Dynamic and Stochastic Brenier Transport via Hopf-Lax formulae on Wasserstein Space

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With many discussions with Yann Brenier and Wilfrid Gangbo

Brenierfest, IHP, January 9-13, 2017


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Optimal transport problem for *the "ballistic cost function"*, which is defined on phase space $M^* \times M$ by,

$$b_T(v,x) := \inf\{\langle v, \gamma(0)\rangle + \int_0^T L(t, \gamma(t), \dot\gamma(t))\, dt; \gamma \in C^1([0,T), M); \gamma(T) = x\},$$

where $L : [0,T] \times M \times M^* \to \mathbf{R} \cup \{+\infty\}$ is a suitable Lagrangian.

- Why this as opposed to the "fixed-state space cost"?
- Existence of Optimal maps
- Duality and Hamilton-Jacobi equations
- Corresponding Benamou-Brenier type formulas
- Hopf-Lax Type formulae on Wasserstein space
- Hamilton-Jacobi equations on Wasserstein space
- Connection to mean field games
- Stochastic mass transport with ballistic cost.

Why the ballistic cost?

For a given function g, the value function

$$\begin{aligned} V_g(t,x) &= \inf\Big\{g(\gamma(0)) + \int_0^t L(s,\gamma(s),\dot\gamma(s))\,ds; \gamma \in C^1([0,T),M); \gamma(t)=x\Big\}, \\ V(0,x) &= g(x), \end{aligned}$$

is formally a solution of the Hamilton-Jacobi equation,

$$(HJ) \qquad \partial_t V + H(t,x,\nabla_x V) = 0 \quad \text{on } [0,T]\times M,$$

where H is the associated Hamiltonian on $[0,T]\times T^*M$, i.e.,

$$H(t,y,x) = \sup_{v\in TM}\{\langle v,x\rangle - L(t,y,v)\}.$$

Both the ballistic cost,

$$b_T(v,x) := \inf\{\langle v,\gamma(0)\rangle + \int_0^T L(t,\gamma(t),\dot\gamma(t))\,dt; \gamma \in C^1([0,T),M); \gamma(T)=x\},$$

and the fixed-end cost

$$c_T(y,x) := \inf\{\int_0^T L(t,\gamma(t),\dot\gamma(t))\,dt; \gamma\in C^1([0,T),M); \gamma(0)=x, \gamma(T)=y\}$$

are formally solutions to (HJ).

Both costs can be seen as "Kernels" that can be used to generate general solutions for (HJ).

- General Hopf-Lax–Lower kernel:

$$V_g(t,x) = \inf\{g(y) + c(t,y,x);\ y \in M\}$$

- General Dual Hopf-Lax formula–Upper kernel:

$$V_g(t,x) = \sup\{b(t,v,x) - g^*(v);\ v \in M^*\}$$

  provided the Lagrangian L is jointly convex and the initial function g is convex.

- Classical Hopf-Lax and Dual Hopf-Lax formulae

  If $L(x,v) = L_0(v)$ and L_0 convex, then

$$c_t(y,x) = tL_0(\frac{1}{t}|x-y|) \quad \text{and} \quad b_t(v,x) = \langle v,x\rangle - tH_0(v).$$

  and

$$V_g(t,x) = \inf\{g(y) + tL_0(\frac{1}{t}|x-y|);\ y \in M\} = (g^* + tH_0)^*.$$

When defined, the upper kernel is much better than the lower kernel.

Hopf-Lax and Dual Hopf-Lax formula

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When defined, the upper kernel is much better than the lower kernel.

If $L(x,v) = L_0(v - Ax)$, where L_0 is convex, lsc and A a matrix, then

$$b(t,v,x) = \langle e^{-tA}x, v\rangle - \psi(t,v),$$

where $\Psi(t,v) = \int_0^t H_0(e^{-sA^*}v)\,ds$. While

$$c(t,y,x) = \Psi^*(t, e^{-tA}y - x).$$

The value function is then, using the fundamental kernel

$$V_g(t,x) = \inf_y \{g(y) + \Psi^*(t, e^{-tA}y - x)\}.$$

While by using the dualizing kernel

$$V_g(t,x) = (g^* + \Psi(t,\cdot)^*(e^{-tA}x).$$

A dual cost function

- Introduce another cost functional

$$\tilde{c}_T(u,v) := \inf\{\int_0^T \tilde{L}(t,\gamma(t),\dot{\gamma}(t))\,dt; \gamma \in C^1([0,T),M); \gamma(0)=u, \gamma(T)=v\},$$

