Martingale Optimal Transport

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Bon anniversaire Yann !

Outline

1. Formulation and Duality
2. Motivation
   - Skorohod Embedding Problem
   - Robust Hedging of Financial Derivatives
3. Quasi-Sure Formulation

Martingale Optimal Transport on the line

Let $\Omega := \mathbb{R} \times \mathbb{R}$, and introduce the canonical process

$$X(\omega) = x, \quad Y(\omega) = y \quad \text{for all } \omega = (x, y) \in \Omega.$$

Transport plans :

$$\Pi(\mu, \nu) \quad := \quad \{\mathbb{P} \in \mathrm{Prob}(\Omega) : \; \mathbb{P} \circ X^{-1} = \mu, \; \mathbb{P} \circ Y^{-1} = \nu\}$$

Martingale Transport plans : μ, ν have finite first moment,

$$\mathcal{M}(\mu, \nu) \quad := \quad \{\mathbb{P} \in \Pi(\mu, \nu) : \; \mathbb{E}^{\mathbb{P}}[Y|X] = X\}$$

i.e. $\mathbb{P}(d\omega) = \mu(dx)\mathbb{P}_x(dy)$, whose desintegration $\mathbb{P}_x$ has barycenter x

Martingale Optimal Transport problem

$$\sup_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}\big[c(X, Y)\big]$$

Martingale restriction

- $\mathbb{E}^{\mathbb{P}}[Y|X] = X$ iff $\mathbb{E}^{\mathbb{P}}[h(x)(Y - X)] = 0$ for all $h \in C_b^0$

$\Longrightarrow$ h will act as Lagrange multipliers... Denote

$$h^{\otimes}(x, y) := h(x)(y - x), \quad x, y \in \mathbb{R}$$

[complementing the standard notations $\varphi \oplus \psi$]

- Strassen '65 : $\mathcal{M}(\mu, \nu) \neq \emptyset$ iff $\mu \preceq \nu$ in convex order :

$$\mu[f] \leq \nu[f] \quad \text{for all} \quad f : \mathbb{R} \longrightarrow \mathbb{R} \textit{ convex}$$

- $\mathcal{M}(\mu, \nu)$ closed convex subset of $\Pi(\mu, \nu)$...

Kantorovitch dual formulation

Martingale Optimal Transport : $c : \Omega \longrightarrow \mathbb{R}$ measurable

$$\mathsf{P}(\mu,\nu) := \sup_{\mathbb{P}\in\mathcal{M}(\mu,\nu)} \mathbb{E}^{\mathbb{P}}[c], \quad \mathcal{M}(\mu,\nu) := \left\{\mathbb{P}\in\Pi(\mu,\nu) : \ \mathbb{E}^{\mathbb{P}}[Y|X] = X\right\}$$

Pointwise Dual Problem :

$$\mathsf{D}(\mu,\nu) := \sup_{(\varphi,\psi,h)\in\mathcal{D}(c)} \mu[\varphi] + \nu[\psi]$$

where

$$\mathcal{D}(c) \ := \ \left\{(\varphi,\psi,h) : \ \varphi\oplus\psi + h^{\otimes} \geq c \text{ on } \Omega\right\}$$

Continuous-time Transport Plans

Let $\Omega := C^0([0,T],\mathbb{R})$ or $\Omega := \mathrm{RCLL}([0,T],\mathbb{R})$, with canonical process and filtration

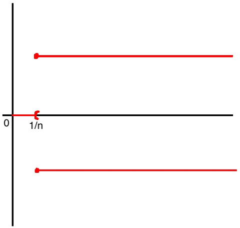
$$X_t(\omega) = \omega(t), \quad \mathcal{F}_t := \sigma(X_s, s \le t) \quad \text{for all} \quad 0 \le t \le T$$

Transport plans :

$$\Pi(\mu,\nu) \quad := \quad \{\mathbb{P} \in \mathrm{Prob}(\Omega) : \ \mathbb{P} \circ X_0^{-1} = \mu, \ \mathbb{P} \circ X_T^{-1} = \nu\}$$

A first difficulty : $\Pi(\mu,\nu)$ is not weakly compact

$\mu := \delta_0$
$\nu := \frac{1}{2}(\delta_{-1} + \delta_1)$



Continuous-time Martingale Transport

Martingale Transport plans : μ, ν have finite first moment,

$$\mathcal{M}(\mu,\nu) \quad := \quad \{\mathbb{P} \in \Pi(\mu,\nu) : \; X \text{ is } \mathbb{P}-\text{martingale}\}$$

i.e. $\mathbb{E}^{\mathbb{P}}[X_t|\mathcal{F}_s] = X_s$ for all $0 \le s \le t \le T$, or "equivalently" :

