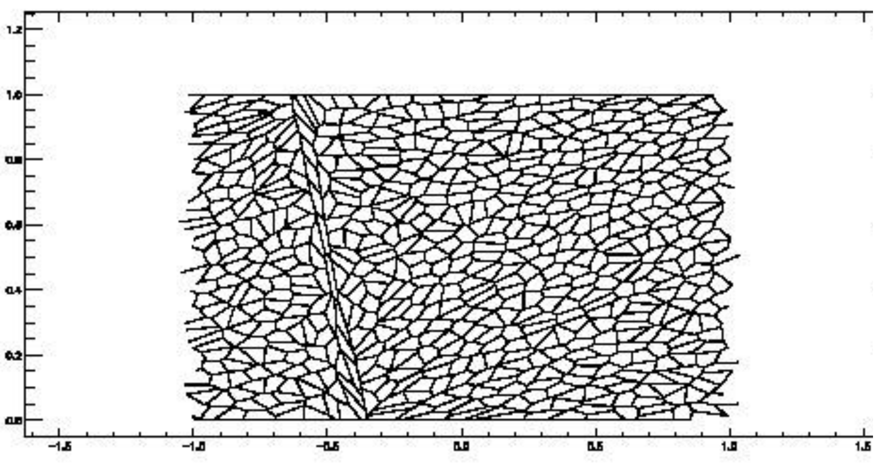
Solve optimal map and transport masses with velocity $\mathbf{U}_i$.

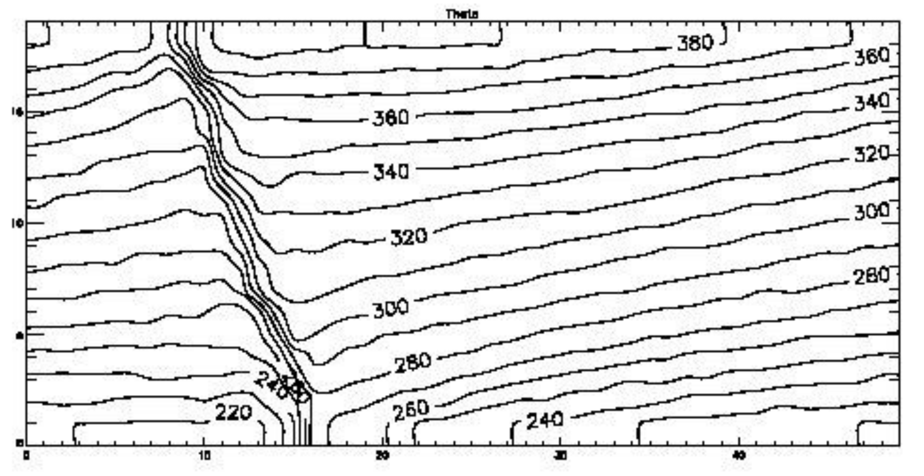
Proved to converge to correct solution by Cullen, Gangbo and Pisante.

Example shown from 2 dimensional problem which has singular solutions corresponding to convex but non-differentiable *P*.

These solutions are a simple model of atmospheric fronts.

Illustration





Element picture

ρ field

Solutions of 2 dimensional SG in vertical cross section after singularity has formed.

Comments on solution

Singularities in the solution can be expected. Those shown in the example correspond to atmospheric fronts.

These singularities do not invalidate the Lagrangian form of the governing equations, or the assumptions used to derive them from the Euler equations.

Given that it is impracticable to use fully Lagrangian methods in production models, the generation of these singularities is an important issue.

Further developments

Issues

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In order to justify the model, need to map the solution into physical space to compare with the Euler solution.

The dual space ‘velocity’ **U** is non-divergent in **X**. Thus the mass density σ is bounded by its initial values. In particular if it is absolutely continuous wrt Lebesgue, it remains so.

Ambrosio proved that the transport equation generates a Lagrangian flow map Φ. This is because **U** is the rotated gradient of a convex function. Cullen and Feldman showed that a Lagrangian map in physical space can then be generated as $F=T \circ \Phi \circ T^{-1}$, giving a weak Lagrangian solution in physical space.

Boundary conditions

In the atmosphere, SG would naturally be applied with periodic boundary conditions in the horizontal, pressure (i.e. mass) as the vertical coordinate, and a rigid boundary at the top (p=0) and a free surface at the bottom (p satisfies a prognostic equation).

