Calculus of Variations and Optimal Transportation

Institut Henri Poincaré
Amphithéâtre Hermite

Paris, January 10 – 13, 2017

	Tuesday	Wednesday	Thursday	Friday
09 :00 – 09 :45	*Welcome Coffee*	*Coffee*	*Coffee*	*Coffee*
09 :45 – 10 :00	*Opening*			
10 :00 – 10 :50	**Pierre-Louis Lions**	**Luigi Ambrosio** New estimates on the matching problem	**François-Xavier Vialard** From unbalanced optimal transport to the Camass-Holm equation	**Benoît Perthame** From transport collapse to kinetic formulations
11 :00 – 11 :50	**Laure Saint-Raymond** Entropy and irreversibility in gas dynamics	**Alexander Mielke** Optimal transport vs. growth and decay	**Ivar Ekeland** A local surjection theorem	**Felix Otto**
12 :00 – 13 :30	*Lunch*	*Lunch*	*Lunch*	*Lunch*
13 :30 – 14 :20	**Nizar Touzi** Martingale Optimal Transport	**Mike Cullen** Applications of optimal transport to weather and climate	**François Golse** On the mean-field and semi-classical limit of the quantum N-body problem	**Sylvia Serfaty** Mean-Field Limits for Ginzburg-Landau vortices
14 :30 – 15 :20	**Christian Léonard** Some results about entropic transport	**Filippo Santambrogio** Regularity via duality in variational MFG and degenerate elliptic PDEs	**Nassif Ghoussoub** Dynamic and Stochastic Brenier Transport via Hopf-Lax formulae on Wasserstein Space	**Cédric Villani**
15 :30 – 16 :00	*Tea Break*	*Tea Break*	*Tea Break*	*Tea Break*
16 :00 – 16 :50	**Quentin Mérigot** Numerical resolution of Euler's equations through semi-discrete optimal transport	**Wilfrid Gangbo** Paths of minimal lengths on the set of exact differential k-forms	**Björn Engquist** Seismic imaging and optimal transport	
17 :00 – 17 :50	**Robert McCann** Optimal transportation between unequal dimensions	**Guillaume Carlier** On multi-marginal optimal transport problem	**Alessio Figalli** Global estimates for local and nonlocal porous medium type equations on bounded domains	
			Cocktail Reception *Musée Bourdelle* *19 :00 – 23 :30*	

Abstracts

Luigi Ambrosio
(Scuola Normale di Pisa, Italy)
New estimates on the matching problem

The matching problem consists in finding the optimal coupling between a random distribution of N points in a d-dimensional domain and another (possibly random) distribution. There is a large literature on the asymptotic behaviour as N tends to infinity of the expectation of the minimum cost, and the results depend on the dimension d and the choice of cost, in this random optimal transport problem. In a recent work, Caracciolo, Lucibello, Parisi and Sicuro proposed an ansatz for the expansion in N of the expectation. I will illustrate how a combination of semigroup smoothing techniques and Dacorogna-Moser interpolation provide first rigorous results for this ansatz.

Guillaume Carlier
(Université Paris Dauphine, France)
On multi-marginal optimal transport problem

In this talk, I will survey a few aspects of multi-marginal optimal transport. Such problems arise for instance when one discretizes in time Brenier's relaxed formulation of incompressible Euler as the equation of geodesics in the group of measure-preserving diffeomorphisms. I will also describe other instances arising in economics and in the study of Wasserstein barycenters and discuss a central limit theorem for the latter.

Mike Cullen
(Met Office, UK)
Applications of optimal transport to weather and climate

The atmospheric motions that govern extratropical weather and the global general circulation that maintains the climate are both strongly determined by the effects of the Earth's rotation. Under these conditions, the governing equations of motion and thermodynamics can in principle be accurately approximated by the semi-geostrophic equations developed by Eliassen in the 1930s. However, in order to justify the use of these equations, it is necessary to show that they can be solved and that their solutions are indeed the limit of solutions of the Euler equations in an appropriate asymptotic limit. Yann Brenier has made crucial contributions to both these problems. He showed how weak solutions of the semi-geostrophic equations could be characterized as solutions of a transport equation coupled to a Monge-Kantorovich problem, which allowed a proof of long time existence of the solutions. Separately he showed that an important special case of the solutions could be proved to be the limit of Euler solutions by using a relative entropy. I will review this work, and briefly describe some of the further work that is proceeding in this area.

Ivar Ekeland
(Université Paris Dauphine, France)
A local surjection theorem

Inverse function theorems, in their soft and hard versions, come with a very small domain of invertibility. I will state soft and hard local surjection theorems which give a much wider range, and give application to singular perturbation problems.
This is joint work with Eric Séré.

Björn Engquist
(University of Texas at Austin, USA)
Seismic imaging and optimal transport

Full Waveform Inversion is a recent technique in seismic imaging. We will discuss the application of optimal transport and the Wasserstein distance as a measure of mismatch between computed seismic waves and measured data. This is a central step in full waveform inversion. The advantage of the Wasserstein distance over more traditional measures will be presented in analytical results as well as by numerical examples.

