Adaptive hp-finite elements with guaranteed error contraction and inexact multilevel solvers

A THESIS PRESENTED AT

SORBONNE UNIVERSITY

DOCTORAL SCHOOL: MATHEMATICAL SCIENCES OF CENTRAL PARIS (ED 386)

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Paris, France, March 22, 2019

informatics mathematics

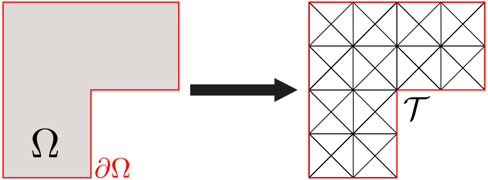
École des Ponts ParisTech

European Research Council

Introduction

Context

- various physical phenomena can be described by PDEs
- need of **efficient**, **accurate**, **reliable** and robust numerical methods
- finite element method (FEM)



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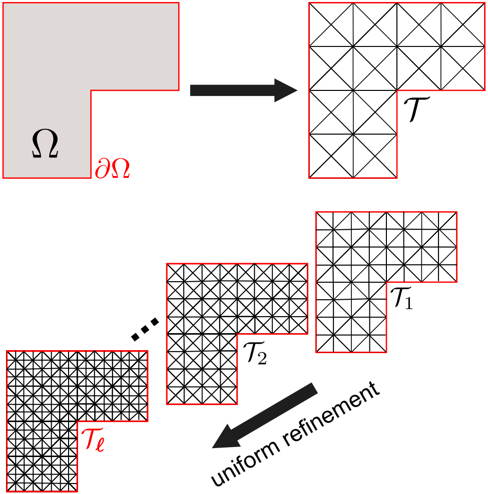
Adaptivity

- **FEM** $\to$ h-**AFEM** $\to$ hp-**AFEM**
- sequence of **nested meshes** $\{\mathcal{T}_\ell\}_{\ell\geq0}$ and polynomial degrees $\{\mathbf{p}_\ell\}_{\ell\geq0}$
- iteration counter ℓ
- a posteriori driven hp-**AFEM**



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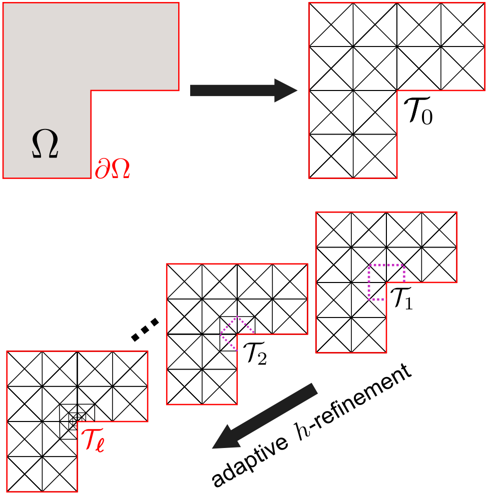

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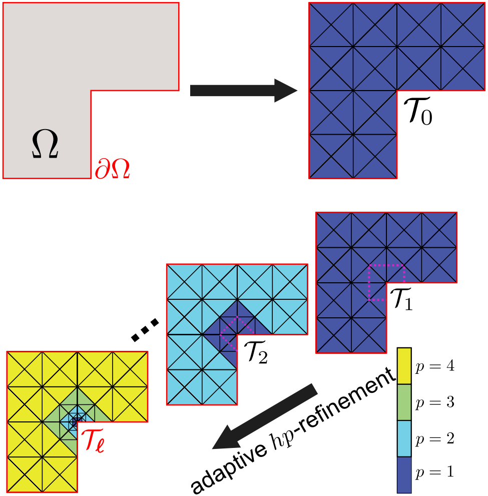

Motivation

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1980s 1990s 2000s 2010s

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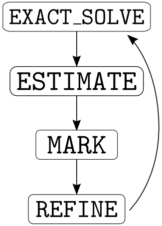

Error reduction property

$$\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\| \le C_{\ell,\mathrm{red}} \|\nabla(u - u_{\ell}^{\mathrm{ex}})\|$$

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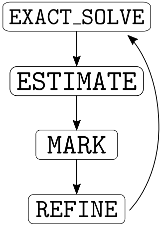

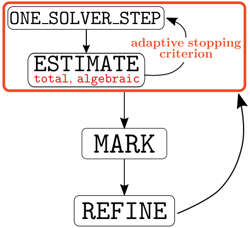

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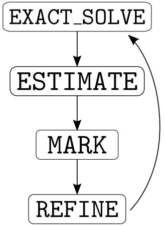

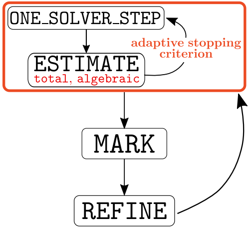

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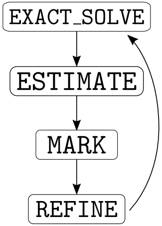

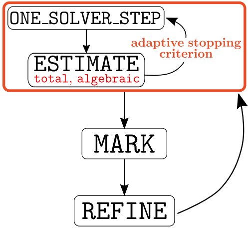

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$$0 < C_{\ell,\mathrm{red}} \lneq 1$$

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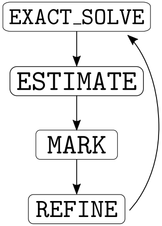

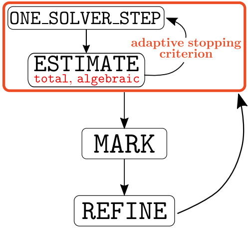

- theoretical analysis
- convergence
- additional assumptions

$$0 < C_{\ell,\mathrm{red}} < 1$$

Error reduction property

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$$\Big\} \; 0 < C_{\ell,\mathrm{red}} \leq 1$$

Outline

1. An adaptive hp-refinement strategy **with exact solver**
2. An adaptive hp-refinement strategy **with inexact solver**
3. Convergence of adaptive hp-refinement strategies

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Setting – model problem

Poisson equation with (homogeneous) Dirichlet boundary conditions

For $f \in L^2(\Omega)$, seek $u \in H_0^1(\Omega)$

$$(\nabla u, \nabla v) = (f, v) \qquad \forall v \in H_0^1(\Omega)$$

Conforming hp–finite element method (initialize $\ell := 0$)

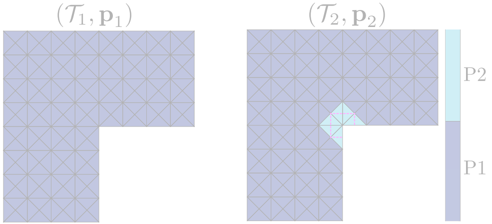
$$(\nabla u_\ell^{\mathrm{ex}}, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell, \qquad V_\ell := \mathbb{P}_{\mathbf{p}}(\mathcal{T}_\ell) \cap H_0^1(\Omega), \quad \forall \ell \geq 0$$

Nested hierarchy of spaces

$$V_\ell \subset V_{\ell+1}, \qquad \forall \ell \geq 0$$

Built up on the pair $(\mathcal{T}_\ell, \mathbf{p}_\ell)$, $\ell \geq 0$

- matching simplicial mesh $\mathcal{T}_\ell$
- $\mathbf{p}_\ell := \{p_{\ell,K}\}_{K \in \mathcal{T}_\ell}$



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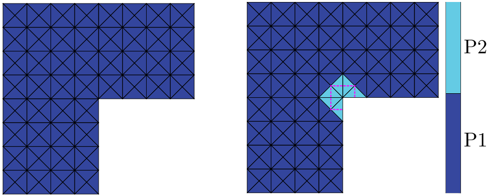
$(\mathcal{T}_1, \mathbf{p}_1)$ $(\mathcal{T}_2, \mathbf{p}_2)$



Modules of the adaptive loop

$$\begin{tikzcd} \texttt{SOLVE} \arrow[r] & \texttt{ESTIMATE} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[lll, bend right] \end{tikzcd}$$

Module SOLVE

FEM

$$(\nabla u_\ell^{\text{ex}}, \nabla v_\ell) = (f, v_\ell) \qquad \forall v_\ell \in V_\ell \qquad \Longleftrightarrow \qquad \mathbb{A}_\ell \mathrm{U}_\ell^{\text{ex}} = \mathrm{F}_\ell$$

- $\mathrm{U}_\ell^{\text{ex}}$ corresponds to the exact FEM solution $u_\ell^{\text{ex}} = \sum_{n=1}^{N_\ell} (\mathrm{U}_\ell^{\text{ex}})_n \, \psi_\ell^n$

Galerkin orthogonality (**exact setting**)

$$\left\|\nabla(u - u_{\ell+1}^{\text{ex}})\right\|^2 = \left\|\nabla(u - u_\ell^{\text{ex}})\right\|^2 - \left\|\nabla(u_{\ell+1}^{\text{ex}} - u_\ell^{\text{ex}})\right\|^2$$

