Mean Field Limits for Ginzburg-Landau Vortices

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The Ginzburg-Landau equations

$u : \Omega \subset \mathbb{R}^2 \to \mathbb{C}$

$$\boxed{-\Delta u = \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{Ginzburg-Landau equation (GL)}$$

$$\boxed{\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{parabolic GL equation (PGL)}$$

$$\boxed{i\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2)} \quad \text{Gross-Pitaevskii equation (GP)}$$

Associated energy

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2}$$

Models: superconductivity, superfluidity, Bose-Einstein condensates, nonlinear optics

Vortices

- in general $|u| \leq 1$, $|u| \simeq 1$ = superconducting/superfluid phase, $|u| \simeq 0$ = normal phase
- u has zeroes with nonzero degrees = **vortices**
- $u = \rho e^{i\varphi}$, characteristic length scale of $\{\rho < 1\}$ is ε = vortex core size
- degree of the vortex at x_0:

$$\frac{1}{2\pi} \int_{\partial B(x_0, r)} \frac{\partial \varphi}{\partial \tau} = d \in \mathbb{Z}$$

- In the limit $\varepsilon \to 0$ vortices become *points*, (or curves in dimension 3).

Solutions of (GL), bounded number N of vortices

- [Bethuel-Brezis-Hélein '94]
  u_ε minimizing E_ε has vortices all of degree $+1$ (or all -1) which converge to a minimizer of

  $$W((x_1, d_1), \dots, (x_N, d_N)) = -\pi \sum_{i \neq j} d_i d_j \log |x_i - x_j| + \text{boundary terms...}$$

  "renormalized energy", Kirchhoff-Onsager energy (in the whole plane)
  minimal energy

  $$\min E_\varepsilon = \pi N |\log \varepsilon| + \min W + o(1) \quad \text{as} \quad \varepsilon \to 0$$

- Some boundary condition needed to obtain nontrivial minimizers
- nonminimizing solutions: u_ε has vortices which converge to a critical point of W:

  $$\nabla_i W(\{x_i\}) = 0 \quad \forall i = 1, \cdots N$$

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Dynamics, bounded number N of vortices

- For well-prepared initial data, $d_i = \pm 1$, solutions to (PGL) have vortices which converge (after some time-rescaling) to solutions to

$$\frac{dx_i}{dt} = -\nabla_i W(x_1, \dots, x_N)$$

[Lin '96, Jerrard-Soner '98, Lin-Xin '99, Spirn '02, Sandier-S '04]

- For well-prepared initial data, $d_i = \pm 1$, solutions to (GP)

$$\frac{dx_i}{dt} = -\nabla_i^\perp W(x_1, \dots, x_N) \qquad \nabla^\perp = (-\partial_2, \partial_1)$$

[Colliander-Jerrard '98, Spirn '03, Bethuel-Jerrard-Smets '08]

- All these hold up to collision time
- For (PGL), extensions beyond collision time and for ill-prepared data [Bethuel-Orlandi-Smets '05-07, S. '07]

Vorticity

- In the case $N_\varepsilon \to \infty$, describe the vortices via the **vorticity** :

  supercurrent

  $$j_\varepsilon := \langle iu_\varepsilon, \nabla u_\varepsilon \rangle \qquad \langle a, b \rangle := \frac{1}{2}(a\bar{b} + \bar{a}b)$$

  vorticity

  $$\mu_\varepsilon := \mathrm{curl}\, j_\varepsilon$$

- $\simeq$ vorticity in fluids, but quantized: $\mu_\varepsilon \simeq 2\pi \sum_i d_i \delta_{a_i^\varepsilon}$
- $\frac{\mu_\varepsilon}{2\pi N_\varepsilon} \to \mu$ signed measure, or probability measure,

Mean-field limit for stationary solutions

If u_ε is a solution to (GL) and $N_\varepsilon \gg 1$ then $\mu_\varepsilon / N_\varepsilon \to \mu$ solution to

$$\mu \nabla h = 0 \qquad h = -\Delta^{-1} \mu$$

in a suitable weak sense ($\simeq$ Delort):

