Some results about entropic transport

Christian Léonard

Université Paris Ouest

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Co-authors

- Ivan Gentil
- Luigia Ripani (PhD. student)
- Johannes Zimmer

Schrödinger problem

- state space: $\mathbb{R}^n$
- path space: $\Omega = C([0,T], \mathbb{R}^n), \quad \Omega = C([0,\infty), \mathbb{R}^n)$

Heat bath

- evolution: $R \in \mathrm{P}(\Omega), \qquad L = (-\nabla V \cdot \nabla + \Delta)/2$
- equilibrium: $m = e^{-V}\,\mathrm{vol}$

- $Z = (Z_t)_{t \geq 0}$: $\quad Z_t = Z_0 - \int_0^t \nabla V(Z_s)/2\, ds + W_t,$
  W: Brownian motion
- R^{μ_0}: $\quad Z_0 \sim \mu_0 \in \mathrm{P}(\mathbb{R}^n), \quad Z \sim R^{\mu_0} \in \mathrm{P}(\Omega)$
- $R^m = R$ is reversible

Heat flow

- $R_t^{\mu_0} = \mu_0 e^{tL} \in \mathrm{P}(\mathbb{R}^n), \quad t \geq 0$

Notation: $P \in \mathrm{P}(\Omega), \quad P_t \in \mathrm{P}(\mathbb{R}^n)$: law of the position at time t

Schrödinger problem

- N particles travel in the heat bath
- no interaction
- $N \to \infty$

Particles

- $(Z^1, \ldots, Z^N) \sim (R^{\mu_0})^{\otimes N}$
- $\widehat{Z}^N := N^{-1} \sum_i \delta_{Z^i}$, random values in $\mathrm{P}(\Omega)$

- law of large numbers: $\lim_{N\to\infty} \widehat{Z}^N = R^{\mu_0}$, a.s.
  $\lim_{N\to\infty} \widehat{Z}^N_t = R^{\mu_0}_t = \mu_0 e^{tL}$, a.s.

Schrödinger problem

Schrödinger's question

Suppose that at time $t = T$, you observe $\widehat{Z}_T^N(\mathrm{obs}) \simeq \mu_T$, with μ_T far from the expected profile $\mu_0 e^{TL}$.
What is the most likely behavior of $(\widehat{Z}_t^N)_{0 \le t \le T}$?

Schrödinger's answer

1. Solve: $H(P|R^{\mu_0}) \to \min, \qquad P \in \mathrm{P}(\Omega): \ P_T = \mu_T$
2. $\mu_t = P_t$, $0 \le t \le T$.

- relative entropy: $H(\mathrm{p}|\mathrm{r}) := \int \log(d\mathrm{p}/d\mathrm{r})\, d\mathrm{p} \in [0, \infty]$
- $H(P|R^{\mu_0}) = H(P|R) - H(\mu_0|R_0)$

Schrödinger's problem

$$H(P|R) \to \min, \qquad P \in \mathrm{P}(\Omega): \ P_0 = \mu_0, \ P_T = \mu_T \qquad \text{(S)}$$

Schrödinger problem

Idea of proof

- Sanov's thm:
  $$\mathbb{P}(\widehat{Z}^N \in \mathcal{G}) \underset{N\to\infty}{\asymp} \exp\big(- N \inf_{P\in\mathcal{G}} H(P|R)\big), \quad \mathcal{G} \subset \mathrm{P}(\Omega)$$
- $\mathcal{G} \subset \mathcal{C} := \{P \in \mathrm{P}(\Omega) : \ P_0 = \mu_0, \ P_1 = \mu_1\}$
- $\mathbb{P}(\widehat{Z}^N \in \mathcal{G} \mid \widehat{Z}^N \in \mathcal{C}) \underset{N\to\infty}{\asymp} \exp\Big(- N \left[\inf_{P\in\mathcal{G}} H(P|R) - H(\mathcal{C}|R)\right]\Big)$

Entropic interpolation

$$[\mu_0, \mu_T]^R = (\mu_t)_{0\le t\le T}, \qquad \mu_t := P_t,\ 0 \le t \le T, \quad P = \mathrm{sol(S)}$$

Concentration of measure

- $R^x(d\omega) := R(d\omega \mid X_0 = x), \qquad d\omega \subset \Omega, \quad x \in \mathbb{R}^n$

Hypothesis: $\nabla^2 V \geq K\,\mathrm{Id}, \quad K \in \mathbb{R}$

Concentration of R^x_T

$\forall A \subset \mathbb{R}^n$: $R^x_T(A) \geq 1/2$, $\forall r \geq 0$,

$$R^x_T(A^r) \geq 1 - \exp\left(-\frac{[r - \mathrm{r}(K,T)]_+^2}{2\mathrm{t}(K,T)}\right)$$