  The new Lagrangian $\tilde{L}$ is defined on $M \times M^*$ by

$$\tilde{L}(t,x,p) := L^*(t,p,x) = \sup\{\langle p,y\rangle + \langle x,q\rangle - L(t,y,q);\ (y,q) \in M \times M^*\}.$$

  The corresponding Hamiltonian is $H_{\tilde{L}}$ is then given by

$$H_{\tilde{L}}(x,y) = -H(y,x).$$

- Recall Bolza's duality: $(\mathcal{P}) = -(\tilde{\mathcal{P}})$, where

  $(\mathcal{P})$ $\quad \inf\{\int_0^T L(\gamma(s),\dot{\gamma}(s))\,ds + \ell(\gamma(0),\gamma(T))$ over all $\gamma \in C^1([0,T),M)\}$

  and its dual

  $(\tilde{\mathcal{P}})$ $\quad \inf\{\int_0^T \tilde{L}(\gamma(s),\dot{\gamma}(s))\,ds + \ell^*(\gamma(0),-\gamma(T))$ over all $\gamma \in C^1([0,T),M).\}$

  This has several consequences

Ballistic cost satisfies two H-J equations

- One consequence is that the Legendre transform of the value functional $x \to V_g(t,x) := \inf\{g(y) + c(t,y,x);\ y \in M\}$ is another value functional

$$\tilde{V}_{g^*}(t,w) = inf\{g^*(v) + \tilde{c}(t,v,w);\ v \in M^*\},$$

  which yields that

$$b(t,v,x) = \inf\{\langle v,y\rangle + c(t,y,x);\ y \in M\} = \sup\{\langle w,x\rangle - \tilde{c}(t,v,w);\ w \in M^*\}.$$

- So, $x \to b(v,x)$ was a "solution" of the HJ equation

$$\begin{aligned} \partial_t b + H(t,x,\nabla_x b) &= 0 \quad \text{on} \quad [0,T] \times M, \\ b_0(x) &= \langle v,x\rangle. \end{aligned}$$

- Now $v \to b(v,x)$ is also a solution for another H-J equation:

$$\begin{aligned} \partial_t b - H(t,\nabla_v b, v) &= 0 \quad \text{on} \quad [0,T] \times M, \\ b_T(v) &= \langle v,x\rangle. \end{aligned}$$

The associated transport problems will be

$$\overline{B}_T(\mu_0, \nu_T) := \sup\{\int_{M^* \times M} b_T(v, x)\, d\pi;\ \pi \in \mathcal{K}(\mu_0, \nu_T)\},$$

$$\underline{B}_T(\mu_0, \nu_T) := \inf\{\int_{M^* \times M} b_T(v, x)\, d\pi;\ \pi \in \mathcal{K}(\mu_0, \nu_T)\},$$

where μ_0 (resp., ν_T) is a probability measure on M^* (resp., M), and $\mathcal{K}(\mu_0, \nu_T)$ is the set of probability measures π on $M^* \times M$ whose marginal on M^* (resp. on M) is μ_0 (resp., ν_T) *(the transport plans)*.

Note that when $T = 0$, we have $b_0(x, v) = \langle v, x\rangle$, which is exactly the case considered by Brenier, that is

$$\overline{W}(\mu_0, \nu_0) := \sup\{\int_{M^* \times M} \langle v, x\rangle\, d\pi;\ \pi \in \mathcal{K}(\mu_0, \nu_0)\},$$

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This is the dynamic version of the Wasserstein distance.

The Ballistic Optimal Transport Problem

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This is the dynamic version of the Wasserstein distance.