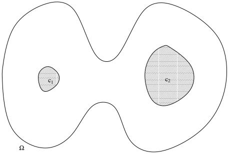
$$T_\mu := -\nabla h \otimes \nabla h + \frac{1}{2} |\nabla h|^2 \delta_i^j$$

Weak relation is

$$\mathrm{div}\ T_\mu = 0 \quad \text{in “finite parts"}$$

[Sandier-S '04]

$\rightsquigarrow$ h is constant on the support of μ



Dynamics in the case $N_\varepsilon \gg 1$

Back to

$$\frac{N_\varepsilon}{|\log \varepsilon|}\partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2 \qquad \text{(PGL)}$$

$$iN_\varepsilon \partial_t u = \Delta u + \frac{u}{\varepsilon^2}(1-|u|^2) \qquad \text{in } \mathbb{R}^2 \qquad \text{(GP)}$$

- For (GP), by Madelung transform, the limit dynamics is expected to be the 2D incompressible Euler equation. Vorticity form

$$\partial_t \mu - \operatorname{div}(\mu \nabla^\perp h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(EV)}$$

- For (PGL), formal model proposed by [Chapman-Rubinstein-Schatzman '96], [E '95]: if $\mu \geq 0$

$$\partial_t \mu - \operatorname{div}(\mu \nabla h) = 0 \qquad h = -\Delta^{-1}\mu \quad \text{(CRSE)}$$

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Study of the Chapman-Rubinstein-Schatzman-E equation

- [Lin-Zhang '00, Du-Zhang '03, Masmoudi-Zhang '05] existence of weak solutions (à la Delort) by vortex approximation method, existence and uniqueness of L^∞ solutions, which decay in $1/t$ (uses pseudo-differential operators)
- [Ambrosio-S '08] variational approach in the setting of a bounded domain. The equation is formally the gradient flow of $F(\mu) = \frac{1}{2}\int_\Omega |\nabla\Delta^{-1}\mu|^2$ for the 2-Wasserstein metric (à la [Otto, Ambrosio-Gigli-Savaré]).
  Existence of weak solutions for bounded energy initial data (i.e. $\mu \in H^{-1}$) via the minimizing movement scheme of De Giorgi, uniqueness in the class L^∞, propagation of L^p regularity. Takes into account possible entrance / exit of mass via the boundary ($\int_\Omega \mu$ not preserved).
  Extension by [Ambrosio-Mainini-S '11] for signed measures.
- [S-Vazquez '13] PDE approach in all dimension. Existence via limits in fractional diffusion $\partial_t\mu + \mathrm{div}\,(\mu\nabla\Delta^{-s}\mu)$ when $s \to 1$, uniqueness in the class L^∞, propagation of regularity, asymptotic self-similar profile

$$\mu(t) = \frac{1}{\pi t}\mathbf{1}_{B_{\sqrt{t}}}$$

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Previous rigorous convergence results

- (PGL) case : [Kurzke-Spirn '14] convergence of $\mu_\varepsilon/(2\pi N_\varepsilon)$ to μ solving (CRSE) under assumption $N_\varepsilon \le (\log\log|\log\varepsilon|)^{1/4}$ + well-preparedness
- (GP) case: [Jerrard-Spirn '15] convergence to μ solving (EV) under assumption $N_\varepsilon \le (\log|\log\varepsilon|)^{1/2}$ + well-preparedness
- both proofs "push" the fixed N proof (taking limits in the evolution of the energy density) by making it more quantitative
- difficult to go beyond these dilute regimes without controlling distance between vortices, possible collisions, etc

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Alternative method: the "modulated energy"

- Exploits the regularity and stability of the solution to the limit equation
- Works for dissipative as well as conservative equations
- Works for gauged model as well

Let $\mathrm{v}(t)$ be the expected limiting velocity field (such that $\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \rightharpoonup \mathrm{v}$ and $\mathrm{curl}\, \mathrm{v} = 2\pi\mu$). Define the modulated energy

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2},$$

modelled on the Ginzburg-Landau energy.
Analogy with "relative entropy" and "modulated entropy" methods [Dafermos '79] [DiPerna '79] [Yau '91] [Brenier '00]....