- $A^r := A + B(0, r)$
- $\mathrm{t}(K,T) = (1 - e^{-KT})/K, \qquad \mathrm{r}(K,T) = \sqrt{2 \log 2\ \mathrm{t}(K,T)}$

Concentration of m $\quad (T \to \infty,\ K > 0)$

$\forall A \subset \mathbb{R}^n$: $m(A) \geq 1/2$, $\forall r \geq 0$,

$$m(A^r) \geq 1 - \exp\left(-K[r - \mathrm{r}_K]_+^2/2\right)$$

Concentration of measure

$$H(P|R) \to \min; \qquad P \in \mathrm{P}(\Omega): \; P_0 = \delta_x, \; P_T(A) = 1$$

Result

- solution: $P^A = \rho_A(X_T)\, R^x$ with: $\rho_A := \frac{\mathbf{1}_A}{R^x_T(A)}$
- equivalent to (S) with: $\mu_0 = \delta_x, \quad \mu_T = \rho_A\, R^x_T = R^x_T(\cdot|A)$
- $H(P^A|R^x) = H(P^A_T|R^x_T) = -\log R^x_T(A)$

- P^A is a Doob h-transform of R^x

Concentration of measure

Girsanov theory

- R^x : $dZ_t = b(Z_t)\,dt + dW_t$, $b := -\nabla V/2$
- P^A : $\to u^A$, $dY_t^A = [b + u_t^A](Y_t^A)\,dt + dW_t$
- $H(P^A|R^x) = \mathbb{E}\int_0^T |u_t^A|^2/2\ dt$

- $B := \mathbb{R}^n \setminus A^r$
- P^B : $\to u^B$, $dY_t^B = [b + u_t^B](Y_t^B)\,dt + dW_t$
- coupling: same W
- $-\log R_T^x(\mathbb{R}^n \setminus A^r) = H(P^B|R^x) = \mathbb{E}\int_0^T |u_t^B|^2/2\ dt \geq\ ?$

The higher the cost $\mathbb{E}\int_0^T |u_t^B|^2/2\ dt$, the stronger the concentration

Concentration of measure

- $-\log R_T^x(\mathbb{R}^n \setminus A^r) = \mathbb{E}\int_0^T |u_t^B|^2/2\ dt \geq ?$

End of the proof

- $|u^B|^2/2 \geq |u^B - u^A|^2/2(1+\alpha) - |u^A|^2/2\alpha$
- $\mathbb{E}\int_0^T |u_t^A|^2/2\ dt = \log(1/R_T^x(A)) \leq \log 2$
- $|Y_T^B - Y_T^A| \geq r$ implies: $\int_0^T |u_t^B - u_t^A|^2\, dt \geq r^2/\mathrm{t}(K,T)$

Lemma

- hypothesis: $\nabla^2 V \geq K\mathrm{Id}, \quad K \in \mathbb{R}$
- $$\infty > \int_0^T |u_t^B - u_t^A|^2\, dt \geq |Y_T^B - Y_T^A|^2/\mathrm{t}(K,T)$$
  with $\mathrm{t}(K,T) := (1-e^{-KT})/K$

- proof. $d(Y^B - Y^A)_t = [b(Y_t^B) - b(Y_t^A)]\, dt + [u_t^B - u_t^A]\, dt$
  $Y_0^B - Y_0^A = 0.$

Brunn-Minkowski inequality

- Ref.: Borell, Lehec, Cordero-Maurey
- $\gamma_T \in \mathrm{P}(\mathbb{R}^n)$: Gaussian$(0, T\mathrm{Id})$
- $A, B \subset \mathbb{R}^n, \quad C := (1-\lambda)A + \lambda B, \quad 0 < \lambda < 1$

Brunn-Minkowski inequality

$$\gamma_T(C) \ge \gamma_T(A)^{1-\lambda}\gamma_T(B)^{\lambda}, \qquad \mathrm{vol}(C) \ge \mathrm{vol}(A)^{1-\lambda}\mathrm{vol}(B)^{\lambda}$$

- idea of proof.
  - $-\log\gamma(C) \le -(1-\lambda)\log\gamma(A) - \lambda\log\gamma(B)$
  - $-\log\gamma(A) = H(P^A|R^{x=0})$
  - $Y^C := (1-\lambda)Y^A + \lambda Y^B,$ (coupling)
  - $\mathrm{u}^C = (1-\lambda)\mathrm{u}^A + \lambda\mathrm{u}^B$
  - $|(1-\lambda)\mathrm{u}^A + \lambda\mathrm{u}^B|^2 \le (1-\lambda)|\mathrm{u}^A|^2 + \lambda|\mathrm{u}^B|^2$
  - $T \to \infty$