Module SOLVE

FEM

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Residual liftings for modules ESTIMATE and REFINE

Residual $\mathcal{R}_\ell \in H^{-1}(\Omega)$

$$\langle \mathcal{R}_\ell, v \rangle := (f, v) - (\nabla u_\ell^{\mathrm{ex}}, \nabla v) \qquad \forall v \in H_0^1(\Omega)$$

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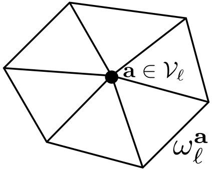

For each vertex $\mathbf{a} \in \mathcal{V}_\ell$ **and its corresponding patch sub-domain** $\omega_\ell^{\mathbf{a}}$

1. $\langle \mathcal{R}_\ell, \psi_\ell^{\mathbf{a}} v\rangle = (f, \psi_\ell^{\mathbf{a}} v)_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla(\psi_\ell^{\mathbf{a}} v))_{\omega_\ell^{\mathbf{a}}} \qquad \forall v \in H_*^1(\omega_\ell^{\mathbf{a}})$

2. $\langle \mathcal{R}_\ell, v\rangle = (f, v)_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v)_{\omega_\ell^{\mathbf{a}}} \qquad \forall v \in H_0^1(\omega_\ell^{\mathbf{a}})$

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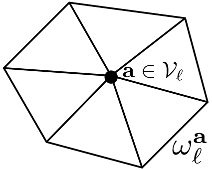

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   $\rightarrow$ discrete $\mathbf{H}(\mathrm{div}, \omega_\ell^{\mathbf{a}})$-conforming lifting $\boldsymbol{\sigma}_\ell^{\mathbf{a}}$

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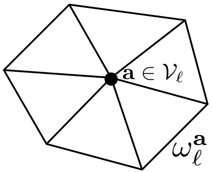

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Residual liftings for modules ESTIMATE and REFINE

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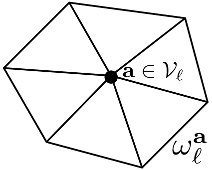

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   $\}$ $\rightarrow$ ESTIMATE

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Residual liftings for modules $\boxed{\texttt{ESTIMATE}}$ and $\boxed{\texttt{REFINE}}$

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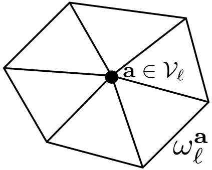

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Module ESTIMATE

Guaranteed upper bound on the energy error

$$\|\nabla(u - u_\ell^{\mathrm{ex}})\| \le \eta(u_\ell^{\mathrm{ex}}, \mathcal{T}_\ell) := \Big\{ \sum_{K\in\mathcal{T}_\ell} \eta_K^2 \Big\}^{\frac{1}{2}}, \qquad \eta_K := \|\nabla u_\ell^{\mathrm{ex}} + \boldsymbol{\sigma}_\ell\|_K + \frac{h_K}{\pi}\|f - \nabla{\cdot}\boldsymbol{\sigma}_\ell\|_K$$

Equilibrated flux reconstruction $\boldsymbol{\sigma}_\ell := \sum_{\mathbf{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_\ell^{\mathbf{a}} \in \mathbf{H}(\mathrm{div},\Omega)$

For each vertex $\mathbf{a} \in \mathcal{V}_\ell$, we solve a small minimization problem

$$\boldsymbol{\sigma}_\ell^{\mathbf{a}} := \arg \min_{\mathbf{v}_\ell \in \mathbf{V}_\ell^{\mathbf{a}},\, \nabla\cdot\mathbf{v}_\ell = \Pi_{Q_\ell^{\mathbf{a}}}(f\psi_\ell^{\mathbf{a}} - \nabla u_\ell^{\mathrm{ex}}\cdot\nabla\psi_\ell^{\mathbf{a}})} \|\psi_\ell^{\mathbf{a}}\nabla u_\ell^{\mathrm{ex}} + \mathbf{v}_\ell\|_{\omega_\ell^{\mathbf{a}}}$$

with properly chosen local *Raviart–Thomas–Nédélec* mixed finite element spaces $\mathbf{V}_\ell^{\mathbf{a}} \times Q_\ell^{\mathbf{a}}$ of order $p_{\mathbf{a}} := \max_{K\in\mathcal{T}_\ell^{\mathbf{a}}} p_K$

Remark: $\boldsymbol{\sigma}_\ell^{\mathbf{a}} \leftarrow$ local discrete $\mathbf{H}(\mathrm{div},\omega_\ell^{\mathbf{a}})$-conforming residual lifting

D. Braess, J. Schöberl, *Equilibrated residual error estimator for edge elements*, Math. Comp. (2008)

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$$\|\nabla(u-u_\ell^{\mathrm{ex}})\| \le \eta(u_\ell^{\mathrm{ex}},\mathcal{T}_\ell) := \Big\{\sum_{K\in\mathcal{T}_\ell}\eta_K^2\Big\}^{\frac12}, \qquad \eta_K := \|\nabla u_\ell^{\mathrm{ex}} + \boldsymbol{\sigma}_\ell\|_K + \frac{h_K}{\pi}\|f-\nabla\cdot\boldsymbol{\sigma}_\ell\|_K$$

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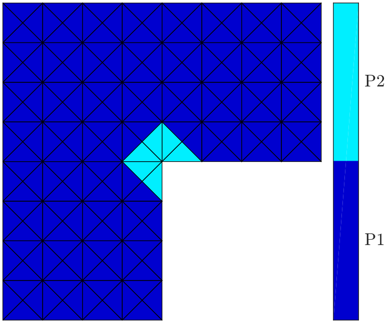
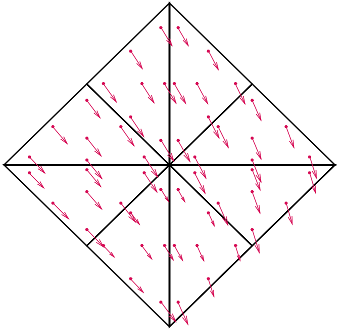
A posteriori error ESTIMATE

Flux reconstruction: illustration on a single patch $\omega_2^{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_2$



Global position of the patch $\omega_2^{\mathbf{a}}$



The exact flux
$-\nabla u \in \mathbf{H}(\mathrm{div}, \Omega)$

A posteriori error ESTIMATE

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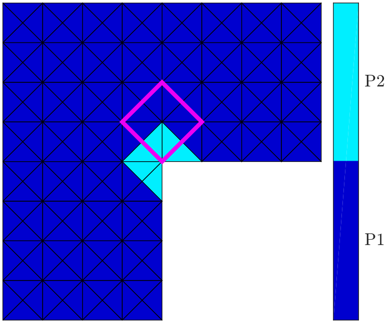

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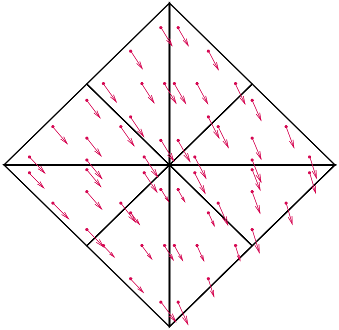

The exact flux $-\nabla u \in \mathbf{H}(\mathrm{div}, \Omega)$

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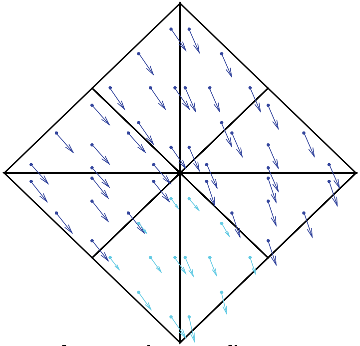
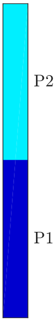
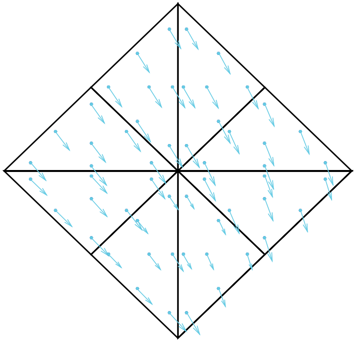


Approximate flux

$-\nabla u_2^{\mathrm{ex}} \notin \mathbf{H}(\mathrm{div}, \Omega)$,

$[\![\nabla u_2 \cdot \mathbf{n}_F]\!] \neq 0$





Flux reconstruction

$\boldsymbol{\sigma}_2 \in \mathbf{V}_2 \subset \mathbf{H}(\mathrm{div}, \Omega)$