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Main result: Gross-Pitaevskii case

Theorem (S. '15)

Assume u_ε *solves (GP) and let* N_ε *be such that* $|\log\varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon}$. *Let* v *be a* $L^\infty(\mathbb{R}_+, C^{0,1})$ *solution to the incompressible Euler equation*

$$\begin{cases} \partial_t \mathrm{v} = 2\mathrm{v}^\perp \mathrm{curl}\,\mathrm{v} + \nabla \mathrm{p} & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases} \quad \text{(IE)}$$

with $\mathrm{curl}\,\mathrm{v} \in L^\infty(L^1)$.
Let $\{u_\varepsilon\}_{\varepsilon>0}$ *be solutions associated to initial conditions* u_ε^0, *with* $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \le o(N_\varepsilon^2)$. *Then, for every* $t \ge 0$, *we have*

$$\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \to \mathrm{v} \quad \text{in } L^1_{loc}(\mathbb{R}^2).$$

Implies of course the convergence of the vorticity $\mu_\varepsilon / N_\varepsilon \to \mathrm{curl}\,\mathrm{v}$
Works in 3D

Main result: parabolic case

Theorem (S. '15)

Assume u_ε *solves (PGL) and let* N_ε *be such that* $1 \ll N_\varepsilon \leq O(|\log\varepsilon|)$. *Let* v *be a* $L^\infty([0,T],C^{1,\gamma})$ *solution to*

- *if* $N_\varepsilon \ll |\log\varepsilon|$

$$\begin{cases} \partial_t \mathrm{v} = -2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} + \nabla \mathrm{p} & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases} \tag{L1}$$

- *if* $N_\varepsilon \sim \lambda|\log\varepsilon|$

$$\partial_t \mathrm{v} = \frac{1}{\lambda}\nabla \mathrm{div}\ \mathrm{v} - 2\mathrm{v}\,\mathrm{curl}\,\mathrm{v} \quad \text{in } \mathbb{R}^2. \tag{L2}$$

Assume $\mathcal{E}_\varepsilon(u_\varepsilon^0, 0) \leq \pi N_\varepsilon|\log\varepsilon| + o(N_\varepsilon^2)$ *and* $\mathrm{curl}\,\mathrm{v}(0) \geq 0$. *Then* $\forall t \leq T$ *we have*

$$\frac{1}{N_\varepsilon}\langle \nabla u_\varepsilon, iu_\varepsilon\rangle \to \mathrm{v} \quad \text{in } L^1_{loc}(\mathbb{R}^2).$$

Taking the curl of the equation yields back the (CRSE) equation if $N_\varepsilon \ll |\log\varepsilon|$, but *not* if $N_\varepsilon \propto |\log\varepsilon|$!
Long time existence proven by [Duerinckx '16].

Proof method

- Go around the question of minimal vortex distances by using instead the modulated energy and showing a Gronwall inequality on $\mathcal{E}$.
- the proof relies on algebraic simplifications in computing $\frac{d}{dt}\mathcal{E}_\varepsilon(u_\varepsilon(t))$ which reveal only quadratic terms
- Uses the regularity of v to bound corresponding terms
- An insight is to think of v as a spatial gauge vector and $\mathrm{div}\ \mathrm{v}$ (resp. p) as a temporal gauge