In the ocean it is naturally applied with rigid boundaries in the horizontal, a free surface at the top and a rigid boundary at the bottom.

If 3d periodic boundaries are assumed, Ambrosio, Figalli, de Phillippis and Colombo showed that SG solutions remain smooth, no singularities form. Thus the boundary is crucial in forming atmospheric fronts.

Free boundary

Cullen, Gangbo and Sedjro solved the problem of an axisymmetric vortex in an ambient rotating fluid at rest. This is done by finding the mass in angular momentum and potential temperature coordinates (Y,Z). This is mapped onto a subset of a physical domain Ω_2x[0,H] where the subset is defined as [$r \leq \rho(z)$]x[0,H] and r is a radial coordinate.

The map is found by solving an optimal transport problem. For a given $\rho(z)$, the energy density takes the form (-sY-zZ), where s is a function of radius, plus terms independent of the map. This is minimised for fixed $\rho(z)$ by choosing (s,z)=grad Ψ with Ψ convex.

Free boundary II

Then have to minimise the energy over choices of $\rho(z)$. This was achieved. In particular, the solution has $\rho(z)$ monotonically increasing in z.

A similar method shows that the 3d problem Ω_2x$[0,h(x_1,x_2)]$ with a free boundary in pressure coordinates can be solved (Cullen, Pelloni, Gilbert and Kuna).

The latter result gives a solution for the physically appropriate choice of boundary conditions.

Convergence of Euler to SG

Convergence of Euler to SG

Physical applicability requires a proof that solutions of the Euler equations converge to solutions of SG at the predicted $O(\varepsilon^2)$ rate. In the general 3d case the SG solution depends on ε.

Loeper proved such a result in the horizontal plane where only $O(\varepsilon)$ convergence is expected.

Brenier and Cullen proved a result for a simplified problem in a vertical cross-section (the Eady model). The SG solution does not depend on ε in this case.

Eady model scaled

With $\varepsilon = U/fL$, $\partial/\partial x_2 = 0$ and $u_1 = \varepsilon u_2$.

$$\varepsilon^2 D_t u_1^\varepsilon + \frac{\partial p^\varepsilon}{\partial x_1} - u_2^\varepsilon = 0$$

$$D_t u_2^\varepsilon + u_1^\varepsilon = U\left(x_3 - \frac{1}{2}\right)$$

$$\varepsilon^2 D_t u_3^\varepsilon + \frac{\partial p^\varepsilon}{\partial x_3} + \rho^\varepsilon = 0$$

$$D_t \rho^\varepsilon + u_2^\varepsilon U = 0$$

$$\nabla \cdot \mathbf{u}^\varepsilon = 0$$

Eady model in new variables

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$$\mathbf{y}^{\varepsilon} = \left(x_1 + u_2^{\varepsilon}, 0, -\rho^{\varepsilon}\right), P^{\varepsilon} = p^{\varepsilon} + \frac{1}{2} x_1^2$$

$$\varepsilon^2 D_t u_1^{\varepsilon} + \frac{\partial P^{\varepsilon}}{\partial x_1} - y_1^{\varepsilon} = 0$$

$$D_t y_1^{\varepsilon} = U\left(x_3 - \frac{1}{2}\right)$$

$$\varepsilon^2 D_t u_3^{\varepsilon} + \frac{\partial P^{\varepsilon}}{\partial x_3} - y_3^{\varepsilon} = 0$$

$$D_t y_3^{\varepsilon} = U\left(y_1^{\varepsilon} - x_1\right)$$

$$\nabla \cdot \mathbf{u}^{\varepsilon} = 0$$

SG model

$$\mathbf{y} = \left(x_1 + u_2, 0, -\rho\right), P = p + \frac{1}{2}x_1^2$$

$$\nabla P = \mathbf{y}$$

$$D_t y_1 = U\left(x_3 - \frac{1}{2}\right)$$

$$D_t y_3 = U\left(y_1 - x_1\right)$$

$$\nabla \cdot \mathbf{u} = 0$$

Convergence of Eady model to SG (Brenier)