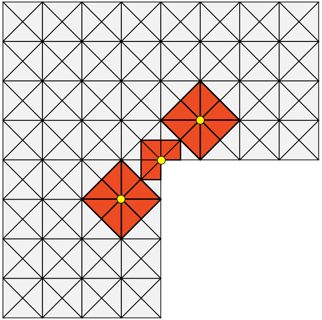
$[\![\nabla u_2 \cdot \mathbf{n}_F]\!] = 0$

Module MARK

A bulk chasing criterion for marking vertices

For a fixed threshold parameter $\theta \in (0,1]$, the set of **marked vertices** $\mathcal{V}_\ell^\theta \subset \mathcal{V}_\ell$ is selected in such a way that

$$\underbrace{\eta\Big(u_\ell^{\mathrm{ex}}, \bigcup_{\mathbf{a}\in\mathcal{V}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}\Big)}_{\text{estimated error in marked region}} \geq \underbrace{\theta\, \eta(u_\ell^{\mathrm{ex}}, \mathcal{T}_\ell)}_{\text{bulk of the estimated total error}}$$



● → **marked vertices** $\mathcal{V}_\ell^\theta$

▲ → consequently set $\omega_\ell := \cup_{\mathbf{a}\in\mathcal{V}_\ell^\theta} \omega_\ell^{\mathbf{a}}$, the open subdomain corresponding to a set of marked elements $\mathcal{M}_\ell^\theta := \bigcup_{\mathbf{a}\in\mathcal{V}_\ell^\theta} \mathcal{T}_\ell^{\mathbf{a}}$

W. Dörfler, *A convergent adaptive algorithm for Poisson's equation*, SIAM J. Numer. Anal. (1996)

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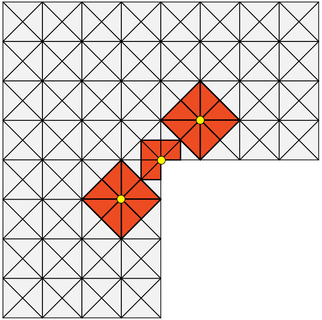

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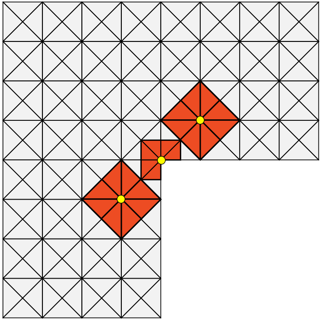

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Error reduction factors

Overview - error reduction factors

$\texttt{SOLVE} \longrightarrow \texttt{ESTIMATE} \longrightarrow \texttt{MARK}$

Overview - error reduction factors

SOLVE → ESTIMATE → MARK → REFINE

Assume **for the moment:** $\mathcal{T}_{\ell+1}$ and $\mathbf{p}_{\ell+1}$ are determined

Overview - error reduction factors

$$
\begin{tikzcd}
\texttt{SOLVE} \arrow[r] & \texttt{ESTIMATE} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[lll, bend right, "C_{\ell,\mathrm{red}}"']
\end{tikzcd}
$$

Assume **for the moment:** $\mathcal{T}_{\ell+1}$ and $\mathbf{p}_{\ell+1}$ <u>are determined</u>

Goal: $\|\nabla(\boldsymbol{u} - \boldsymbol{u}^{\mathbf{ex}}_{\ell+1})\| \leq \quad C_{\ell,\mathbf{red}} \quad \|\nabla(\boldsymbol{u} - u^{\mathrm{ex}}_{\ell})\|$

For each marked vertex $\mathbf{a} \in \mathcal{V}_\ell^\theta$ only, construct residual lifting $r^{\mathbf{a},hp}$

Overview - error reduction factors

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Goal: $\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\ell+1}^{\mathbf{ex}})\| \leq \underbrace{C_{\ell,\mathrm{red}}}_{\textit{fully computable}} \|\nabla(\boldsymbol{u} - u_\ell^{\mathrm{ex}})\|$

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Overview - error reduction factors

$$\texttt{SOLVE} \rightarrow \texttt{ESTIMATE} \rightarrow \texttt{MARK} \rightarrow \texttt{REFINE} \xrightarrow{C_{\ell,\mathrm{red}}} \texttt{SOLVE}$$

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Goal: $$\underbrace{\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\ell+1}^{\mathbf{ex}})\|}_{\textit{both unknown}} \leq \underbrace{C_{\ell,\mathrm{red}}}_{\textit{fully computable}} \|\nabla(\boldsymbol{u} - u_\ell^{\mathrm{ex}})\|$$

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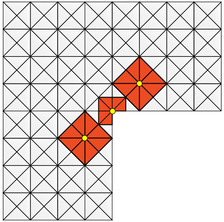
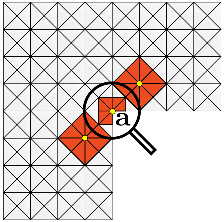
Residual lifting

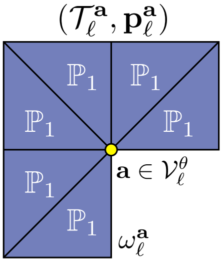


Notation: for each marked vertex $\mathbf{a} \in \mathcal{V}_\ell^\theta$ (○) and the associated coarse patch $\mathcal{T}_\ell^{\mathbf{a}}$

- the local submesh refinement $\mathcal{T}_{\ell+1}|_{\omega_\ell^{\mathbf{a}}}$
- the local polynomial degrees $\mathbf{p}_{\ell+1}|_{\omega_\ell^{\mathbf{a}}}$
- the local space $V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H_0^1(\omega_\ell^{\mathbf{a}})$

Residual lifting





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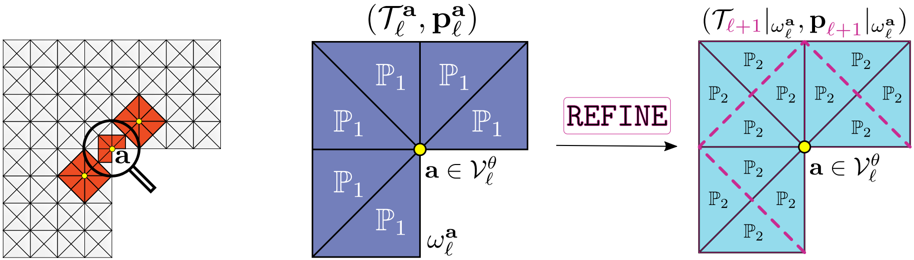

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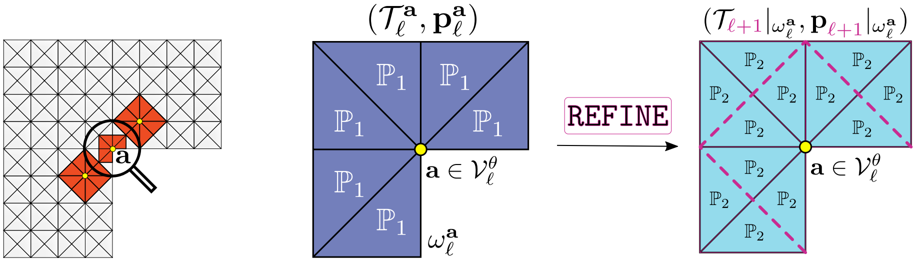
Residual lifting



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Residual lifting II.

One local problem per marked vertex $\mathbf{a} \in \mathcal{V}_\ell^\theta$ (residual lifting)

Given a local space $V_\ell^{\mathbf{a},hp} := V_{\ell+1}|_{\omega_\ell^{\mathbf{a}}} \cap H_0^1(\omega_\ell^{\mathbf{a}})$, we solve

$$(\nabla r^{\mathbf{a},hp}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}} \qquad \forall v^{\mathbf{a},hp} \in V_\ell^{\mathbf{a},hp}$$

Then, if $\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} r^{\mathbf{a},hp} \neq 0$, we have the **guaranteed lower bound**

$$\|\nabla(\underbrace{u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}}}_{\text{only } u_\ell^{\mathrm{ex}} \text{ is known}})\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} \left\|\nabla r^{\mathbf{a},hp}\right\|_{\omega_\ell^{\mathbf{a}}}^2}{\left\|\nabla\left(\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}} =: \eta_{\mathcal{M}_\ell^\theta}$$

Remark: $r^{\mathbf{a},hp} \leftarrow$ local discrete $H_0^1(\omega_\ell^{\mathbf{a}})$-conforming residual lifting

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Remark: $r^{\mathbf{a},hp} \leftarrow$ local discrete $H_0^1(\omega_\ell^{\mathbf{a}})$-conforming residual lifting

Guaranteed lower bound on $\left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}})\right\|_{\omega_\ell}$

Discrete local lower bound $\underline{\eta}_{\mathcal{M}_\ell^\theta}$

$$\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}})\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} \left\|\nabla r^{\mathbf{a},hp}\right\|_{\omega_\ell^{\mathbf{a}}}^2}{\left\|\nabla\left(\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}} =: \underline{\eta}_{\mathcal{M}_\ell^\theta}$$