The disordered case

- In real superconductors one wants to flow currents and prevent the vortices from moving because that dissipates energy
- Model pinning and applied current by pinning potential $0 < a(x) \leq 1$ and force F
- equation reduces to

$$(\alpha + i|\log\varepsilon|\beta)\partial_t u_\varepsilon = \Delta u_\varepsilon + \frac{a u_\varepsilon}{\varepsilon^2}(1-|u_\varepsilon|^2) + \frac{\nabla a}{a}\cdot\nabla u_\varepsilon + i|\log\varepsilon|F^\perp\cdot\nabla u_\varepsilon + f u_\varepsilon$$

  competition between vortex interaction, pinning force $\nabla h := -\nabla \log a$ and applied force F
- Case of finite number of vortices treated in [Tice '10], [S-Tice '11], [Kurzke-Marzuola-Spirn '15]

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Convergence to fluid-like equations

Gross-Pitaevskii case

Theorem (Duerinckx-S)

In the regime $|\log \varepsilon| \ll N_\varepsilon \ll \frac{1}{\varepsilon}$, *convergence of* $j_\varepsilon / N_\varepsilon$ *to solutions of*

$$\begin{cases} \partial_t \mathrm{v} = \nabla \mathrm{p} + (-F + 2\mathrm{v}^\perp)\mathrm{curl}\,\mathrm{v} & \textit{in } \mathbb{R}^2 \\ \mathrm{div}\,(a\mathrm{v}) = 0 & \textit{in } \mathbb{R}^2, \end{cases}$$

Theorem (Duerinckx-S)

- $N_\varepsilon \ll |\log \varepsilon|, \ \lambda_\varepsilon := \frac{N_\varepsilon}{|\log \varepsilon|}, \ F_\varepsilon = \lambda_\varepsilon F, a_\varepsilon = a^{\lambda_\varepsilon} \ (h_\varepsilon = \lambda_\varepsilon h)$

$j_\varepsilon / N_\varepsilon$ converges to

$$\begin{cases} \partial_t \mathrm{v} = \nabla \mathrm{p} + (-\nabla^\perp h - F^\perp - 2\mathrm{v}) \mathrm{curl}\, \mathrm{v} & \text{in } \mathbb{R}^2 \\ \mathrm{div}\ \mathrm{v} = 0 & \text{in } \mathbb{R}^2, \end{cases}$$

- $N_\varepsilon = \lambda |\log \varepsilon| \ (\lambda > 0)$

$j_\varepsilon / N_\varepsilon$ converges to

$$\partial_t \mathrm{v} = \frac{1}{\lambda} \nabla (\frac{1}{a} \mathrm{div}\ (a\mathrm{v})) + (-\nabla^\perp h - F^\perp - 2\mathrm{v}) \mathrm{curl}\, \mathrm{v} \quad \text{in } \mathbb{R}^2.$$

$\rightsquigarrow$ vorticity evolves by

$$\partial_t \mu = \mathrm{div}\ (\Gamma \mu)$$

with $\Gamma = \textit{pinning} + \textit{applied force} + \textit{interaction}$

Homogenization questions

- we want to consider rapidly oscillating pinning force

$$\eta_\varepsilon h(x, \frac{x}{\eta_\varepsilon}) \quad \eta_\varepsilon \ll 1$$

and scale η_ε with ε

- too difficult to take the diagonal limit $\eta_\varepsilon \to 0$ directly from GL eq.
- Instead homogenize the limiting equations

$$\partial_t \mu = \mathrm{div}\,(\Gamma \mu) \qquad \Gamma = -\nabla^\perp h - F^\perp - 2\mathrm{v}$$

$\sim$ homogenization of nonlinear transport equations.

- easier when interaction is negligible $\rightsquigarrow$ Γ independent of μ, *washboard model*
- Understand *depinning current* and velocity law (in $\sqrt{F - F_c}$)
- Understand thermal effects by adding noise to such systems $\rightsquigarrow$ *creep, elastic effects*

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Sketch of proof: quantities and identities