One of the Kantorovich Potentials is nice

By standard Kantorovich duality,

$$\begin{aligned}\underline{B}_T(\mu_0,\nu_0) := & \inf\{\int_{M^*\times M} b(v,x))\,d\pi;\pi\in\mathcal{K}(\mu_0,\nu_T)\}\\ = & \sup\{\int_M \phi_1(x)\,d\nu_T(x)-\int_{M^*}\phi_0(v)\,d\mu_0(v);\ \phi_1,\phi_0\in\mathcal{K}(b)\},\end{aligned}$$

where $\mathcal{K}(b)$ is the set of functions $\phi_1\in L^1(M,\nu_T)$, $\phi_0\in L^1(M^*,\mu_0)$ such that

$$\phi_1(x)-\phi_0(v)\le b(v,x)\quad\text{for all } (v,x)\in M^*\times M.$$

Kantorovich functions in $\mathcal{K}(c)$ can be assumed to satisfy

$$\phi_1(x)=\inf_{v\in M^*} b(v,x)+\phi_0(v)\quad\text{and}\quad\phi_0(v)=\sup_{x\in M}\phi_1(x)-b(v,x).$$

Say that ϕ_0 (resp., ϕ_1) is an initial (resp., final) Kantorovich potential.

Main advantages of the ballistic cost

- $b_T(v, x)$ is concave in v and convex in x. It is also Lipschitz continuous.
- In the case of $\underline{B}_T(\mu_0, \nu_T)$, the initial Kantorovich potential ϕ_0 is convex, though nothing can be said about ϕ_1.
- For $\overline{B}_T(\mu_0, \nu_T)$, the final potential is convex and nothing can be said about ϕ_1.
- Even though $c(y, x)$ is jointly convex, nothing can be said about the Kantorovich potentials of

$$C_T(\mu_0, \mu_T) := \inf\{\int_{M\times M} c_T(y, x)\, d\pi;\ \pi \in \mathcal{K}(\mu_0, \mu_T)\},$$

including the case where $L(x, v) = |v|^p$ ($p \geq 1$), that is when $c_1(y, x) = |x - y|^p$.
Gangbo-McCann worked with c-convexity in order to deal with the regularity of Kantorovich potentials.

Minimizing Map for Ballistic Cost

Theorem (A): Under suitable assumptions on the Lagrangian L. Let μ_0, ν_T be probabilities on M^*, M such that μ_0 is absolutely continuous with respect to Lebesgue measure. Then,

1. The following duality holds:

$$\underline{B}_T(\mu_0, \nu_T) = \sup \Big\{ \int_M \phi_T(x)\, d\nu_T(x) + \int_M \tilde{\phi}_0(v)\, d\mu_0(v); \\ \phi_0 \text{ concave } \& \ \phi_t \text{ solution of (HJ)} \Big\}.$$

$$\begin{cases} \partial_t \phi + H(t, x, \nabla_x \phi) = 0 \text{ on } [0, T] \times M, \\ \phi(0, x) = \phi_0(x), \end{cases}$$

2. There exists a concave function $\phi_0 : M \to \mathbf{R}$ and a bounded locally Lipschitz vector field $X(x, t) : M \times]0, T[\longrightarrow M)$ such that, if Φ_s^t, $(s, t) \in]0, T[^2$ is the flow of X from time s to time t, then

$$\underline{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \Phi_0^T \circ \nabla \tilde{\phi}_0(v) d\mu_0(v).$$

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$$\begin{cases} \partial_t \phi + H(t, x, \nabla_x \phi) = 0 \text{ on } [0, T] \times M, \\ \phi(0, x) = \phi_0(x), \end{cases}$$

2. There exists a concave function $\phi_0 : M \to \mathbf{R}$ and a bounded locally Lipschitz vector field $X(x, t) : M \times]0, T[\longrightarrow M)$ such that, if $\Phi_s^t, (s, t) \in]0, T[^2$ is the flow of X from time s to time t, then

$$\underline{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \Phi_0^T \circ \nabla \tilde{\phi}_0(v) d\mu_0(v).$$

Maximizing Map for Ballistic Cost

Theorem (B): Under suitable assumptions on the Lagrangian L. Let μ_0, ν_T be probabilities on M^*, M such that ν_T is absolutely continuous with respect to Lebesgue measure. Then,