Proof

$$\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} = \sup_{v_{\ell+1}\in V_{\ell+1}(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

$$\geq \sup_{v_{\ell+1}\in V_{\ell+1}^0(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

$$= \sup_{v_{\ell+1}\in V_{\ell+1}^0(\omega_\ell)} \frac{(f, v_{\ell+1})_{\omega_\ell} - (\nabla u_\ell, \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}} \geq \frac{\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} (\nabla r^{\mathbf{a},hp}, \nabla r^{\mathbf{a},hp})_{\omega_\ell^{\mathbf{a}}}}{\left\|\nabla\left(\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}}$$

To finish take $\left(\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} r_{\mathbf{a}}^{hp}\right)$ as test function $v_{\ell+1}$

Guaranteed lower bound on $\left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}})\right\|_{\omega_\ell}$

Discrete local lower bound $\underline{\eta}_{\mathcal{M}_\ell^\theta}$

$$\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell^{\mathrm{ex}})\|_{\omega_\ell} \geq \frac{\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} \left\|\nabla r^{\mathbf{a},hp}\right\|^2_{\omega_\ell^{\mathbf{a}}}}{\left\|\nabla\left(\sum_{\mathbf{a}\in\mathcal{V}_\ell^\theta} r^{\mathbf{a},hp}\right)\right\|_{\omega_\ell}} =: \underline{\eta}_{\mathcal{M}_\ell^\theta}$$

Proof

$$\|\nabla(u_{\ell+1} - u_\ell)\|_{\omega_\ell} = \sup_{v_{\ell+1}\in V_{\ell+1}(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

$$\geq \sup_{v_{\ell+1}\in V^0_{\ell+1}(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}}$$

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&\geq \sup_{v_{\ell+1}\in V_{\ell+1}^0(\omega_\ell)} \frac{(\nabla(u_{\ell+1} - u_\ell), \nabla v_{\ell+1})_{\omega_\ell}}{\|\nabla v_{\ell+1}\|_{\omega_\ell}} \\
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Error contraction property

Guaranteed bound on the error reduction factor

The new (*unknown*) numerical solution $u_{\ell+1}^{\mathrm{ex}} \in V_{\ell+1}$ satisfies

$$\left\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\right\| \leq C_{\ell,\mathrm{red}} \left\|\nabla(u - u_{\ell}^{\mathrm{ex}})\right\| \text{ with } 0 < C_{\ell,\mathrm{red}} := \sqrt{1 - \theta^2 \frac{\underline{\eta}^2_{\mathcal{M}_\ell^\theta}}{\eta^2(u_\ell^{\mathrm{ex}}, \mathcal{M}_\ell^\theta)}} \leq 1$$

Proof (sketch)

1. Galerkin orthogonality
   $$\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\|^2 = \|\nabla(u - u_{\ell}^{\mathrm{ex}})\|^2 - \qquad \|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\|^2$$
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→ motivation for our *hp*-**refinement criterion** employed within the module `REFINE`

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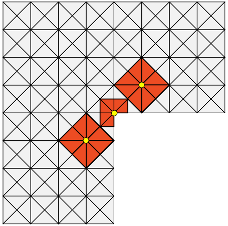
$\rightarrow$ motivation for our ***hp*-refinement criterion** employed within the module $\boxed{\texttt{REFINE}}$

REFINE: residual liftings & local hp-decision criterion $\to (\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$

Two local FE problems on each patch $\mathcal{T}_\ell^{\mathbf{a}}$ attached to a **marked vertex** $\mathbf{a} \in \mathcal{V}_\ell^\theta$



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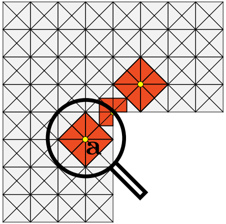

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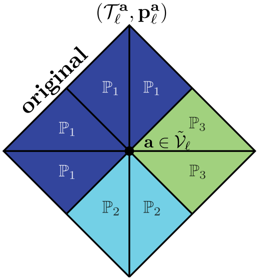

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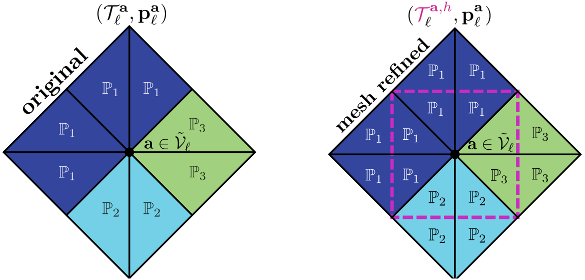

h-refinement residual lifting

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},h} \in \boxed{V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a}}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

p

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},p} \in \boxed{V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a}}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

$\to$ **local hp-decision criterion** is then based on $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}}$ and $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}$

REFINE: residual liftings & local hp-decision criterion $\to (\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$

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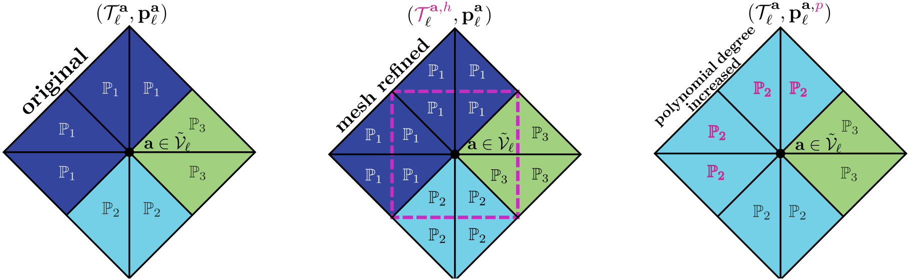


h-refinement residual lifting

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},h} \in \boxed{V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a}}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

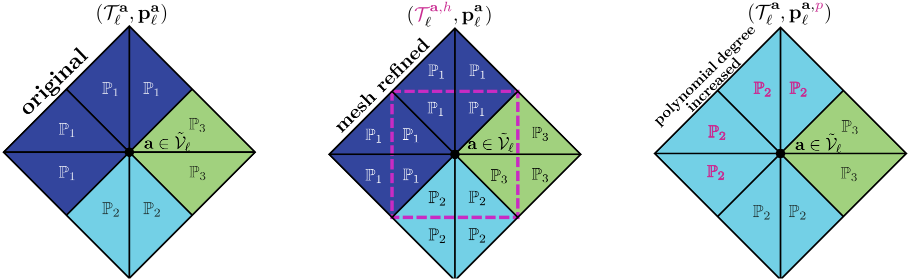
p-refinement residual lifting

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},p} \in \boxed{V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a}}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

$\to$ **local hp-decision criterion** is then based on $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}}$ and $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}$

REFINE: residual liftings & local hp-decision criterion $\to (\mathcal{T}_{\ell+1}, \mathbf{p}_{\ell+1})$

Two local FE problems on each patch $\mathcal{T}_\ell^{\mathbf{a}}$ attached to a **marked vertex $\mathbf{a} \in \mathcal{V}_\ell^\theta$**



h-refinement residual lifting

$$(\nabla r^{\mathbf{a},h}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},h})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},h} \in \boxed{V_\ell^{\mathbf{a},h} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a}}}(\mathcal{T}_\ell^{\mathbf{a},h}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

p-refinement residual lifting

$$(\nabla r^{\mathbf{a},p}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} = (f, v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla v^{\mathbf{a},p})_{\omega_\ell^{\mathbf{a}}} \quad \forall\, v^{\mathbf{a},p} \in \boxed{V_\ell^{\mathbf{a},p} := \mathbb{P}_{\mathbf{p}_\ell^{\mathbf{a},p}}(\mathcal{T}_\ell^{\mathbf{a}}) \cap H_0^1(\omega_\ell^{\mathbf{a}})}$$

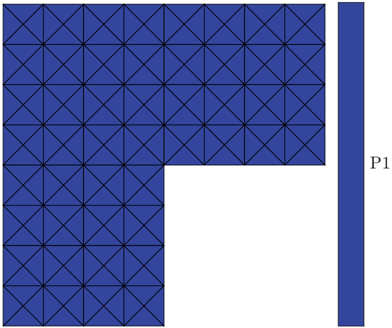
$\to$ **local hp-decision criterion** is then based on $\|\nabla r^{\mathbf{a},h}\|_{\omega_\ell^{\mathbf{a}}}$ and $\|\nabla r^{\mathbf{a},p}\|_{\omega_\ell^{\mathbf{a}}}$

Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$
- $\theta = 0.5$

$(\mathcal{T}_1, \mathbf{p}_1)$



Numerics

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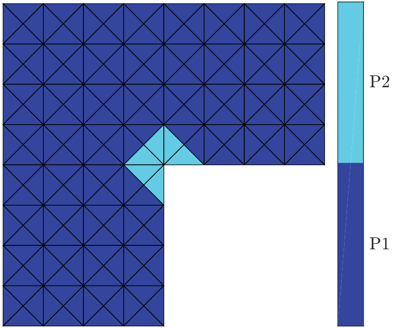
$(\mathcal{T}_2, \mathbf{p}_2)$



Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

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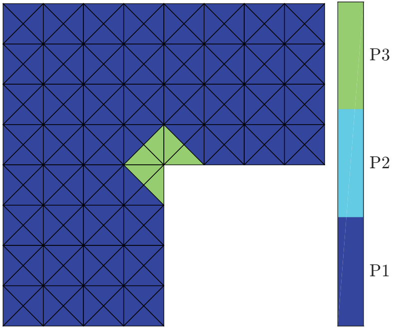
$(\mathcal{T}_3, \mathbf{p}_3)$



Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

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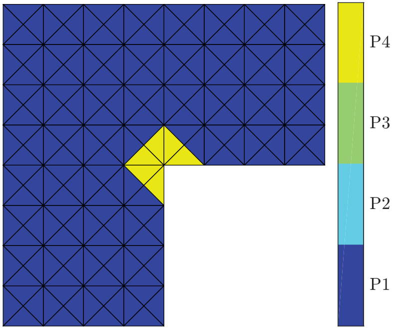
- $\theta = 0.5$

$(\mathcal{T}_4, \mathbf{p}_4)$



Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

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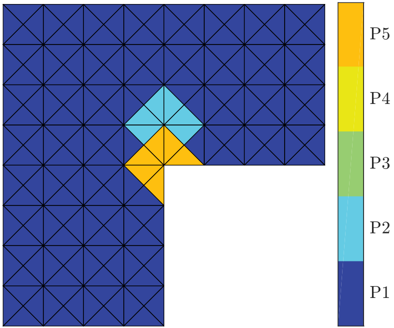
- $\theta = 0.5$

$(\mathcal{T}_5, \mathbf{p}_5)$

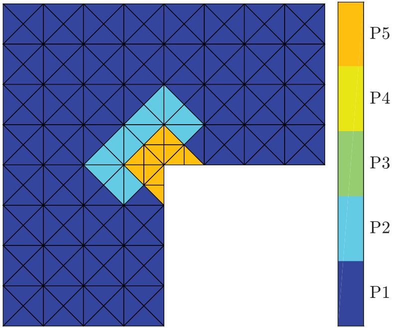


Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$
- $\theta = 0.5$

$(\mathcal{T}_6, \mathbf{p}_6)$



Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

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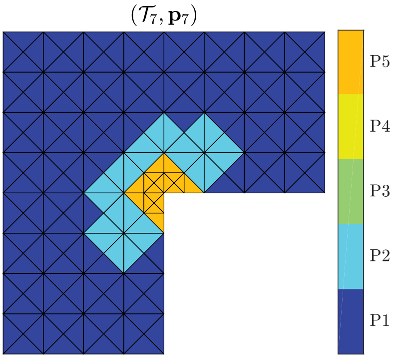

Numerics

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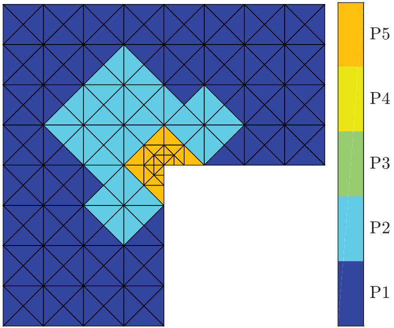
- $\theta = 0.5$

$(\mathcal{T}_8, \mathbf{p}_8)$



Numerics

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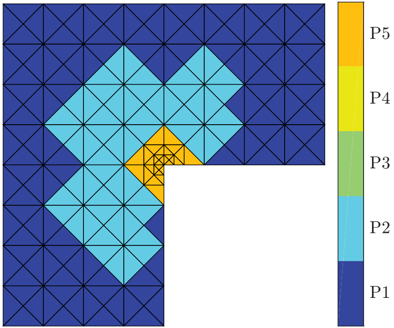
- $\theta = 0.5$

$(\mathcal{T}_9, \mathbf{p}_9)$



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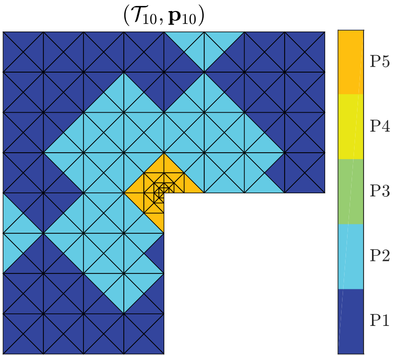

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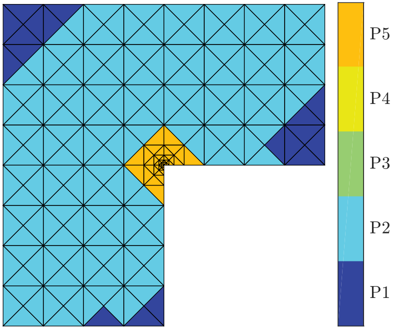
$(\mathcal{T}_{15}, \mathbf{p}_{15})$



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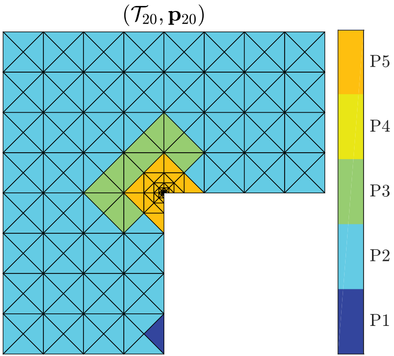

Numerics

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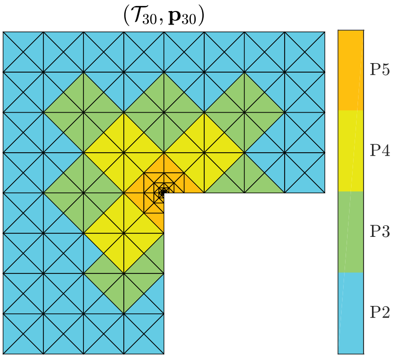

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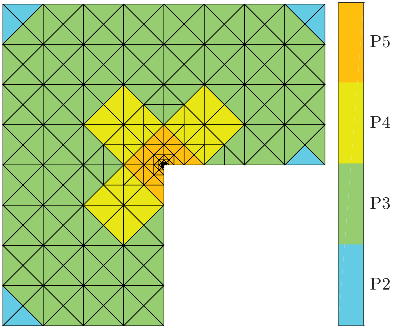
- $\theta = 0.5$

$(\mathcal{T}_{40}, \mathbf{p}_{40})$

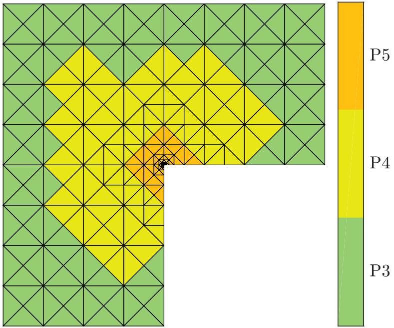


Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$
- $\theta = 0.5$

$(\mathcal{T}_{50}, \mathbf{p}_{50})$



Numerics

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

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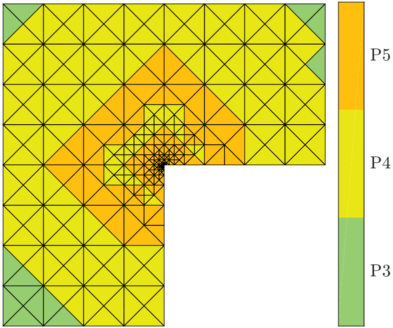
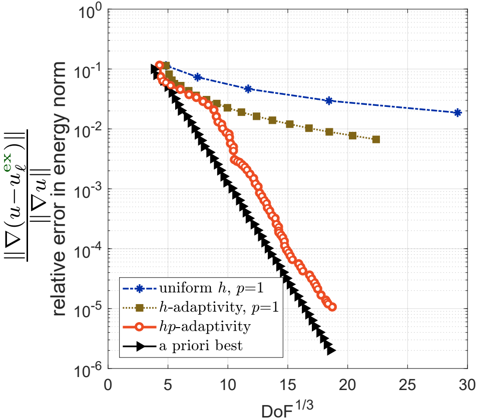
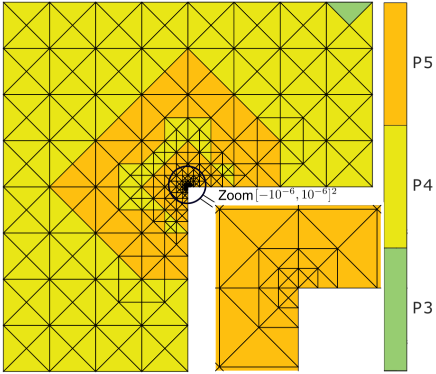
- $\theta = 0.5$