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2} \quad \text{(modulated energy)}$$

$$j_\varepsilon = \langle iu_\varepsilon, \nabla u_\varepsilon\rangle \qquad \mathrm{curl}\, j_\varepsilon = \mu_\varepsilon \quad \text{(supercurrent and vorticity)}$$

$$V_\varepsilon = 2\langle i\partial_t u_\varepsilon, \nabla u_\varepsilon\rangle \quad \text{(vortex velocity)}$$

$$\partial_t j_\varepsilon = \nabla\langle iu_\varepsilon, \partial_t u_\varepsilon\rangle + V_\varepsilon$$

$$\partial_t \mathrm{curl}\, j_\varepsilon = \partial_t \mu_\varepsilon = \mathrm{curl}\, V_\varepsilon \quad (V_\varepsilon^\perp \text{ transports the vorticity}).$$

$$S_\varepsilon := \langle \partial_k u_\varepsilon, \partial_l u_\varepsilon\rangle - \frac{1}{2}\left(|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\delta_{kl} \quad \text{(stress-energy tensor)}$$

$$\tilde{S}_\varepsilon = \langle \partial_k u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_k, \partial_l u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}_l\rangle - \frac{1}{2}\left(|\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}|^2 + \frac{1}{2\varepsilon^2}(1-|u_\varepsilon|^2)^2\right)\delta_{kl} \quad \text{“modulated stress tensor”}$$

Sketch of proof: quantities and identities

$$\mathcal{E}_\varepsilon(u,t) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u - iuN_\varepsilon \mathrm{v}(t)|^2 + \frac{(1-|u|^2)^2)}{2\varepsilon^2} \quad \text{(modulated energy)}$$

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The Gross-Pitaevskii case

Time-derivative of the energy (if u_ε solves (GP) and v solves (IE))

$$\frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t),t))}{dt} = \int_{\mathbb{R}^2} N_\varepsilon \underbrace{(N_\varepsilon \mathrm{v} - j_\varepsilon)}_{\text{linear term}} \cdot \underbrace{\partial_t \mathrm{v}}_{2\mathrm{v}^\perp \mathrm{curl}\,\mathrm{v} + \nabla \mathrm{p}} - N_\varepsilon V_\varepsilon \cdot \mathrm{v}$$

linear term a priori controlled by $\sqrt{\mathcal{E}} \rightsquigarrow$ unsufficient

But

$$\mathrm{div}\ \tilde{S}_\varepsilon = -N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon)^\perp \mathrm{curl}\,\mathrm{v} - N_\varepsilon \mathrm{v}^\perp \mu_\varepsilon + \frac{1}{2} N_\varepsilon V_\varepsilon$$

Multiply by $2\mathrm{v}$

$$\int_{\mathbb{R}^2} 2\mathrm{v} \cdot \mathrm{div}\ \tilde{S}_\varepsilon = \int_{\mathbb{R}^2} -N_\varepsilon (N_\varepsilon \mathrm{v} - j_\varepsilon) \cdot 2\mathrm{v}^\perp \mathrm{curl}\,\mathrm{v} + N_\varepsilon V_\varepsilon \cdot \mathrm{v}$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\text{controlled by } \mathcal{E}_\varepsilon} : \underbrace{\nabla \mathrm{v}}_{\text{bounded}}$$

$\rightsquigarrow$ Gronwall OK: if $\mathcal{E}_\varepsilon(u_\varepsilon(0)) \le o(N_\varepsilon^2)$ it remains true (vortex energy is $\pi N_\varepsilon |\log \varepsilon| \ll N_\varepsilon^2$ in the regime $N_\varepsilon \gg |\log \varepsilon|$)

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The parabolic case

If u_ε solves (PGL) and v solves (L1) or (L2)

$$\frac{d\mathcal{E}_\varepsilon(u_\varepsilon(t),t))}{dt} = -\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon|^2 + \int_{\mathbb{R}^2} (N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)\cdot\partial_t \mathrm{v} - N_\varepsilon V_\varepsilon \cdot \mathrm{v})$$

$$\mathrm{div}\ \tilde{S}_\varepsilon = \frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle + N_\varepsilon(N_\varepsilon \mathrm{v} - j_\varepsilon)^\perp \mathrm{curl}\,\mathrm{v} - N_\varepsilon \mathrm{v}^\perp \mu_\varepsilon.$$