1. The following duality holds:

$$\overline{B}_T(\mu_0, \nu_T) = \inf\Big\{\int_M \psi_T^*(x)\, d\nu_T(x) + \int_M \psi_0(v)\, d\mu_0(v); \\ \psi_T \text{ convex } \& \ \psi_t \text{ solution of (dual-HJ)}\Big\}.$$

$$\begin{cases} \partial_t\psi - H(\nabla_v\psi, v) = 0 \text{ on } [0,T]\times M, \\ \psi(T,v) = \psi_T(v), \end{cases}$$

2. There exists a convex function $\psi : M \to \mathbf{R}$ and a bounded locally Lipschitz vector field $Y(x,t) : M\times]0,T[\longrightarrow M)$ such that, if $\Psi_s^t, (s,t) \in]0,T[^2$ is the flow of Y from time s to time t, then

$$\overline{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \nabla\psi^* \circ \Psi_0^T(v)) d\mu_0(v).$$

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$$\begin{cases} \partial_t \psi - H(\nabla_v \psi, v) &= 0 \text{ on } [0, T] \times M, \\ \psi(T, v) &= \psi_T(v), \end{cases}$$

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$$\overline{B}_T(\mu_0, \nu_T) = \int_{M^*} b_T(v, \nabla \psi^* \circ \Psi_0^T(v)) d\mu_0(v).$$

Our assumptions on the Lagrangian

- (A1) $L : M \times M^* \to \mathbf{R}$ is convex, proper and lower semi-continuous.
- (A2) The set $F(x) := \{v; L(x,v) < \infty\}$ is non-empty for all $x \in M$, and for some $\rho > 0$, we have for all $x \in M$,

$$\mathrm{dist}(0, F(x)) \leq \rho(1 + |x|).$$

- (A3) For all $(x,v) \in M \times M^*$, we have

$$L(x,v) \geq \theta(\max\{0, |v| - \alpha|x|\}) - \beta|x|,$$

  where α, β are constants, and θ is coercive, non-decreasing on $[0, \infty)$.

Equivalently, for the corresponding Hamiltonian H,

- (A1) $H(x,y)$ is finite and concave in x convex in y.
- (A2) $H(x,y) \leq \phi(y) + (\alpha|y| + \beta)|x|$, where α, β constants and ϕ convex.
- (A3) $H(x,y) \geq \psi(y) - (\gamma|x| + \delta)|y|$, where γ, δ constants and ψ concave.

Hopf-Lax Formula for the minimal cost

Theorem (B): Assume $M = \mathbf{R}^d$ and that L satisfies hypothesis (A1), (A2) and (A3), and let μ_0 (resp. ν_T) be a probability measure on M^* (resp., M). If μ_0 is absolutely continuous with respect to Lebesgue measure, then

1. The following Hopf-Lax formula holds:

$$\underline{B}_T(\mu_0, \nu_T) = \inf\{\underline{W}(\mu_0, \nu) + C_T(\nu, \nu_T);\ \nu \in \mathcal{P}(M)\}.$$

2. The infimum is attained at some probability measure ν_0 on M.
3. The initial Kantorovich potential for $C_T(\nu_0, \nu_T)$ is concave.

• Worth noting: If $L(x, v) = \frac{1}{2}|v|^2$ (i.e., $c(y, x) = \frac{1}{2}|x - y|^2$), the initial Kantorovich potential for $C_T(\nu_0, \nu_T)$ is then of the form

$$\phi_0(y) = g(y) - \frac{1}{2}|y|^2 \quad \text{where } g \text{ is a convex function.}$$

But ϕ_0 can still be concave if $0 \leq D^2 g \leq I$, which is what occurs above in (3).

A consequence of Hopf-Lax formula

By the Hopf-Lax inequality, there is ν_0 on M such that

$$\underline{B}_T(\mu_0,\nu_T) = C_T(\nu_0,\nu_T) + \underline{W}(\mu_0,\nu_0).$$