$(\mathcal{T}_{60}, \mathbf{p}_{60})$



Numerics II. – exponential convergence

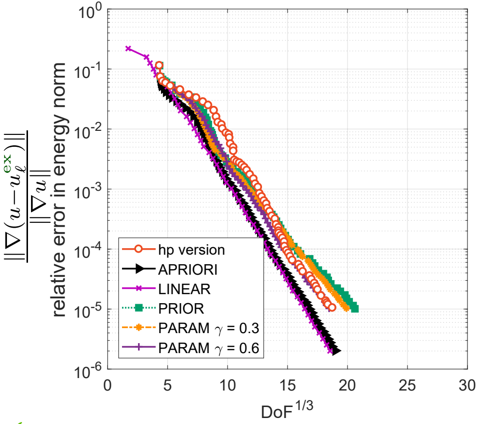


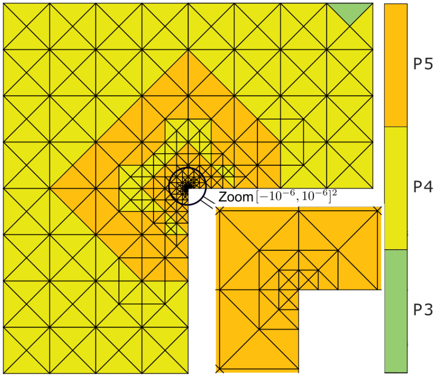


✓ Obtained exponential convergence, **comparison with classical approaches** and the final mesh with polynomial degree distribution $(\mathcal{T}_{65}, \mathbf{p}_{65})$.

W. F. Mitchel, M. A. McClain, *A comparison of hp-adaptive strategies for elliptic partial differential equations (long version)*, NISTIR (2011)

Numerics II. – exponential convergence





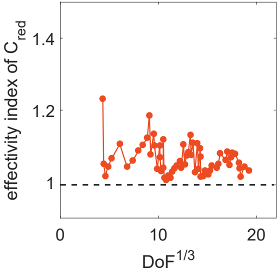
✓ Obtained exponential convergence, **comparison with other hp-adaptive approaches** and the final mesh with polynomial degree distribution $(\mathcal{T}_{65}, \mathbf{p}_{65})$

W. F. Mitchel, M. A. McClain, *A comparison of hp-adaptive strategies for elliptic partial differential equations (long version)*, NISTIR (2011)

Numerics III.

Effectivity indices of the estimated error reduction factor $C_{\ell,\mathrm{red}}$ and $\underline{\eta}_{\mathcal{M}_\ell^\theta}$

$$I_{\mathrm{red}}^{\mathrm{eff}} = \frac{C_{\ell,\mathrm{red}}}{\|\nabla(u-u_{\ell+1}^{\mathrm{ex}})\| \,/\, \|\nabla(u-u_{\ell}^{\mathrm{ex}})\|}$$



$$I_{\mathrm{LB}}^{\mathrm{eff}} = \frac{\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell^\theta}}$$

$$I_{\eta}^{\mathrm{eff}} = \frac{\eta(u_{\ell}^{\mathrm{ex}},\mathcal{T}_\ell)}{\|\nabla(u-u_{\ell}^{\mathrm{ex}})\|}$$

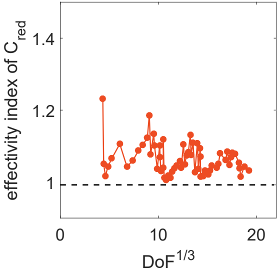
Numerics III.

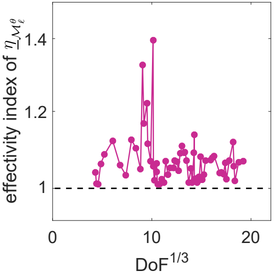
Effectivity indices of the estimated error reduction factor $C_{\ell,\mathrm{red}}$ and $\underline{\eta}_{\mathcal{M}_\ell^\theta}$

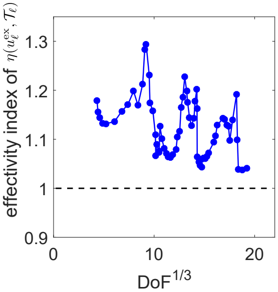
$$I_{\mathrm{red}}^{\mathrm{eff}} = \frac{C_{\ell,\mathrm{red}}}{\|\nabla(u-u_{\ell+1}^{\mathrm{ex}})\| \,/\, \|\nabla(u-u_{\ell}^{\mathrm{ex}})\|}$$

$$I_{\mathrm{LB}}^{\mathrm{eff}} = \frac{\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_{\ell}^{\mathrm{ex}})\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell^\theta}}$$

$$I_{\eta}^{\mathrm{eff}} = \frac{\eta(u_\ell^{\mathrm{ex}},\mathcal{T}_\ell)}{\|\nabla(u-u_{\ell}^{\mathrm{ex}})\|}$$







Outline

1. An adaptive hp-refinement strategy **with exact solver**
2. An adaptive hp-refinement strategy **with inexact solver**
3. Convergence of adaptive hp-refinement strategies

Extension to the **inexact algebraic solver setting**

Goal

1. avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$

SOLVE → ESTIMATE → MARK → REFINE

$$\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\| \le C_{\ell,\mathrm{red}} \|\nabla(u - u_\ell^{\mathrm{ex}})\|, \quad 0 \le C_{\ell,\mathrm{red}} \le 1$$

→ only *approximate* solution $\mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell$ (corresponding $u_\ell \approx u_\ell^{\mathrm{ex}}$)

ONE_SOLVER_STEP → ESTIMATE → MARK → REFINE

adaptive stopping criterion

2. recover the **contraction property** also in the inexact setting

Extension to the **inexact algebraic solver setting**

Goal

1. avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$

$$\begin{tikzcd} \mathtt{SOLVE} \arrow[r] & \mathtt{ESTIMATE} \arrow[r] & \mathtt{MARK} \arrow[r] & \mathtt{REFINE} \arrow[lll, bend right] \end{tikzcd}$$

$$\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\| \leq C_{\ell,\mathrm{red}} \|\nabla(u - u_\ell^{\mathrm{ex}})\|, \quad 0 \leq C_{\ell,\mathrm{red}} \leq 1$$

$\rightarrow$ only *approximate* solution $\mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell$ (corresponding $u_\ell \approx u_\ell^{\mathrm{ex}}$) ✓

$$\begin{tikzcd} \mathtt{ONE\_SOLVER\_STEP} \arrow[r] & \underset{\mathtt{total,\ algebraic}}{\mathtt{ESTIMATE}} \arrow[l, bend left, "\text{adaptive stopping criterion}"] \arrow[r] & \mathtt{MARK} \arrow[r] & \mathtt{REFINE} \arrow[ll, bend right] \end{tikzcd}$$

2. recover the **contraction property** also in the inexact setting

Extension to the **inexact algebraic solver setting**

Goal

1. avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$

$$\texttt{SOLVE} \longrightarrow \texttt{ESTIMATE} \longrightarrow \texttt{MARK} \longrightarrow \texttt{REFINE}$$

$$\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\| \le C_{\ell,\mathrm{red}} \|\nabla(u - u_\ell^{\mathrm{ex}})\|, \quad 0 \le C_{\ell,\mathrm{red}} \le 1$$

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$$\boxed{\texttt{ONE\_SOLVER\_STEP} \longrightarrow \underset{\texttt{total, algebraic}}{\texttt{ESTIMATE}}} \longrightarrow \texttt{MARK} \longrightarrow \texttt{REFINE}$$

adaptive stopping criterion

2. recover the **contraction property** also in the inexact setting

Extension to the **inexact algebraic solver setting**

Goal

1. avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$

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\boxed{\mathtt{SOLVE}} \arrow[r] & \boxed{\mathtt{ESTIMATE}} \arrow[r] & \boxed{\mathtt{MARK}} \arrow[r] & \boxed{\mathtt{REFINE}} \arrow[lll, bend right, "\boxed{C_{\ell,\mathrm{red}}}"']
\end{tikzcd}$$

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\ell+1}^{\mathbf{ex}})\| \leq C_{\ell,\mathrm{red}} \|\nabla(\boldsymbol{u} - u_\ell^{\mathrm{ex}})\|, \quad 0 \leq C_{\ell,\mathrm{red}} \leq 1$$