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T h_t dX_t\Big] = 0 \quad \text{for} \quad \mathbb{F}-\text{meas. bdd } h : [0,T] \times \Omega \longrightarrow \mathbb{R}$$

Martingale Optimal Transport : $c : (\Omega, \mathcal{F}_T) \longrightarrow \mathbb{R}$ measurable

$$\mathrm{P}(\mu,\nu) \quad := \quad \sup_{\mathbb{P} \in \mathcal{M}(\mu,\nu)} \quad \mathbb{E}^{\mathbb{P}}[c(X_\cdot)]$$

Continuous-time Martingale Optimal Transport

Martingale Optimal Transport : $c : (\Omega, \mathcal{F}_T) \longrightarrow \mathbb{R}$ measurable

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Dual Problem :

$$\mathbf{D}(\mu,\nu) := \sup_{(\varphi,\psi,h)\in\mathcal{D}(c)} \mu[\varphi]+\nu[\psi]$$

where

$$\mathcal{D}(c) \; := \; \Big\{(\varphi,\psi,h) : \; \varphi(X_0)+\psi(X_T)+\underbrace{\int_0^T h_t dX_t}_{!!!} \geq c \text{ on } \Omega\Big\}$$

Extensions

- Martingale optimal transport in $\mathbb{R}^d$
- Multiple marginals
- Full marginals,

e.g. fake Brownian motion : $\mu_t = \mathcal{N}(0, t)$ for all $t \geq 0$

[Hamza, Klebaner... Yor and co-authors]

Some References

- Introduced by Pierre Henry-Labordère,

Discrete-time : Beiglböck, Davis, De March, Ghoussoub, Griessler, Henry-Labordère, Hobson, Kim, Klimmek, Lim, Neuberger, Nutz, Penkner, Juillet, Schachermayer, NT

Continuous-time : Beiglböck, Bayraktar, Claisse, Cox, Davis, Dolinsky, Galichon, Guo, Hu, Henry-Labordère, Hobson, Huesmann, Perkowski, Proemel, Kallblad, Klimmek, Obloj, Siorpaes, Soner, Spoida, Stebegg, Tan, NT, Zaev

Outline

Formulation of the SEP

$(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ filtered probability space, B Brownian motion

SEP(ν) : Find a stopping time τ such that

$$\mathbb{P} \circ (B_\tau)^{-1} = \nu \quad \text{and} \quad B_{.\wedge\tau} \quad \text{UI}$$

SEP(μ, ν) : Find a stopping time τ such that

$$\mathbb{P} \circ (B_0)^{-1} = \mu, \quad \mathbb{P} \circ (B_\tau)^{-1} = \nu \quad \text{and} \quad B_{.\wedge\tau} \quad \text{UI}$$

Possibly under weak formulation...

Motivation of the SEP

Weak law of large numbers $\Longrightarrow$ Central Limit Theorem

$X_i \sim iid\mu$ centered measure

$X_i = B^i_{\tau_i}$, with $\tau_i \sim$ iid τ, and B^i iid BM. Then

$$\frac{1}{\sqrt{n}}\sum_{i=1}^n X_i = B_{T_n}, \quad \text{where} \quad T_n := \frac{1}{n}\sum_{i=1}^n \tau_i \xrightarrow{\mathbb{P}} \mathbb{E}[\tau] = \mathbb{E}[X_i^2]$$

Some solutions of the SEP

- Doob
- Root
- Azéma-Yor
- Vallois

Many many more

Some solutions of the SEP

- Doob
- Root $\Longrightarrow \min_\tau \mathbb{E}[\tau]$ (Rost)
- Azéma-Yor $\Longrightarrow \max_\tau \mathbb{E}[B^*_\tau]$
- Vallois $\Longrightarrow \max_\tau \mathbb{E}[L_\tau]$

Many many more

Connection with Martingale Transport

The process $\left\{X_t := X_0 + B_{\frac{t}{T-t}\wedge\tau}, t \in [0,T]\right\}$ martingale with $X_T = B_\tau \sim \nu$

Conversely, every martingale is a time-changed Brownian motion

Martingale Transport $\Longrightarrow$ find a solution τ of the SEP for each given optimality criterion...

Even if not explicit, open to numerical techniques

Beiglböck, Cox & Huesmann [Invent. Math.]