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Define

$$P^* = \mathbf{x}\cdot\mathbf{y} - P$$

Define relative entropy

$$\eta_{P^*}(t,\mathbf{y},\mathbf{y}') = P^*(t,\mathbf{y}') - P^*(t,\mathbf{y}) - \nabla P^*(t,\mathbf{y})\cdot(\mathbf{y}'-\mathbf{y})$$

$$\sim |\mathbf{y}'-\mathbf{y}|^2$$

Derive estimate of

$$e(t) = \int \varepsilon^2 \left\{ \frac{1}{2}\left|\mathbf{u}^\varepsilon(t,\mathbf{x}) - \mathbf{u}(t,\mathbf{x})\right|^2 + \eta_{P^*}(t,\mathbf{y},\mathbf{y}^\varepsilon) \right\} \mathrm{d}\mathbf{x}$$

of the form

$$\frac{d}{dt}\left(e(t) + O(\varepsilon^2)\right) \le c\left(e(t) + O(\varepsilon^2)\right)$$

Remarks

Gives convergence at O(ε) of |**y**-$\mathbf{y}^{\varepsilon}$| for finite time interval where solutions are smooth.

Not optimal.

Extension to 3d requires control of second time derivative of $\mathbf{u}^{\varepsilon}$, much more difficult.

Uniqueness

Uniqueness

Loeper proved short time existence of smooth solutions in periodic geometry. These solutions are unique in the class of smooth solutions.

Formal arguments show that time of existence tends to ∞ as ε tends to zero. Does not hold with fixed boundaries.

Feldman and Tudorascu show that if strong solutions exist, they are unique in the class of weak Lagrangian renormalised solutions. Their method is based on Brenier’s method for comparing Euler with SG.

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Variable rotation

Variable rotation

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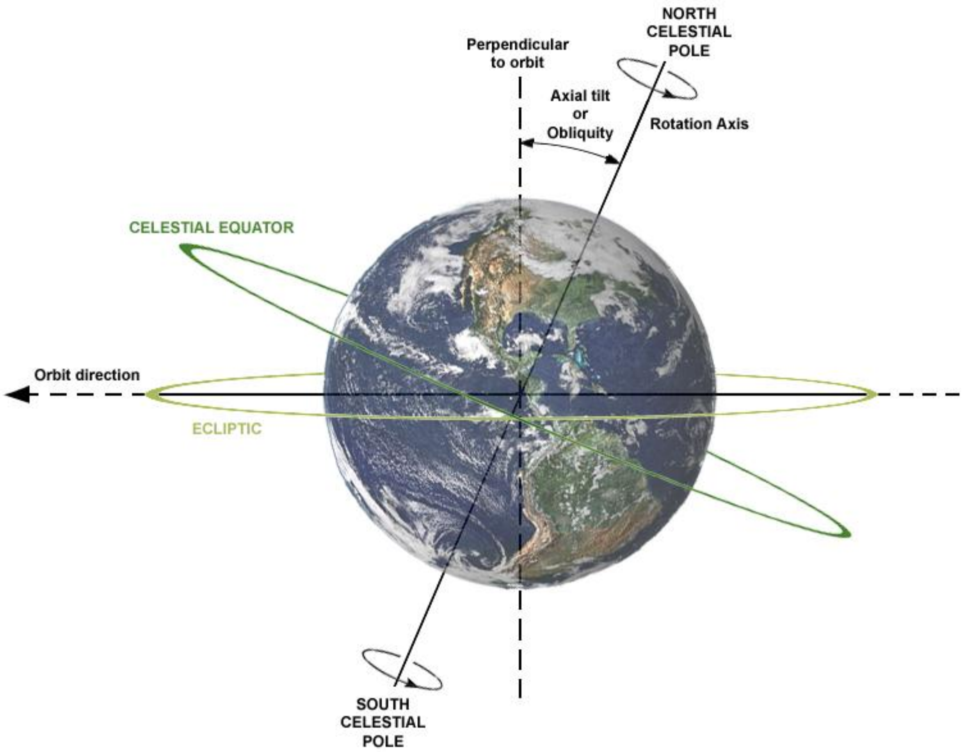
SG valid in atmosphere on scales >1000km. In the ocean the requirement is scales >30km. In the atmosphere, SG only useful on scales comparable to the earth's radius, and the resulting variation of the vertical component of the rotation vector must be included (Rossby waves).

The strongest symmetry in the problem is the radial symmetry, the aspect ratio H/L is $O(\varepsilon^2)$. This breaks the symmetry about the rotation axis.

Thus a loss of conservation laws associated with rotational symmetry is to be expected.