Prekopa-Leindler inequality

- $0 < \lambda < 1$
- $\theta\big((1-\lambda)x + \lambda x'\big) \le (1-\lambda)\psi(x) + \lambda\psi'(x'), \quad \forall x, x'$

Prekopa-Leindler inequality

- $\int_{\mathcal{X}} e^{-\theta}\, d\gamma_T \ge \left(\int_{\mathcal{X}} e^{-\psi}\, d\gamma_T\right)^{1-\lambda} \left(\int_{\mathcal{X}} e^{-\psi'}\, d\gamma_T\right)^{\lambda}$
- $\int_{\mathcal{X}} e^{-\theta}\, d\mathrm{vol} \ge \left(\int_{\mathcal{X}} e^{-\psi}\, d\mathrm{vol}\right)^{1-\lambda} \left(\int_{\mathcal{X}} e^{-\psi'}\, d\mathrm{vol}\right)^{\lambda}$

- idea of proof
  - $e^{-\psi} = \mathbf{1}_A, \quad \psi = \iota_A$
  - $\log \int_{\mathcal{Z}} e^f\, d\mathrm{r} = \sup\left\{\int_{\mathcal{Z}} f\, d\mathrm{p} - H(\mathrm{p}|\mathrm{r});\ \mathrm{p} \in \mathrm{P}(\mathcal{Z}) : \int_{\mathbb{R}^n} f_+\, d\mathrm{p} < \infty\right\}$
    - ⋆ $f = -\iota_A - \log \mathrm{r}(A)$ gives $\inf_{\mathrm{p}\in \mathrm{P}(A)} H(\mathrm{p}|\mathrm{r}) = H(\mathrm{r}(\cdot|A)|\mathrm{r})$
  - $|(1-\lambda)\mathrm{u}^{\psi} + \lambda \mathrm{u}^{\psi'}|^2 \le (1-\lambda)|\mathrm{u}^{\psi}|^2 + \lambda|\mathrm{u}^{\psi'}|^2$
- works also for m with $K > 0$

Monge-Kantorovich problem (the Schrödinger way)

- N non-interacting particles travel in the void (no heat bath)
- $N \to \infty$
- $d\langle Y^i \rangle_t := (dY^i_t)^2 = 0, \quad \forall 1 \leq i \leq N, \qquad$ (zero temperature)

Hamilton principle

$N^{-1} \sum_i \int_{[0,1]} |\dot{Y}^i_t|^2/2 \, dt \to \min; \qquad \widehat{Y}^N : \quad \widehat{Y}^N_0 \simeq \mu_0, \ \widehat{Y}^N_1 \simeq \mu_1$

"Monge-Kantorovich question"

Find $\lim_{N\to\infty} \widehat{Y}^N_t \in \mathrm{P}(\mathbb{R}^n), \quad 0 \leq t \leq 1$

Brenier + McCann's answer

$E_P \int_{[0,1]} |\dot{X}_t|^2 \, dt \to \min; \qquad P \in \mathrm{P}(\Omega) : \quad P_0 = \mu_0, \ P_1 = \mu_1 \quad$ (MK)

- $[\mu_0, \mu_1] = (P_t)_{0 \leq t \leq 1} : \qquad$ displacement interpolation

Monge-Kantorovich problem

- where is the optimal transport?
- $E_P \int_{[0,1]} |\dot{X}_t|^2\, dt \to \min; \qquad P \in \mathrm{P}(\Omega): \quad P_0 = \mu_0,\ P_1 = \mu_1$
- $\gamma_t^{xy} := (1-t)x + ty,\ 0 \le t \le 1: \quad$ geodesic

Results

- $P^{xy} = \delta_{\gamma^{xy}}, \quad \forall (x,y) \in \mathbb{R}^n,\ P_{01}\text{-a.e.}$
- (MK) is equivalent to its static version:
  $\int_{\mathbb{R}^n \times \mathbb{R}^n} |y-x|^2\, \pi(dxdy) \to \min; \quad \pi \in \mathrm{P}(\mathbb{R}^n \times \mathbb{R}^n): \pi_0 = \mu_0, \pi_1 = \mu_1$
  $$P(\cdot) = \int_{\mathbb{R}^n \times \mathbb{R}^n} \delta_{\gamma^{xy}}(\cdot)\, \pi(dxdy)$$

- idea of proof
  - $E_P \int_{[0,1]} |\dot{X}_t|^2\, dt = \int_{\mathbb{R}^n \times \mathbb{R}^n} [E_{P^{xy}} \int_{[0,1]} |\dot{X}_t|^2\, dt]\, P_{01}(dxdy)$
  - $[E_{P^{xy}} \int_{[0,1]} |\dot{X}_t|^2\, dt \ge \int_{[0,1]} |\dot{\gamma}_t^{xy}|^2\, dt = |y-x|^2$

- inf(MK)$= W_2^2(\mu_0, \mu_1)$

Monge-Kantorovich problem

- $\ddot{X}_t = 0, \quad \forall 0 < t < 1, \quad P\text{-a.s.}$
- $\dot{X}_t = X_1 - X_0 = (X_t - X_0)/t, \quad \forall 0 < t < 1, \quad P\text{-a.s.}$

Question

Is P Markov?