→ only *approximate* solution $\mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell$ (corresponding $u_\ell \approx u_\ell^{\mathrm{ex}}$) ✓

$$\begin{tikzcd}
\boxed{\mathtt{ONE_SOLVER_STEP}} \arrow[r] & \boxed{\mathtt{ESTIMATE}_{\mathtt{total,\ algebraic}}} \arrow[l, bend left, "\text{adaptive stopping criterion}"] \arrow[r] & \boxed{\mathtt{MARK}} \arrow[r] & \boxed{\mathtt{REFINE}} \arrow[lll, bend right]
\end{tikzcd}$$

2. recover the **contraction property** also in the inexact setting

Extension to the **inexact algebraic solver setting**

Goal

1. avoid the *unrealistic* exact solution of $\mathbb{A}_\ell \mathrm{U}_\ell^{\mathrm{ex}} = \mathrm{F}_\ell$

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$$\begin{tikzcd} \boxed{\texttt{ONE\_SOLVER\_STEP} \rightarrow \underset{\texttt{total, algebraic}}{\texttt{ESTIMATE}} \quad \text{adaptive stopping criterion}} \arrow[r] & \texttt{MARK} \arrow[r] & \texttt{REFINE} \arrow[ll, bend right, "\boxed{C_{\ell,\mathrm{red}}}"'] \end{tikzcd}$$

2. recover the **contraction property** also in the inexact setting ✓

$$\|\nabla(\boldsymbol{u} - \boldsymbol{u}_{\ell+1})\| \leq C_{\ell,\mathrm{red}} \|\nabla(\boldsymbol{u} - u_\ell)\|, \quad 0 \leq C_{\ell,\mathrm{red}} \leq 1$$

Adaptive sub-loop: $\boxed{\texttt{ONE\_SOLVER\_STEP}} \rightleftarrows \boxed{\texttt{ESTIMATE}}$

- we apply one iteration of the iterative solver to the resulting algebraic problem

$(\nabla u_\ell, \nabla v_\ell) \approx (f, v_\ell) \quad \forall v_\ell \in V_\ell \iff \mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell \to$ inexact approximation u_ℓ

- a posteriori error bounds on the **total** and also **algebraic** errors

Guaranteed total energy error upper bound

$$\|\nabla(u - u_\ell)\| \le \eta(u_\ell, \mathcal{T}_\ell) = \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_K^2(u_\ell) \right\}^{\frac{1}{2}}$$

$$\eta(u_\ell, \mathcal{T}_\ell) := \underbrace{\|\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\mathrm{dis}}\|}_{\eta_{\mathrm{dis}}(u_\ell, \mathcal{T}_\ell)} + \underbrace{\left\{ \sum_{K \in \mathcal{T}_\ell} \frac{h_K}{\pi} \left\| f - \Pi_{\mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell)} f \right\|_K^2 \right\}^{\frac{1}{2}}}_{\eta_{\mathrm{osc}}(u_\ell, \mathcal{T}_\ell)} + \underbrace{\|\boldsymbol{\sigma}_{\ell,\mathrm{alg}}\|}_{\eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell)}$$

A. Ern, M. Vohralík, *Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs*, SISC, '13

Adaptive sub-loop: ONE_SOLVER_STEP $\rightleftarrows$ ESTIMATE

- we apply one iteration of the iterative solver to the resulting algebraic problem

$$(\nabla u_\ell, \nabla v_\ell) \approx (f, v_\ell) \quad \forall v_\ell \in V_\ell \quad \Longleftrightarrow \quad \mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell \to \text{ inexact approximation } u_\ell$$

- a posteriori error bounds on the **total** and also **algebraic** errors

Guaranteed total energy error upper bound

$$\underbrace{\|\nabla(u - u_\ell)\|}_{\text{total error}} \leq \eta(u_\ell, \mathcal{T}_\ell) = \left\{ \sum_{K \in \mathcal{T}_\ell} \eta_K^2(u_\ell) \right\}^{\frac{1}{2}}$$

$$\eta(u_\ell, \mathcal{T}_\ell) := \underbrace{\|\nabla u_\ell + \boldsymbol{\sigma}_{\ell,\mathrm{dis}}\|}_{\eta_{\mathrm{dis}}(u_\ell, \mathcal{T}_\ell)} + \underbrace{\left\{ \sum_{K \in \mathcal{T}_\ell} \frac{h_K}{\pi} \left\| f - \Pi_{\mathbb{P}_{\mathbf{p}_\ell}(\mathcal{T}_\ell)} f \right\|_K^2 \right\}^{\frac{1}{2}}}_{\eta_{\mathrm{osc}}(u_\ell, \mathcal{T}_\ell)} + \underbrace{\|\boldsymbol{\sigma}_{\ell,\mathrm{alg}}\|}_{\eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell)}$$

A. Ern, M. Vohralík, *Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs*, SISC, '13

Adaptive sub-loop: ONE_SOLVER_STEP $\rightleftarrows$ ESTIMATE

- we apply one iteration of the iterative solver to the resulting algebraic problem

$$(\nabla u_\ell, \nabla v_\ell) \approx (f, v_\ell) \quad \forall v_\ell \in V_\ell \iff \mathbb{A}_\ell \mathrm{U}_\ell \approx \mathrm{F}_\ell \to \text{ inexact approximation } u_\ell$$

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A. Ern, M. Vohralík, *Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs*, SISC, '13

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A. Ern, M. Vohralík, *Adaptive inexact Newton methods with a posteriori stopping criteria for nonlinear diffusion PDEs*, SISC, '13

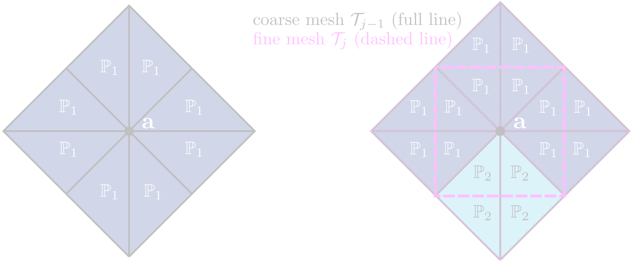
Crucial ingredients: $\mathbf{H}(\mathrm{div},\Omega)$-conforming flux reconstructions

Algebraic error flux reconstruction $\boldsymbol{\sigma}_{\ell,\mathrm{alg}}$

- $\boldsymbol{\sigma}_{\ell,\mathrm{alg}} := \sum_{j=1}^{\ell} \sum_{\mathbf{a}\in\mathcal{V}_{j-1}} \boldsymbol{\sigma}_{j,\mathrm{alg}}^{\mathbf{a}} \leftarrow$ **multilevel approach** $\leftarrow \mathcal{T}_{j,j-1}^{\mathbf{a}},\ 1 \le j \le \ell$

Discretization flux reconstruction $\boldsymbol{\sigma}_{\ell,\mathrm{dis}}$

- $\boldsymbol{\sigma}_{\ell,\mathrm{dis}} := \sum_{\mathbf{a}\in\mathcal{V}_\ell} \boldsymbol{\sigma}_{\ell,\mathrm{dis}}^{\mathbf{a}} \leftarrow$ local problems posed on the finest patches $\mathcal{T}_\ell^{\mathbf{a}}$, $\mathbf{a} \in \mathcal{V}_\ell$



Guaranteed algebraic error upper bound

$$\|\nabla(u_\ell^{\mathrm{ex}} - u_\ell)\| \le \eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell) := \left\{ \sum_{K\in\mathcal{T}_\ell} \eta_{\mathrm{alg},K}^2(u_\ell) \right\}^{\frac{1}{2}}$$

J. Papež, U. Rüde, M. Vohralík, and B. Wohlmuth, *Sharp algebraic and total a posteriori error bounds for h and p finite elements via a multilevel approach*. HAL preprint 01662944, Dec. 2017.