Alessio Figalli
(ETH, Suisse)
Global estimates for local and nonlocal porous medium type equations on bounded domains

The behavior of solutions to the classical porous medium equation is by now well understood : the support of the solution expands at finite speed, and for large times it behaves as the separate-variable solution. When the Laplacian is replaced by a nonlocal diffusion, completely new and surprising phenomena arise depending on the power of the nonlinearity and the one of the diffusion. The aim of the talk is to present a self-contained overview of this theory.

Wilfrid Gangbo
(University of California LA, USA)
Paths of minimal lengths on the set of exact differential k-forms

We initiate the study of optimal transportation of exact differential k-forms and introduce various distances as minimal actions. Our study involves dual maximization problems with constraints on the codifferential of k-forms. When $k < n$ only some directional derivatives of a vector field are controlled. This is in contrast with prior studies of optimal transportation of volume forms ($k = n$), where the full gradient of a scalar function is controlled. Furthermore, our study involves paths of bounded variations on the set of k-currents. This talk is based a joint work with B. Dacorogna and O. Kneuss.

François Golse

(Ecole Polytechnique, France)

On the mean-field and semi-classical limit of the quantum N-body problem

The Hartree equation is derived from the N-body Schrödinger equation in the mean-field limit, assuming that $N \gg 1$ and that the coupling constant (i.e. the strength of the interaction potential) is of order $O(1/N)$. When the interaction potential is bounded with Lipschitz continuous gradient, one can prove that the convergence rate for the mean-field limit is uniform in the Planck constant. This convergence rate estimate is based on a kind of interpolation argument involving two very different bounds : (a) a convergence rate estimate in trace norm for the Dyson series representing the solution to the BBGKY hierarchy, and (b) an estimate involving a quantum analogue of the quadratic Monge-Kantorovich or Vasershtein distance used in optimal transport. (Work in collaboration with C. Mouhot, T. Paul, M. Pulvirenti).

Nassif Ghoussoub

(University of British Columbia, Canada)

Dynamic and Stochastic Brenier Transport via Hopf-Lax formulae on Wasserstein Space

I investigate deterministic and stochastic dynamic optimal mass transport problems associated to ballistic-related cost functionals on phase space. This leads to Hopf-Lax formulae on Wasserstein Space, and to links with Mean Field Games.

Christian Léonard

(Université Paris Ouest)

Some results about entropic transport

Optimal transport is a powerful tool for proving several functional inequalities such as : concentration inequalities, geometric inequalities (Prekopa-Leindler inequality) or entropy-entropy production inequalities related to rates of convergence to equilibrium of heat flows, seen as gradient flows of the entropy with respect to the Wasserstein metric. This approach is fully efficient in geodesic spaces but it is uneasy to transfer it to discrete graphs. Recently Maas and Mielke discovered a Riemmanian geometry on the set of probability measures (not related to optimal transport) on a graph with respect to which reversible random walks are gradient flows of the entropy. This opened a successful way to implement the "gradient flow strategy" on graphs. In this talk, another approach will be sketched. It relies on entropic transport rather than standard optimal transport. Displacement interpolations are replaced by entropic interpolations solving the Schrödinger problem : an entropy minimization problem on the set of probability measures with marginal constraints. The large deviation problem leading to Schrödinger problem will be introduced. Some easy proofs of functional inequalities, based on this entropy minimization problem, will be sketched. The gradient flow structure of heat flows will also be presented in this spirit : heat flows are "gradient flows" of the entropy with respect to some large deviation cost rather than the Wasserstein metric.

Pierre-Louis Lions
(Collège de France, France)
TBA

Robert McCann
(University of Toronto, Canada)
Optimal transportation between unequal dimensions

Following the seminal contributions of Yann Brenier, the theory of optimal transportation blossomed into a powerful tool for exploring applications both within and outside mathematics. Its impact is felt in such far flung areas as geometry, analysis, dynamics, partial differential, equations, economics, machine learning, weather prediction, and computer vision. The basic problem is to transport one probability density onto other, while minimizing a given cost $c(x, y)$ per unit transported. In the vast majority of applications, the probability densities live on spaces with the same (finite) dimension. After briefly surveying a few highlights from this theory, we focus our attention on what can be said when the densities instead live on spaces with two different (yet finite) dimensions. Although the answer can still be characterized as the solution to a fully nonlinear differential equation, it now becomes badly nonlocal in general. Remarkably however, one can identify conditions under which the equation becomes local, elliptic, and amenable to further analysis.