- $dX_t = v_t(X_t)\, dt, \quad 0 \le t \le 1, \quad P\text{-a.s. ?}$
- no shocks

Answer (Brenier + McCann)

- yes, P is Markov because
- $dX_t = \nabla\Psi_t(X_t)\, dt, \quad 0 \le t \le 1, \quad P\text{-a.s.}$ where
- $\begin{cases} \partial_t \Psi + |\nabla\Psi|^2/2 = 0, & 0 \le t < 1, \\ \Psi(1, \cdot) = \Psi_1, & t = 1, \end{cases}$

Benamou-Brenier formula

$$E_Q \int_{[0,1]} |\dot{X}_t|^2\, dt \to \min; \qquad Q \in \mathrm{P}(\Omega): \quad Q_0 = \mu_0,\ Q_1 = \mu_1 \quad \text{(MK)}$$

- $\inf(\text{MK}) = W_2^2(\mu_0, \mu_1)$
- $\dot{X}_t = v_t(X_t), \quad P\text{-a.s.}, \qquad P := \text{sol(MK)}$
- if: $\dot{X}_t = u_t(X_t), \quad Q\text{-a.s.},$ then:
  $\partial_t Q_t + \nabla \cdot (Q_t u_t) = 0$
  $E_Q \int_{[0,1]} |\dot{X}_t|^2\, dt = E_Q \int_{[0,1]} |u_t(X_t)|^2\, dt = \int_{[0,1]\times\mathbb{R}^n} |u_t(z)|^2\, dt Q_t(dz)$

Benamou-Brenier formula

$$W_2^2(\mu_0, \mu_1) = \inf_{(\nu,u)} \int_{[0,1]\times\mathbb{R}^n} |u_t(z)|^2\, dt \nu_t(dz)$$

- $(\nu, u): \quad \partial_t + \nabla \cdot (\nu u) = 0, \qquad \nu_0 = \mu_0,\ \nu_1 = \mu_1$

Γ-convergence

Slowing down

- $0 < \tau \leq 1, \qquad \tau \to 0$
- $Z^\tau_t := Z_{\tau t},\ 0 \leq t \leq 1$

- R: $\qquad dZ_t = -\nabla V(Z_t)/2\ dt + dW_t, \qquad\qquad Z_0 \sim m$
- R^τ : $\quad dZ^\tau_t = -\tau \nabla V(Z^\tau_t)/2\ dt + \sqrt{\tau} dW_t, \quad Z^\tau_0 \sim m$
- $L^{R^\tau} = \tau L = \tau(-\nabla V \cdot \nabla + \Delta)/2$

Schrödinger problem

$$\tau H(P|R^\tau) \to \min; \quad P \in \mathrm{P}(\Omega) :\ P_0 = \mu_0,\ P_1 = \mu_1 \qquad (\mathsf{S}^\tau)$$

Γ-convergence

- $\mathrm{sol}(\mathrm{S}^\tau) =: P^\tau, \qquad \mathrm{sol}(\mathrm{MK}) =: P$

Result (Mikami, L.)

$\Gamma\text{-}\lim_{\tau\to 0}(\mathrm{S}^\tau) = (\mathrm{MK})/2$

- $\lim_{\tau\to 0^+} \inf(\mathrm{S}^\tau) = \inf(\mathrm{MK})/2 = W_2^2(\mu_0, \mu_1)/2$
- $\lim_{\tau\to 0^+} P^\tau := P \in \mathrm{sol}(\mathrm{MK}), \qquad$ (subsequence)
- hints
    - $\tau b(X_t)\, dt + \sqrt{\tau} dW_t \underset{\tau\to 0}{\simeq} \sqrt{\tau} dW_t, \quad$ take $V = 0$
    - $\lim_{\tau\to 0^+} R^{\tau,xy} = \delta_{\gamma^{xy}}$
    - $R^\tau_{01}(dxdy) \propto dx\ \exp(-\tau^{-1}|y-x|^2/2)\, dy$
    - $\tau H(\pi|R^\tau_{01}) = \tau \int_{\mathbb{R}^n\times\mathbb{R}^n} \log(d\pi/dR^\tau_{01})\, d\pi$
      $\underset{\tau\to 0}{\longrightarrow} \int_{\mathbb{R}^n\times\mathbb{R}^n} |y-x|^2/2\ \pi(dxdy)$