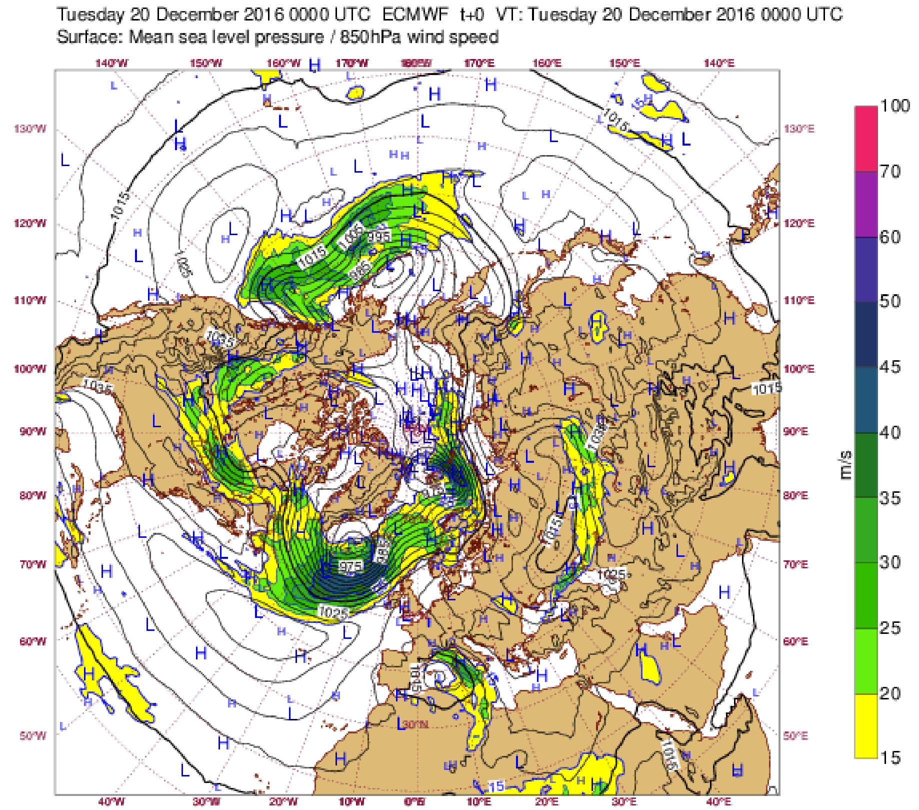
Variable rotation

The vertical component of the Earth's rotation vector depends on latitude.



Variable rotation

Tropical and extratropical weather are different



Dimensionless equations with variable rotation

$$\nabla p = \left(\alpha u_{g2}, -\alpha u_{g1}, -\rho\right)$$

$$u = u_g + \mathrm{O}\left(\varepsilon\right)$$

$$D_t u_{g1} + \frac{\partial p}{\partial x_1} - \alpha u_2 = \mathrm{O}\left(\varepsilon^2\right)$$

$$D_t u_{g2} + \frac{\partial p}{\partial x_1} + \alpha u_1 = \mathrm{O}\left(\varepsilon^2\right)$$

$$\frac{\partial p}{\partial x_3} + \rho = \mathrm{O}\left(\varepsilon^2\right)$$

$$D_t \rho = 0$$

$$\nabla \cdot u = 0$$

Remarks

α is a dimensionless smooth function of position. Consider the case α strictly >0 (excludes equator where SG will be highly degenerate).

Optimal transport method calculates a measure-preserving mapping μ which represents the integral of $\mathbf{u}$ over a timestep δt.

However $\alpha\mathbf{u}$ is non-integrable, so this method will be ill-posed.

Cheng, Cullen and Feldman proved short-time existence of solutions by using the implicit function theorem instead of optimal transport.

Remarks II

Can allow the use of optimal transport by conformal rescaling of the space by a factor α, (Cullen, Douglas, Roulstone, Sewell 2005). Under this $\alpha \mathrm{d}\mathbf{z}$, where $\mathrm{d}\mathbf{z}$ is a line element in the original space, becomes $\mathrm{d}\mathbf{x}$ in the transformed space. If the original space is Euclidean, the transformed space will be a Riemannian manifold.

Optimal transport can then be used locally to construct a solution. However the non-integrability remains an issue in constructing a global solution-work in progress.

Questions