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From options prices to marginals

• X : price of a risky asset at time 1, interest rate $= 0$

• Prices of options defined by payoff $(X-K)^+$, and $(K-X)^+$ are available for all strikes K

$$C(K) \text{ and } P(K)$$

Assume Evaluation function $\pi : \mathbb{L}^0 \longrightarrow \mathbb{R}$ linear and "continuous"

By no arbitrage considerations,

- $K \longmapsto \pi((X-K)^+) = C(K)$ non-decreasing, convex, $C'(\infty) = 0$, and $C'(-\infty) = 1$

$$\mu := C'' \text{ is a probability measure}$$

- $C(K) - P(K) = C(0) - K$

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- $C(K) - P(K) = C(0) - K$

Vanilla options prices induced by call prices

- For $\varphi \in C^2$, we have

$$\varphi(X) = \varphi(x_0) + (X - x_0)\varphi'(x_0) + \int_{x_0}^{\infty} (X-K)^+ \varphi''(K)dK + \int_{-\infty}^{x_0} (K-x)^+ \varphi''(K)dK$$

Then, by linearity and continuity :

$$\pi\big(\varphi(X)\big) = \varphi(x_0) + \big(\pi(X) - x_0\big)\varphi'(x_0) + \int_{x_0}^{\infty} \pi\big((X-K)^+\big) \varphi''(K)dK + \int_{-\infty}^{x_0} \pi\big((K-x)^+\big) \varphi''(K)dK$$

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$\Longrightarrow$ after two integrations by parts

$$\pi(\varphi(X)) = \int \varphi(K)C''(dK) = \int \varphi(K)\mu(dK) = \mu(\varphi)$$

- By density, $\pi(\varphi(X)) = \mu(\varphi)$ for all φ, up to measurability...

Hedgin instruments

X and Y prices of an asset at times 1 and 2, Interest rate $= 0$

Prices C_1, C_2 of options of maturities $1, 2$ available for all strikes

Then, prices of Vanilla options with maturities 1 and 2

$$\pi\big(\varphi(X)\big) = \mu(\varphi), \quad \pi\big(\psi(Y)\big) = \nu(\psi), \quad \mu := C_1'' \text{ and } \nu := C_2''$$

In addition, dynamic trading for zero cost

$$\underbrace{h_0(X - X_0)}_{\rightsquigarrow\varphi(X)} + h_1(X)(Y - X) \quad \Longrightarrow \quad h_1(X)(Y - X) =: h_1^{\otimes}(X, Y)$$

Robust / Model-Free Superhedging Problem

- Exotic option defined by the payoff $c(X, Y)$ at time 2 :

$$c : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

Robust super hedging problem naturally formulated as :

$$\mathbf{D}(\mu, \nu) \quad := \quad \inf_{(\varphi,\psi,h)\in\mathcal{D}} \{\mu(\varphi) + \nu(\psi)\}$$

where

$$\mathcal{D} := \{(\varphi, \psi, h) \in \mathbb{L}^1(\mu) \times \mathbb{L}^1(\nu) \times \mathbb{L}^0 : \ \varphi \oplus \psi + h^{\otimes} \geq c\}$$

$\equiv$ Kantorovitch dual

Outline

Duality for USC claim

Theorem (Beiglböck, Henry-Labordère, Penkner)

Assume $c \in$ USC and bounded from above. Then $\mathbf{P} = \mathbf{D}$, *and existence holds for* $\mathbf{P}(\mu, \nu)$ *for all* $\mu \preceq \nu$

• There are easy examples where existence for the dual fails, even for bounded c, bounded support... (Beiglböck, Henry-Labordère & Penkner, Beiglböck, Nutz & NT)

• The condition $c \in$ USC is not innocent, e.g. duality fails for the LSC function $c(x, y) := \mathbb{1}_{\{x \neq y\}}$ on $[0, 1] \times [0, 1]$

Quasi-sure robust superhedging

Definition

$\mathcal{M}(\mu,\nu)-$q.s. (quasi surely) means $\mathbb{P}-$a.s. for all $\mathbb{P}\in\mathcal{M}(\mu,\nu)$

- The quasi-sure robust superhedging cost

$$\mathbf{D}^{qs} := \inf_{(\varphi,\psi,h)\in\mathcal{D}^{qs}} \{\mu(\varphi)+\nu(\psi)\}$$

$$\mathcal{D}^{qs} := \{(\varphi,\psi,h) :\in \hat{L}(\mu,\nu)\times\mathbb{L}^0,\ \varphi\oplus\psi+h^{\otimes}\geq c,\ \mathcal{M}(\mu,\nu)-\text{q.s.}\}$$

is more natural... $(\hat{L}(\mu,\nu)\supset\mathbb{L}^1(\mu)\times\mathbb{L}^1(\nu))$

- Then, $\mathbf{D}(\mu,\nu)\geq\mathbf{D}^{qs}(\mu,\nu)\ \geq\ \mathbf{P}(\mu,\nu)$

so if the duality $\mathbf{P}=\mathbf{D}$ holds, it follows that $\mathbf{D}=\mathbf{D}^{qs}$

Structure of polar sets in (standard) optimal transport

$\mathcal{N}_\mu := \{\nu - \text{null sets}\}$, $\mathcal{N}_\nu$...