$\phi = \mathrm{p}$ if $N_\varepsilon \ll |\log\varepsilon|$ $\qquad$ $\phi = \lambda\,\mathrm{div}\ \mathrm{v}$ if not

Multiply by $\mathrm{v}^\perp$ and insert:

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2\tilde{S}_\varepsilon : \nabla \mathrm{v}^\perp - N_\varepsilon V_\varepsilon \cdot \mathrm{v} - 2N_\varepsilon|\mathrm{v}|^2\mu_\varepsilon$$

$$-\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|}|\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi|^2 + 2\mathrm{v}^\perp\cdot\frac{N_\varepsilon}{|\log\varepsilon|}\langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon\phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle.$$

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The vortex energy $\pi N_\varepsilon |\log \varepsilon|$ is no longer negligible with respect to N_ε^2.
We now need to prove

$$\frac{d\mathcal{E}_\varepsilon}{dt} \le C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|) + o(N_\varepsilon^2).$$

Need all the tools on vortex analysis:

- vortex ball construction [Sandier '98, Jerrard '99, Sandier-S '00, S-Tice '08]: allows to bound the energy of the vortices from below in disjoint vortex balls B_i by $\pi |d_i| |\log \varepsilon|$ and deduce that the energy outside of $\cup_i B_i$ is controlled by the excess energy $\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log \varepsilon|$
- “product estimate" of [Sandier-S '04] allows to control the velocity:

$$\left| \int V_\varepsilon \cdot \mathrm{v} \right| \le \frac{2}{|\log \varepsilon|} \left(\int |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 \int |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 \right)^{\frac{1}{2}}$$

$$\le \frac{1}{|\log \varepsilon|} \left(\frac{1}{2} \int |\partial_t u_\varepsilon - i u_\varepsilon N_\varepsilon \phi|^2 + 2 \int |(\nabla u_\varepsilon - i u_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 \right)$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\le C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log\varepsilon|)} : \underbrace{\nabla \mathrm{v}^\perp}_{\text{bounded}} - \underbrace{N_\varepsilon V_\varepsilon \cdot \mathrm{v}}_{\text{controlled by prod. estimate}} - 2N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$$

$$-\int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|} |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2 + \underbrace{2\mathrm{v}^\perp \cdot \frac{N_\varepsilon}{|\log\varepsilon|} \langle \partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi, \nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}\rangle}_{\text{bounded by Cauchy-Schwarz}} .$$

$$\frac{d\mathcal{E}_\varepsilon}{dt} \le C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log\varepsilon|) + \int_{\mathbb{R}^2} \frac{N_\varepsilon}{|\log\varepsilon|} \left(\frac{1}{2} + \frac{1}{2} - 1\right) |\partial_t u_\varepsilon - iu_\varepsilon N_\varepsilon \phi|^2$$

$$+ \frac{2N_\varepsilon}{|\log\varepsilon|} \int_{\mathbb{R}^2} |(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}^\perp|^2 + |(\nabla u_\varepsilon - iu_\varepsilon N_\varepsilon \mathrm{v}) \cdot \mathrm{v}|^2 - 2N_\varepsilon \int_{\mathbb{R}^2} |\mathrm{v}|^2 \mu_\varepsilon$$

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$\rightsquigarrow$ Gronwall OK

$$\frac{d\mathcal{E}_\varepsilon}{dt} = \int_{\mathbb{R}^2} 2 \underbrace{\tilde{S}_\varepsilon}_{\leq C(\mathcal{E}_\varepsilon - \pi N_\varepsilon |\log\varepsilon|)} : \underbrace{\nabla \mathrm{v}^\perp}_{\text{bounded}} - \underbrace{N_\varepsilon V_\varepsilon \cdot \mathrm{v}}_{\text{controlled by prod. estimate}} - 2N_\varepsilon |\mathrm{v}|^2 \mu_\varepsilon$$

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Joyeux anniversaire, Yann!