Kinematics

Results

- $P^\tau = f_0^\tau(X_0) g_1^\tau(X_1)\ R^\tau$ for some $f_0^\tau, g_1^\tau : \mathbb{R}^n \to [0,\infty)$.
- $\overrightarrow{L}_t^{P^\tau} = L^{R^\tau} + \tau \Gamma(g_t^\tau, \cdot)/g_t^\tau$ (h-transform)

- (f_0^τ, g_1^τ) depends on (μ_0, μ_1)
- Γ : carré du champ
- $g_t^\tau(z) := E_{R^\tau}[g_1^\tau(X_1) \mid X_t = z]$, (solves a backward heat equation)

Kinematics

Kinematics of P^τ

- $\overrightarrow{L}^{P^\tau} := \partial_t + \overrightarrow{v}^\tau \cdot \nabla + \tau\Delta/2$
- $\overrightarrow{v}^\tau_t = -\tau\nabla V/2 + \nabla\Psi^\tau_t, \qquad \Psi^\tau_t := \tau \log g^\tau_t$

- $\begin{cases} \partial_t\Psi^\tau + |\nabla\Psi^\tau|^2/2 + \tau(-\nabla V \cdot \nabla + \Delta)\Psi^\tau/2 = 0, & 0 \le t < 1, \\ \Psi^\tau(1,\cdot) = \tau \log g^\tau_1, & t = 1. \end{cases}$
- $\tau \to 0^+$, viscosity solution (of course, but also. . .)
- $\tau \to 0^+$, Laplace-Varadhan principle + Hopf-Lax formula

Kinematics of P (Brenier's theorem)

- $L^P := \partial_t + v^* \cdot \nabla$
- $v^*_t = \nabla\Psi_t$

- $\begin{cases} \partial_t\Psi + |\nabla\Psi|^2/2 = 0, & 0 \le t < 1, \\ \Psi(1,\cdot) = \Psi_1, & t = 1. \end{cases}$

Stochastic velocities

- Aim: entropic actions
- P: $\overrightarrow{L}^P = \overrightarrow{v}^P \cdot \nabla + \tau\Delta/2$

Velocities

- forward: $\overrightarrow{v}_t^P(z) := \lim_{h\to 0^+} E_P\left(\frac{X_{t+h}-X_t}{h} \mid X_t = z\right)$
- backward: $\overleftarrow{v}_t^P(z) := \lim_{h\to 0^+} E_P\left(\frac{X_t-X_{t-h}}{h} \mid X_t = z\right)$

- $\begin{cases} v^{\mathrm{cu},P} := (\overrightarrow{v}^P + \overleftarrow{v}^P)/2, \\ v^{\mathrm{os},P} := (\overrightarrow{v}^P - \overleftarrow{v}^P)/2. \end{cases}$ $\quad \begin{cases} \overrightarrow{v}^P = v^{\mathrm{cu},P} + v^{\mathrm{os},P}, \\ \overleftarrow{v}^P = v^{\mathrm{cu},P} - v^{\mathrm{os},P}, \end{cases}$

Results

- $\partial_t\mu + \nabla\cdot(\mu v^{\mathrm{cu},P}) = 0, \qquad \mu_t := P_t, \quad \mu_t := dP_t/d\mathrm{vol}$
- $v^{\mathrm{os},P} = \tau\nabla \log\sqrt{\mu},$ (time reversal)

Entropic actions

- $\overrightarrow{L}^P = \overrightarrow{v}^P \cdot \nabla + \tau\Delta/2$
- $\overrightarrow{\beta}^P := \tau^{-1}(\overrightarrow{v}^P - \overrightarrow{v}^{R^\tau}), \quad \overleftarrow{\beta}^P := \tau^{-1}(\overleftarrow{v}^P - \overleftarrow{v}^{R^\tau})$

Results

If $H(P|R^\tau) < \infty$, then:

- $H(P|R^\tau) = H(P_0|m) + \tau E_P \int_{[0,1]} |\overrightarrow{\beta}^P(t, X_t)|^2/2 \, dt$
- $H(P|R^\tau) = H(P_1|m) + \tau E_P \int_{[0,1]} |\overleftarrow{\beta}^P(t, X_t)|^2/2 \, dt$

Actions

- forward: $\overrightarrow{A}(P|R^\tau) := H(P|R^{\tau, P_0 \rightarrow}) = \tau E_P \int_{[0,1]} |\overrightarrow{\beta}^P(t, X_t)|^2/2 \, dt$
- backward: $\overleftarrow{A}(P|R^\tau) := H(P|R^{\tau, \leftarrow P_1}) = \tau E_P \int_{[0,1]} |\overleftarrow{\beta}^P(t, X_t)|^2/2 \, dt$