Let g be the concave function on M^* such that $(\nabla g)_\# \mu_0 = \nu_0$ and

$$\underline{W}(\mu_0,\nu_0) = \int_M \langle \nabla g(v), v\rangle d\mu_0(v).$$

Let Φ_0^T be the flow such that

$$C_T(\nu_0,\nu_T) = \int_M c_T(y,\Phi_0^T y) d\nu_0(y).$$

Since

$$b_T(v,x) \le c_T(\nabla g(v),x) + \langle \nabla g(v), v\rangle \text{ for all } v \in M^*$$

$$\begin{aligned}
\underline{B}_T(\mu_0,\nu_T) &\le \int_M b_T(v,\Phi_0^T \circ \nabla g(v)) d\mu_0(v) \\
&\le \int_{M^*\times M} \left\{ c_T(\nabla g(v),\Phi_0^T \circ \nabla g(v)) + \langle \nabla g(v), v\rangle \right\} d\mu_0(v) \\
&= \int_M c_T(y,\Phi_0^T y) d\nu_0(y) + \int_M \langle \nabla g(v), v\rangle\, d\mu_0(v) \\
&= C_T(\nu_0,\nu_T) + \underline{W}(\mu_0,\nu) = \underline{B}_T(\mu_0,\nu_T).
\end{aligned}$$

However this formula doesn't lift:

$$c(t, y, x) = \sup\{b(t, v, x) - \langle v, y\rangle; v \in M^*\}.$$

Theorem (D): Assume ν_0 and ν_T are probability measures on M such that ν_0 is absolutely continuous with respect to Lebesgue measure. Then, TFAE:

1. The initial Kantorovich potential of $C_T(\nu_0, \nu_T)$ is concave.
2. The following holds:

$$C_T(\nu_0, \nu_T) = \sup\{\underline{B}_T(\mu, \nu_T) - \underline{W}(\nu_0, \mu);\ \mu \in \mathcal{P}(M^*)\}.$$

   and the sup is attained at some probability measure μ_0 on M^*.

Corollary: Consider the cost $c(y, x) = c(x - y)$, where c is a convex function on M and let ν_0, ν_1 be probability measures on M such that the initial Kantorovich potential associated to $C_T(\nu_0, \nu_T)$ is concave. Then, there exist concave functions $\phi_0 : M \to \mathbf{R}$ and $\phi_1 : M^* \to \mathbf{R}$ such that

$$C_1(\nu_0, \nu_1) - K = \int_M c(\nabla\phi_1 \circ \nabla\phi_0(y) - y) d\nu_0(y) = \int_M \langle \nabla\tilde{\phi}_1(y) - \nabla\phi_0(y), y\rangle\, d\nu_0(y),$$

where $K = K(c)$ is a constant and $\tilde{\phi}$ is the concave Legendre transform of ϕ.

Brenier-Benamou Type formula

Theorem (E): For fixed probability measures μ_0 on M^* and ν_T on M,

- As a function of the end measure:

$$\underline{B}_T(\mu_0,\nu_T) = \inf\left\{\underline{W}(\mu_0,\rho_0) + \int_0^T \int_M L(x,w_t(x))\,d\varrho_t(x)dt;\ (\varrho,w)\in P(0,T;\nu_T)\right\}$$

where $P(0,T;\nu_T)$ is the set of pairs (ϱ,w) such that $t\to\varrho_t\in\mathcal{P}(M)$, $t\to w_t\in\mathbf{R}^n$ are paths of Borel vector fields such that

$$\begin{cases} \partial_t\varrho + \nabla\cdot(\varrho w) = 0 & \text{in } \mathcal{D}'((0,T)\times M) \\ \varrho_T = \nu_T. \end{cases}$$

- As a function of the initial measure:

$$\overline{B}_T(\mu_0,\nu_T) = \sup\left\{\overline{W}(\nu_T,\rho_T) - \int_0^T \int_M \tilde{L}(x,w_t(x))\,d\varrho_t(x)dt;\ (\varrho,w)\in P(0,T;\mu_0)\right\}$$

where $P(0,T;\mu_0)$ is the set of pairs (ϱ,w) such that

$$\begin{cases} \partial_t\varrho + \nabla\cdot(\varrho w) = 0 & \text{in } \mathcal{D}'((0,T)\times M) \\ \varrho_0 = \mu_0. \end{cases}$$