Theorem (Kellerer)

For $N \subset \mathbb{R} \times \mathbb{R}$, TFAE :

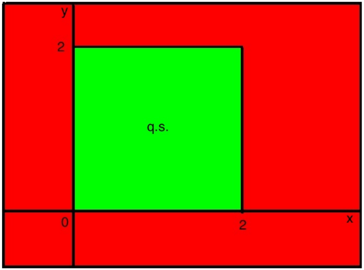
- $\mathbb{P}[N] = 0$ *for all* $\mathbb{P} \in \Pi(\mu, \nu)$
- $N \subset (N_\mu \times \mathbb{R}) \cup (\mathbb{R} \times N_\nu)$ *for some* $N_\mu \in \mathcal{N}_\mu$, $N_\nu \in \mathcal{N}_\nu$

$\Longrightarrow$ no difference between the pointwise and the quasi-sure formulations in standard optimal transport

Pointwise versus Quasi-sure superhedging I

Suppose $\mathrm{Supp}(\mu) = [0,2]$ $=\mathrm{Supp}(\nu) = [0,2]$, then

- $\mathcal{M}(\mu,\nu)-$q.s. only involves the values $(x,y) \in [0,2]^2$
- Pointwise superhedging involves all values $(x,y) \in \mathbb{R}^2$

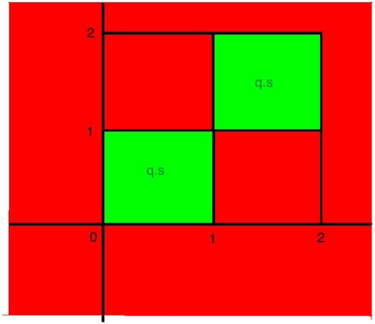


Pointwise versus Quasi-sure superhedging II

Suppose $\text{Supp}(\mu) = \text{Supp}(\nu) = [0,2]$, and $C_\mu(1) = C_\nu(1)$

$$\mathbb{E}\big[(X-1)^+\big] \;=\; \mathbb{E}\big[(Y-1)^+\big] \;\le\; \mathbb{E}\big[(X-1)^+\big]$$

by Jensen's inequality, and then $\{X \ge 1\} = \{Y \ge 1\}$



Structure of polar sets in martingale optimal transport

Potential functions $U^\mu(x) := \int |\xi - x|\mu(d\xi)$, $U^\nu(x) := ...$, then :

$\{U^\mu < U^\nu\} = \cup_{k\geq 0} I_k$, $\quad I_k = (a_k, b_k)$, $\quad J_k := I_k \cup \{\nu - \text{atoms} \in \bar{I}_k\}$

Theorem (Beiglböck, Nutz & NT '15)

For $N \subset \mathbb{R} \times \mathbb{R}$, TFAE :

- $\mathbb{P}[N] = 0$ *for all* $\mathbb{P} \in \mathcal{M}(\mu, \nu)$
- $N \subset (N_\mu \times \mathbb{R}) \cup (\mathbb{R} \times N_\nu) \cup \Delta^c$ *for some* $N_\mu \in \mathcal{N}_\mu$, $N_\nu \in \mathcal{N}_\nu$

$$\Delta \quad := \quad \{(x,x)\} \cup \left[\cup_k \left(I_k \times J_k\right)\right]$$

Integrability

Definition

$f:\mathbb{R}\longrightarrow\mathbb{R}$ **convex** is called a $(\mu,\nu)-$moderator for (φ,ψ) if

$$\varphi - f \in \mathbb{L}^1(\mu), \quad \psi + f \in \mathbb{L}^1(\nu), \quad \text{and} \quad (\nu-\mu)(f) < \infty$$

We denote

$$\hat{L}(\mu,\nu) \quad := \quad \big\{(\varphi,\psi) \text{ admitting some convex moderator}\big\}$$

For $(\varphi,\psi)\in\hat{L}(\mu,\nu)$, the value

$$\mu(\varphi)+\nu(\psi) := \mu(\varphi - f)+\nu(\psi - f)+(\nu-\mu)(f)$$

is independent of the choice of the moderator f

Duality and existence under quasi-sure robust superhedging

Theorem (Beiglböck, Nutz & NT '15)

Let $\mu \preceq \nu$ *and* $c \geq 0$ *measurable. Then*

$$\mathbf{P}(\mu,\nu) \;=\; \mathbf{D}^{qs}(\mu,\nu)$$

and existence holds for $\mathbf{D}^{qs}$*, whenever finite*

Extensions

• Geometry of martingale transport plans on the line : started from Beiglbock & Juillet '15

• Extension to $\mathbb{R}^n$:

- Lim '16 : 1-dim marginals constraints $(\mu_i, \nu_i)_{1 \leq i \leq n}$
- Ghoussoub, Kim & Lim '16, and De March '17...

• Complete duality for continuous-time martingale transport ? ?