Entropic actions

- $\beta^{\mathrm{cu}} := (\overrightarrow{\beta} + \overleftarrow{\beta})/2, \quad \beta^{\mathrm{os}} := (\overrightarrow{\beta} - \overleftarrow{\beta})/2$
- $v = \tau\beta$

Actions

- current: $\quad A^{\mathrm{cu}}(P|R^\tau) := \tau E_P \int_{[0,1]} |\beta^{\mathrm{cu},P}(t, X_t)|^2/2 \; dt$
- osmotic: $\quad A^{\mathrm{os}}(P|R^\tau) := \tau E_P \int_{[0,1]} |\beta^{\mathrm{os},P}(t, X_t)|^2/2 \; dt$

Results

- $A^{\mathrm{cu}}(P|R^\tau) = \tau^{-1} \int_{[0,1]\times\mathbb{R}^n} |v^{\mathrm{cu},P}|^2/2 \, dt dP_t$
- $A^{\mathrm{os}}(P|R^\tau) = \tau^{-1} \int_{[0,1]\times\mathbb{R}^n} |v^{\mathrm{os},P}|^2/2 \, dt dP_t = \tau/8 \; \int_{[0,1]} I(P_t|m) \, dt$

- $I(\mu|m) := \int_{\mathbb{R}^n} |\nabla \log(d\mu/dm)|^2 \; d\mu, \quad$ (Fisher information)

Entropic actions

- entropic action: $A(P|R^\tau) := \left[\overrightarrow{A}(P|R^\tau) + \overleftarrow{A}(P|R^\tau)\right]/2$
- parallelogram identity: $(|\overrightarrow{\beta}^P|^2 + |\overleftarrow{\beta}^P|^2)/2 = |\beta^{\mathrm{cu},P}|^2 + |\beta^{\mathrm{os},P}|^2$

- $A(P|R^\tau) = A^{\mathrm{cu}}(P|R^\tau) + A^{\mathrm{os}}(P|R^\tau)$
  $= \tau^{-1}/2 \ \int_{[0,1]\times\mathbb{R}^n} |v^{\mathrm{cu},P}|^2 \, dt d\mu_t + \tau/8 \ \int_{[0,1]} I(\mu_t|m) \, dt$
- (S^τ) is equivalent to:
  $\tau A(P|R^\tau) \to \min; \qquad P \in \mathrm{P}(\Omega) : \ P_0 = \mu_0, \ P_1 = \mu_1 \quad (\widetilde{S}^\tau)$

- entropic cost: $C^\tau(\mu_0, \mu_1) := \inf(\widetilde{S}^\tau)$

- $\lim_{\tau\to 0^+} C^\tau = W_2^2/2$
- $\lim_{\tau\to 0^+} \int_{[0,1]\times\mathbb{R}^n} |v^{\mathrm{cu},P^\tau}|^2 \, dt dP_t^\tau = W_2^2(\mu_0, \mu_1)$
- $\lim_{\tau\to 0^+} \tau^2 \int_{[0,1]} I(P_t^\tau|m) \, dt = 0$

Entropic actions

Instructions for use

1. use the two directions of time to compute the time derivatives
2. express the forward and backward entropic actions with $\overrightarrow{\beta}$ and $\overleftarrow{\beta}$
3. use the parallelogram identity to express them in terms of current and osmotic actions

- the osmotic action is a small perturbation of the current action
- it allows to follow Otto's heuristics with rigorous calculations

Equations of motion

- $L^\tau = \tau L, \quad L = (\nabla V \cdot \nabla + \Delta)/2$

If it exists, the unique solution P^τ of (S^τ) satisfies

- P^τ is Markov
- $P^\tau = f_0(X_0) g_1(X_1)\, R^\tau$
- $\mu_t^\tau = f_t g_t\ m, \quad 0 \le t \le 1$

$$\begin{cases} f_t(z) & := \quad E_{R^\tau}[f_0(X_0)|X_t = z] \\ g_t(z) & := \quad E_{R^\tau}[g_1(X_1)|X_t = z] \end{cases}$$

$$\begin{cases} (-\partial_t + \tau L) f = 0 \\ f_{|t=0} = f_0 \end{cases} \qquad \begin{cases} (\partial_t + \tau L) g = 0 \\ g_{|t=1} = g_1 \end{cases}$$

Equations of motion

- define: $\varphi := \log f, \quad \psi := \log g$
- $\overrightarrow{\beta} = \nabla\psi, \quad \overleftarrow{\beta} = -\nabla\varphi$

- f, g solve forward and backward heat equations
- φ, ψ solve forward and backward HJB equations

Conserved quantities

$$t \mapsto \int_{\mathbb{R}^n} \Gamma_n(f_t, g_t)\, dm \quad \text{is constant,} \qquad n \geq 0$$