Lifting a value function to a value function on Wasserstein space

Started with $\phi_0(y) = \langle v, y \rangle$ and defined $b_v(t,x)$ as a Value functional

$$
\begin{aligned}
b_v(t,x) &= \inf\left\{\phi_0(\gamma(0)) + \int_0^t L(s,\gamma(s),\dot\gamma(s))\,ds; \gamma \in C^1([0,T),M); \gamma(t) = x\right\} \\
&= \inf\left\{\phi_0(y) + c_t(y,x); y \in M\right\} \qquad (\textit{Hopf} - \textit{Lax formula}).
\end{aligned}
$$

It satisfies the Hamilton-Jacobi equation on M.

$$\partial_t b + H(t,x,\nabla_x b) = 0 \quad \text{on } [0,T] \times M,$$

We then lifted b_v to Wasserstein space by defining $B_{\mu_0}(t,\nu) = \underline{B}_t(\mu_0,\nu)$.

$$
\begin{aligned}
B_{\mu_0}(t,\nu) &= \inf\{\underline{W}(\mu_0,\tilde\nu) + C_t(\tilde\nu,\nu);\ \nu \in \mathcal{P}(M)\}\ (\textit{Hopf} - \textit{Lax formula}) \\
&= \inf\{\mathcal{U}_{\mu_0}(\varrho_0) + \int_0^t \mathcal{L}(\varrho,w)dt; (\varrho,w) \in P(0,t;\nu)\}(\textit{Value functional})
\end{aligned}
$$

1. Do they satisfy a Hamilton-Jacobi equation on Wasserstein space?
2. Do they provide solutions to mean field games?

Under technical conditions

(Ambrosio-Feng) (at least in a particular case): Value functionals on Wasserstein space yield a unique metric viscosity solution for

$$\begin{cases} \partial_t B + \mathcal{H}(t, \nu, \nabla_\nu B(t, \mu)) = 0, \\ B(0, \nu) = \underline{W}(\mu_0, \nu) \end{cases}$$

Here the Hamiltonian on Wasserstein space is defined as

$$\mathcal{H}(\nu, \zeta) = \sup\{\int \langle \zeta, \xi \rangle d\nu - \mathcal{L}(\nu, \xi); \xi \in T^*_\nu(\mathcal{P}(M))\}$$

(Gangbo-Swiech) Value functions on Wasserstein space with suitable initial data yield solutions to the so-called Master equation for mean field games without diffusion and without potential term.

Theorem (Gangbo-Swiech) Assume $\mathcal{U}_0 : \mathcal{P}(M) \to \mathbb{R}$, $U_0 : M \times \mathcal{P}(M) \to \mathbb{R}$ are such that $\nabla_q U_0(q, \mu) \equiv \nabla_\mu \mathcal{U}_0(\mu)(q) \quad \forall q \in M \ \mu \in \mathcal{P}(M)$, and consider the value functional,

$$\mathcal{U}(t, \nu) = \inf_{(\varrho, w) \in P(0,t;\nu)} \int_0^t \mathcal{L}(\varrho, w) dt + \mathcal{U}_0(\varrho_0)$$

Then, there exists $U : [0, T] \times M \times \mathcal{P}(M) \to \mathbb{R}$ such that

$$\nabla_q U_t(q, \mu) \equiv \nabla_\mu \mathcal{U}_t(\mu)(q) \quad \forall q \in M \ \mu \in \mathcal{P}(M).$$

and U satisfies the Master equation (but without diffusion)

Mean Field Equation

This yields the existence for any probabilities μ_0, ν_T, a function $\beta : [0, T] \times M \times \mathcal{P}(M) \to \mathbb{R}$ such that

$$\nabla_x \beta(t, x, \mu) \equiv \nabla_\mu B_{\mu_0}(t, \mu)(x) \quad \forall x \in M \; \mu \in \mathcal{P}(M).$$

There exists $\rho \in AC^2((0, T) \times \mathcal{P}(M))$ such that

$$\begin{cases} \partial_t \beta + \int \langle \nabla_\mu \beta(t, x, \mu) \cdot \nabla H(x, \nabla_x \beta) \rangle \, d\mu + H(x, \nabla_x \beta(t, x, \mu)) = 0, \\ \partial_t \rho + \nabla(\rho \nabla H(x, \nabla_x \beta)) = 0, \\ \beta(0, \cdot, \cdot) = \beta_0, \quad \rho(T, \cdot) = \nu_T, \end{cases}$$

where $\beta_0(x, \rho) = \phi_\rho(x)$, where ϕ_ρ is the convex function such that $\nabla \phi_\rho$ pushes μ_0 into ρ.