Quentin Mérigot
(Université Paris Sud, France)
Numerical resolution of Euler's equations through semi-discrete optimal transport

In this talk, I will present a numerical method for extracting minimal geodesics along the group of volume preserving maps, equipped with the L^2 metric. As observed by Arnold, these geodesics solve the Euler equations of inviscid incompressible fluids. The method relies on the generalized polar decomposition of Brenier, numerically implemented through semi-discrete optimal transport. It is robust enough to extract non-classical, multi-valued solutions of Euler's equations predicted by Brenier and Schnirelman. Our convergence results encompass this generalized model, and our numerical experiments illustrate it for the first time in two space dimensions (joint work with Jean-Marie Mirebeau). I will also show how this approach leads to a numerical scheme able to approximate regular solutions to the Cauchy problem for Euler equations (joint work with Thomas Gallouet).

Alexander Mielke
(WIAS Berlin)
Optimal transport versus growth and decay

While the theory of optimal transport has experienced huge development over the last decade, a corresponding theory for reactions between different species is still in its infancy. We consider a simple model for one species that diffuses and can react with the background by absorption or desorption, i.e. growth or decay. This results in the so-called Hellinger-Kantorovich distance between arbitrary measures, which can be seen as an ind-convolution of the Wasserstein-Kantorovich distance for transport and the Hellinger-Fisher-Rao distance growth and decay. We discuss properties and variational characterizations of this distance which are generalizations of Yann Brenier's result for optimal transport. (This is joint work with Matthias Liero and Giuseppe Savaré.)

Felix Otto
(Max Planck Institute, Leipzig, Germany)
TBA

Benoît Perthame
(Université Pierre et Marie Curie, France)
From transport collapse to kinetic formulations

Laure Saint-Raymond
(ENS Paris and Université Pierre et Marie Curie, France)
Entropy and irreversibility in gas dynamics

Filippo Santambrogio
(Université Paris Sud, France)
Regularity via duality in variational mean field games and degenerate elliptic PDEs

The starting point of the talk is a technique, first developed by Yann Brenier (and then improved by Luigi Ambrosio and Alessio Figalli) to prove $L^2(BV)$ regularity of the pressure in the variational formulation of the incompressible Euler equation. We used the same technique with Pierre Cardaliaguet and Alpár Mészáros to prove the same regularity result for the pressure appearing in a mean field game with density constraint where, exactly as for the incompressible Euler case, the summability improvement deriving from this result was needed to give a meaning to a Lagrangian formulation, and that point we realized that the technique is indeed a very general and powerful tool in convex optimization. In the talk I will mainly concentrate on some easier cases, and show regularity results for mean field games with density penalizations (instead of constraints, a work in collaboration with Adam Prosinski) and for more classical variational problems, including those corresponding to some degenerate elliptic PDEs such as the p-Laplace equation. This provides a different point of view on some classical results, and allows sometimes to guess how to generalize them : in particular, starting from this intuition, we could prove, with Lorenzo Brasco, a new and sharp regularity estimate for the p-Laplace equation.

Sylvia Serfaty
(Université Pierre et Marie Curie, France and Courant Institute, NYU)
Mean-Field Limits for Ginzburg-Landau vortices

Ginzburg-Landau type equations are models for superconductivity, superfluidity, Bose-Einstein condensation. A crucial feature is the presence of quantized vortices, which are topological zeroes of the complex-valued solutions. This talk will review some results on the derivation of effective models to describe the statics and dynamics of these vortices, with particular attention to the situation where the number of vortices blows up with the parameters of the problem. In particular we will present new results on the derivation of mean field limits for the dynamics of many vortices starting from the parabolic Ginzburg-Landau equation or the Gross-Pitaevskii (= Schrodinger Ginzburg-Landau) equation, as well as results and questions on the situation with random environment.

Nizar Touzi
(Ecole Polytechnique, France)
Martingale Optimal Transport

We review the recent developments in dimension 1 and in higher dimension. In contrast with the standard optimal transport problem, the quasi-sure formulation plays a crucial role in order to obtain the Kantorovitch duality for general measurable couplings, and to justify attainability in the dual problem.

François-Xavier Vialard
(Université Paris Dauphine, France)
From unbalanced optimal transport to the Camassa-Holm equation

The group of diffeomorphisms of a compact manifold endowed with the L^2 metric acting on the space of probability densities gives a unifying framework for the incompressible Euler equation and the theory of optimal mass transport. Recently, several authors including ourselves, have extended optimal transport to the space of positive Radon measures where the Wasserstein-Fisher-Rao distance is a natural extension of the classical L^2-Wasserstein distance. In this talk, our goal is to show a similar relation between this unbalanced optimal transport problem which will be presented in details and the Hdiv right-invariant metric on the group of diffeomorphisms, which corresponds to the Camassa-Holm (CH) equation in one dimension. Among others, a surprising application of this geometric point of view is the following : Solutions to the standard CH equation give particular solutions of the incompressible Euler equation on a group of homeomorphisms of $\mathbb{R}^2$ which preserve a radial density that has a singularity at zero.

Cédric Villani
(IHP and Université Claude Bernard Lyon 1, France)
TBA