- $\Gamma(\varphi) = |\nabla\varphi|^2, \qquad \Gamma_2(\varphi) = \sum_{i,j}(\partial_i\partial_j\varphi)^2 + \nabla^2 V(\nabla\varphi)$
- $n = 0 : \quad \int_{\mathbb{R}^n} \Gamma_0(f_t, g_t)\, dm = \int_{\mathbb{R}^n} f_t g_t\, dm = 1,$ (conservation of mass)
- $n = 1 : \quad \int_{\mathbb{R}^n} \Gamma(f_t, g_t)\, dm = -\int_{\mathbb{R}^n} \overrightarrow{\beta}_t \cdot \overleftarrow{\beta}_t\, d\mu_t$
  $- \int_{\mathbb{R}^n} |\beta_t^{\mathrm{cu}}|^2\, d\mu_t + \int_{\mathbb{R}^n} |\beta_t^{\mathrm{os}}|^2\, d\mu_t$

$$t \mapsto \int_{\mathbb{R}^n} |v_t^{\mathrm{cu},\tau}|^2\, d\mu_t^\tau - \tau^2/4 \int_{\mathbb{R}^n} I(\mu_t^\tau | m)\, d\mu_t^\tau \quad \text{is constant}$$

Derivatives of the entropy

Derivatives of the entropy

- $$\frac{d}{dt}\ H(\mu_t^\tau|m) = \tau/2\ \int_{\mathbb{R}^n} \{\Gamma(\psi_t) - \Gamma(\varphi_t)\}\, d\mu_t$$
- $$\frac{d^2}{dt^2}\ H(\mu_t^\tau|m) = \tau^2/2\ \int_{\mathbb{R}^n} \{\Gamma_2(\psi_t) + \Gamma_2(\varphi_t)\}\, d\mu_t$$

- recall: $\beta^{\mathrm{os}} = \nabla \log \sqrt{d\mu/dm}, \quad \partial_t \mu + \nabla \cdot (\mu v^{\mathrm{cu}}) = 0$
- d/dt
  - $\tau(\Gamma(\psi) - \Gamma(\varphi)) = \tau(|\overrightarrow{\beta}|^2 - |\overleftarrow{\beta}|^2) = \tau \beta^{\mathrm{os}} \cdot \beta^{\mathrm{cu}} = \beta^{\mathrm{os}} \cdot v^{\mathrm{cu}}$
  - $H'(t) = \langle \beta_t^{\mathrm{os}}, v_t^{\mathrm{cu}} \rangle_{\mu_t}/2 = \langle \nabla \log(d\mu_t/dm), v_t^{\mathrm{cu}} \rangle_{\mu_t}$
  - Otto's heuristics: $H'(t) = \left\langle \mathrm{grad}_{\mu_t} H, \dot{\mu}_t \right\rangle_{\mu_t}$
- d^2/dt^2
  - hypothesis: $\nabla^2 V \geq K \mathrm{Id}, \quad K \in \mathbb{R}$
  - $\tau^2/2\ (\Gamma_2(\psi) + \Gamma_2(\varphi)) \geq \tau^2 K/2\ (|\overrightarrow{\beta}|^2 + |\overleftarrow{\beta}|^2)$
    $= \tau^2 K\ (|\beta^{\mathrm{cu}}|^2 + |\beta^{\mathrm{os}}|^2) = K(|v^{\mathrm{cu}}|^2 + \tau^2 |\beta^{\mathrm{os}}|^2)$
  - $H''(t) \geq K \int_{\mathbb{R}^n} |v_t^{\mathrm{cu}}|^2\, d\mu_t + K\tau^2 I(\mu_t|m)/4$

HWI

- Otto-Villani
- hypothesis: $\nabla^2 V \geq K\mathrm{Id}, \quad K \in \mathbb{R}$

HWI inequality

$$H(\mu_1|m) - H(\mu_0|m) \leq W_2(\mu_0, \mu_1)\sqrt{I(\mu_1|m)} - KW_2^2(\mu_0, \mu_1)/2$$

For all $\nu \in \mathrm{P}(\mathbb{R}^n)$:

- HWI: $H(\nu|m) \leq W_2(\nu, m)\sqrt{I(\nu|m)} - KW_2^2(\nu, m)/2$
- Talagrand: $W_2^2(\nu, m)/2 \leq K^{-1}H(\nu|m), \quad (K > 0)$
- log-Sobolev: $H(\nu|m) \leq (2K)^{-1}I(\nu|m), \qquad (K > 0)$

HWI

Sketch of proof

- $h(1) - h(0) = h'(1) - \int_0^1 t h''(t)\, dt$
- $h(t) = H(\mu_t^\tau | m)$
- $\tau \to 0$