What about solutions to mean field games that include diffusions?

$$\underline{B}_T^s(\mu_0, \nu_T) := \inf_{V \sim \mu_0} \inf_{X \in \mathcal{A}, X_T \sim \nu_T} \mathbf{E}_P \left\{ \langle V, X_0 \rangle + \int_0^T L(t, X(t), \beta(t, X))\, dt \right\},$$

where $\mathcal{A}$ is the class of all $\mathbf{R}^d$-valued continuous semimartingales $(X_t)_{0 \le t \le T}$ on a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ such that there exists a Borel measurable $\beta_X : [0, T] \times C([0, T]) \to \mathbf{R}^d$ satisfying

1. $w \to \beta_X(t, w)$ is $\mathcal{B}(C([0, t]))_+$-measurable for all t, where $\mathcal{B}(C([0, t])$ denotes the Borel σ-field on $C([0, t])$.
2. $X(t) = X(0) + \int_0^t \beta_X(s, X)\, ds + W_X(t)$, where $W_X(t)$ is a $\sigma[X(s); 0 \le s \le t]$-Brownian motion.

The fixed end measures cost has been studied by Mikami, Thieulin, Leonard.

$$C_T^s(\nu_0, \nu_T) := \inf \mathbf{E}_P \left\{ \int_0^T L(t, X(t), \beta(t, X))\, dt;\ X \in \mathcal{A},\ X(0) \sim \nu_0, X(T) \sim \nu_T \right\},$$

Theorem (F): Under suitable conditions on L

1. Duality:

$$\underline{B}_T^s(\mu_0,\nu_T) \quad = \quad \sup\left\{\int_M \phi_T(x)\, d\nu_T(x) + \int_M \tilde{\phi}_0(v)\, d\mu_0(v); \right. \\ \left. \phi_0 \text{ concave \& } \phi_t \text{ solution of (HJB)}\right\}.$$

$$\begin{cases} \partial_t\phi + \frac{1}{2}\Delta\phi + H(t,x,\nabla_x\phi) & = & 0 \text{ on } [0,T]\times M, \\ \phi(0,x) & = & \phi_0(x), \end{cases}$$

2. For any probability measure ν on M, we have

$$\underline{B}_T^s(\mu_0,\nu) = \inf\left\{\underline{W}(\mu_0,\rho_0) + \int_0^T\int_M L\big(t,x,b_t(x)\big)\,d\varrho_t(x)dt;\ (\varrho,b)\in P(0,T;\nu)\right\}$$

where $P(0,T;\nu_T)$ is the set of pairs (ϱ,b) such that $t\to\varrho_t\in\mathcal{P}(M)$, $t\to b_t\in\mathbf{R}^n$ are paths of Borel vector fields such that

$$\begin{cases} \partial_t\varrho - \frac{1}{2}\Delta\rho + \nabla\cdot(\varrho b) & = & 0 \quad \text{in } \mathcal{D}'\big((0,T)\times M\big) \\ \varrho_T & = & \nu. \end{cases}$$

There exists $\beta : [0, T] \times M \times \mathcal{P}(M) \to \mathbb{R}$ and $\rho \in AC^2((0, T) \times \mathcal{P}(M))$ such that

$$\begin{cases} \partial_t \beta - \frac{1}{2}\Delta\beta + \int \langle \nabla_\mu \beta(t, x, \mu) \cdot \nabla H(x, \nabla_x \beta) \rangle \, d\mu + H(t, x, \nabla_x \beta(t, x, \mu)) = 0, \\ \partial_t \rho - \frac{1}{2}\Delta\rho + \nabla(\rho \nabla H(t, x, \nabla_x \beta)) = 0, \\ \beta(0, \cdot, \cdot) = \beta_0, \quad \rho(T, \cdot) = \nu_T, \end{cases}$$

where $\beta_0(x, \rho) = \phi_\rho(x)$, where ϕ_ρ is the convex function such that $\nabla\phi_\rho$ pushes μ_0 into ρ.

Many Happy Returns Yann Brenier