- $h'(1) = \langle \nabla \log(d\mu_1/dm), v_1^{\mathrm{cu}} \rangle_{\mu_1} \leq \sqrt{I(\mu_1|m)} \left[\int_{\mathbb{R}^n} |v_1^{\mathrm{cu}}|^2\, d\mu_1 \right]^{1/2}$
- $\int_{[0,1]} t h''(t) \geq K \int_{[0,1]\times\mathbb{R}^n} t |v_t^{\mathrm{cu}}|^2\, dt d\mu_t + K\tau^2 \int_{[0,1]} t I(\mu_t|m)\, dt/4$

- $t \mapsto \int_{\mathbb{R}^n} |v_t^{\mathrm{cu}}|^2\, d\mu_t - \tau^2/4 \ \int_{\mathbb{R}^n} I(\mu_t|m)\, d\mu_t$ is constant
- $\lim_{\tau\to 0^+} \int_{[0,1]\times\mathbb{R}^n} |v_t^{\mathrm{cu}}|^2\, dt d\mu_t = W_2^2(\mu_0, \mu_1)$
- $\lim_{\tau\to 0^+} \tau^2 \int_{[0,1]} I(\mu_t|m)\, dt = 0$

JKO

- Ref. Adams, Dirr, Erbar, Fathi, Maas, Mielke, Peletier, Renger, Zimmer
- $H(P|R^\tau) = \mathcal{F}(P_0) + \mathcal{F}(P_1) + A^{\mathrm{cu}}(P|R^\tau) + A^{\mathrm{os}}(P|R^\tau)$
  - time reversal: $H(P|R^\tau) = H(P^*|R^{\tau,*}) = H(P^*|R^\tau)$
- (half) free energy: $\mathcal{F}(\mu) := H(\mu|m)/2 = [H(\mu|\mathrm{vol}) + \int_{\mathbb{R}^n} V\, d\mu]/2$
- $\overrightarrow{A}(P|R^\tau) := H(P|R^{\tau,P_0\rightarrow}) = H(P|R^\tau) - H(P_0|m)$

Forward action decomposition

$$\overrightarrow{A}(P|R^\tau) = H(P|R^{\tau,P_0\rightarrow}) = \mathcal{F}(P_1) - \mathcal{F}(P_0) + A^{\mathrm{cu}}(P|R^\tau) + A^{\mathrm{os}}(P|R^\tau)$$

- $A^{\mathrm{cu}}(P|R^\tau) = \tau^{-1} \int_{[0,1]\times\mathbb{R}^n} |v^{\mathrm{cu},P}|^2/2\, dtdP_t \simeq (2\tau)^{-1} W_2^2(P_0, P_1)$
- $A^{\mathrm{os}}(P|R^\tau) = \tau/8 \int_{[0,1]} I(P_t|m)\, dt$

JKO

Particles

- $(Z^1, \dots, Z^N) \sim (R^{\mu_0})^{\otimes N}$
- $\widehat{Z}^N := N^{-1} \sum_i \delta_{Z^i}$, random values in $\mathrm{P}(\Omega)$

- time discretization: $\widehat{Z}^{N,\tau}_t := \widehat{Z}^{N,\tau}_{\lfloor t/\tau \rfloor \tau}, \quad t \geq 0$

Large deviation principle

$$\mathbb{P}(\widehat{Z}^{N,\tau} \in G) \underset{N\to\infty}{\asymp} \exp\left(-N \inf_{\mu \in G} \sum_{k \geq 0} J^\tau(\mu_{k\tau}, \mu_{k\tau+\tau})\right)$$

JKO-type function

$$J^\tau(\alpha, \beta) := \mathcal{F}(\beta) - \mathcal{F}(\alpha) + \tau^{-1} C^\tau(\alpha, \beta), \quad \alpha, \beta \in \mathrm{P}(\mathbb{R}^n)$$

- C^τ : entropic cost

JKO

- $JKO^\tau(\alpha, \beta) := \mathcal{F}(\beta) - \mathcal{F}(\alpha) + (2\tau)^{-1} W_2^2(\alpha, \beta)$
- $J^\tau(\alpha, \beta) := \mathcal{F}(\beta) - \mathcal{F}(\alpha) + \tau^{-1} C^\tau(\alpha, \beta)$
  $\simeq JKO^\tau(\alpha, \beta) + \tau/8 \ \inf_{\mu:\alpha,\beta} \int_{[0,1]} I(\mu_t|m)\, dt$

Result. (Erbar, Maas, Renger)

$\lim_{\tau \to 0^+} \tau \ \int_{[0,1]} I(\mu_t^\tau|m)\, dt = 0$

- another proof?: $\frac{d^2}{dt^2} I(\mu_t|m)$ involves Γ_3

Work in progress

- discrete graphs
- $CD(K, N)$

Thank you for your attention