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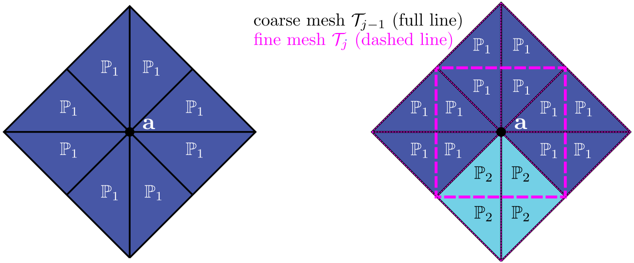

Guaranteed algebraic error upper bound

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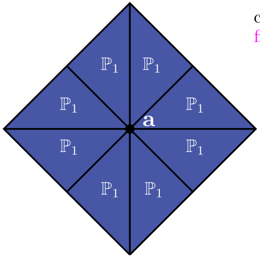

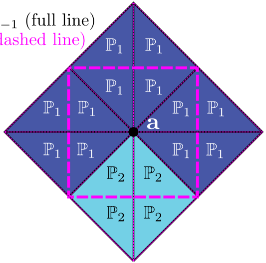

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$$\|\nabla(u - u_\ell)\| \geq \frac{\sum_{\mathbf{a} \in \mathcal{V}_\ell} \left\| \nabla \rho_{\ell,\mathrm{tot}}^{\mathbf{a}} \right\|_{\omega_\ell^{\mathbf{a}}}^2}{\|\nabla \rho_{\ell,\mathrm{tot}}\|} =: \mu(u_\ell)$$

- *total residual lifting* $\rho_{\ell,\mathrm{tot}} := \sum_{\mathbf{a} \in \mathcal{V}_\ell} \psi_\ell^{\mathbf{a}} \rho_{\ell,\mathrm{tot}}^{\mathbf{a}} \in H_0^1(\Omega)$
- $\rho_{\ell,\mathrm{tot}}^{\mathbf{a}} \leftarrow$ obtained by solving a small local primal FE problem

Adaptive stopping criterion for the algebraic solver

$$\boxed{\eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell\, \mu(u_\ell) \qquad 0 < \gamma_\ell < 1 \quad (\text{typically } \gamma_\ell \approx 0.1)}$$

Ensuring the desired balance ✓

$$\underbrace{\|\nabla(u_\ell^{\mathrm{ex}} - u_\ell)\|}_{\text{algebraic error}} \leq \eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell\, \mu(u_\ell) \leq \gamma_\ell \underbrace{\|\nabla(u - u_\ell)\|}_{\text{total error}}$$

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Error reduction factor in presence of inexact solver

- **Galerkin orthogonality** relation between $u_{\ell+1}^{\mathrm{ex}}$ and u_ℓ

$$\left\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\right\|^2 = \left\|\nabla(u - u_\ell)\right\|^2 - \left\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell)\right\|^2$$

- **Intermediate result** $\leftarrow \underline{\eta}_{\mathcal{M}_\ell^\theta}$ & prior results of Chapter 1

$$\left\|\nabla(u - u_{\ell+1}^{\mathrm{ex}})\right\| \leq C_{\ell,\mathrm{red}}^{*} \left\|\nabla(u - u_\ell)\right\|$$

Computable guaranteed bound on the error reduction factor

Using the adaptive stopping criterion at level $\ell+1$ with $0 < \gamma_{\ell+1} \leq (1 - C_{\ell,\mathrm{red}}^{*})$

$$\|\nabla(\underbrace{u - u_{\ell+1}}_{\text{both unknown}})\| \leq C_{\ell,\mathrm{red}} \|\nabla(\underbrace{u - u_\ell}_{\text{only } u_\ell \text{ known}})\|, \qquad 0 \leq C_{\ell,\mathrm{red}} := \frac{\sqrt{1 - \left(\frac{\underline{\eta}_{\mathcal{M}_\ell^\theta}}{\eta(u_\ell,\mathcal{T}_\ell)}\right)^2}}{(1-\gamma_{\ell+1})} \leq 1$$

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Numerics I.

L-shaped domain in 2D: $\Omega := (-1,1) \times (-1,1) \setminus [0,1] \times [-1,0]$, $f = 0$

- singular exact solution (in polar coordinates): $u(r,\varphi) = r^{\frac{2}{3}} \sin \frac{2\varphi}{3}$

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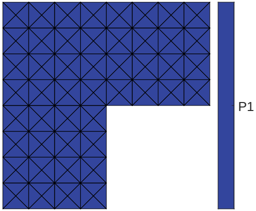
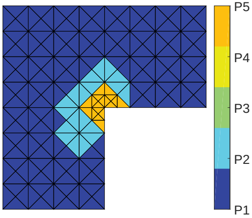
Inexact setting: V-cycle multigrid with Gauss-Seidel as a smoother



$(\mathcal{T}_1, \mathbf{p}_1)$



$(\mathcal{T}_7, \mathbf{p}_7)$

Numerics I.

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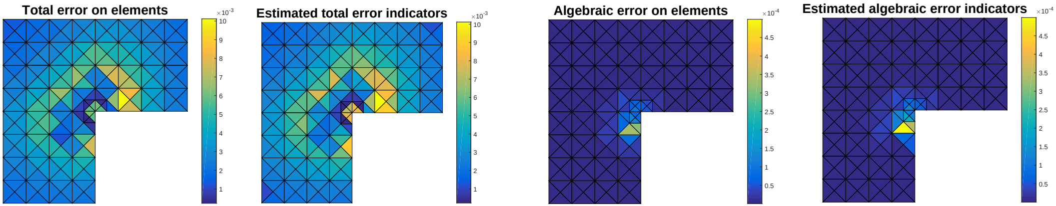
Inexact setting: V-cycle multigrid with Gauss–Seidel as a smoother

$I_{\mathrm{eff}}^{\mathrm{tot}} = 1.096$ $I_{\mathrm{eff}}^{\mathrm{alg}} = 1.365$



$\|\nabla(u-u_7)\|_K$ $\eta_K(u_7)$ $\|\nabla(u_7^{\mathrm{ex}}-u_7)\|_K$ $\eta_{\mathrm{alg},K}(u_7)$

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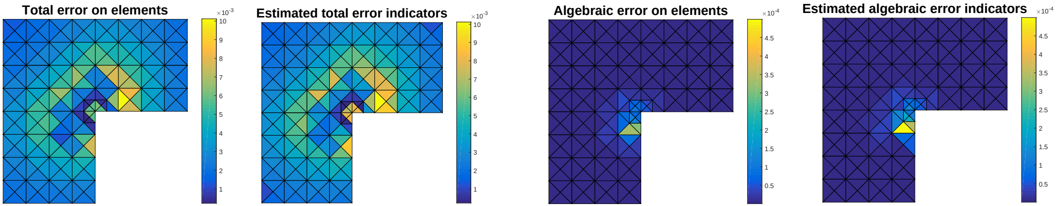
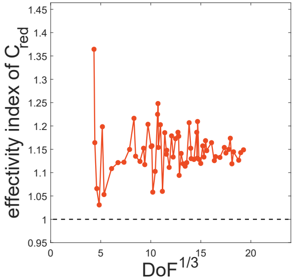


$\|\nabla(u - u_7)\|_K$ $\qquad$ $\eta_K(u_7)$ $\qquad$ $\|\nabla(u_7^{\mathrm{ex}} - u_7)\|_K$ $\qquad$ $\eta_{\mathrm{alg},K}(u_7)$

Numerics II.

Effectivity indices of the estimated error reduction factor $C_{\ell,\mathrm{red}}$ and $\underline{\eta}_{\mathcal{M}_\ell^\theta}$

$$I_{\mathrm{red}}^{\mathrm{eff}} = \frac{C_{\ell,\mathrm{red}}}{\|\nabla(u-u_{\ell+1})\| \,/\, \|\nabla(u-u_\ell)\|} \qquad I_{\mathrm{LB}}^{\mathrm{eff}} = \frac{\|\nabla(u_{\ell+1}^{\mathrm{ex}} - u_\ell)\|_{\omega_\ell}}{\underline{\eta}_{\mathcal{M}_\ell^\theta}} \qquad \gamma_\ell := \min(1 - C_{\ell,\mathrm{red}}^*, 0.1)$$



M. ARIOLI, E. H. GEORGOULIS, AND D. LOGHIN, *Stopping criteria for adaptive finite element solvers*, SISC, 2013

C. CARSTENSEN, M. FEISCHL, M. PAGE, AND D. PRAETORIUS, *Axioms of adaptivity*, CAMWA, 2014

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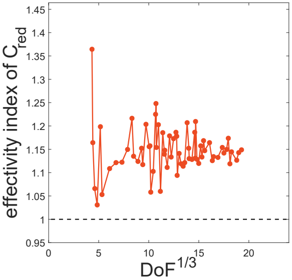

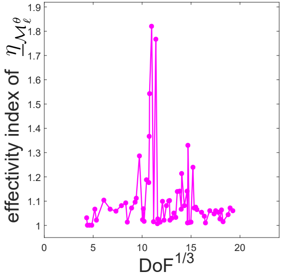

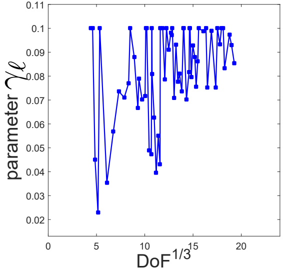

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Numerics III. – adaptivity for algebraic solver

Adaptive stopping criterion $\boxed{\eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell) \le \gamma_\ell\, \mu(u_\ell)}$ **in practice**

Note: *classical (non-adaptive) stopping criterion* $\frac{\|\mathrm{F}_\ell - \mathbb{A}_\ell \mathrm{U}_\ell\|}{\|\mathrm{F}_\ell\|} \le \varepsilon$

Numerics III. – adaptivity for algebraic solver

Adaptive stopping criterion $\boxed{\eta_{\mathrm{alg}}(u_\ell, \mathcal{T}_\ell) \leq \gamma_\ell\, \mu(u_\ell)}$ **in practice**

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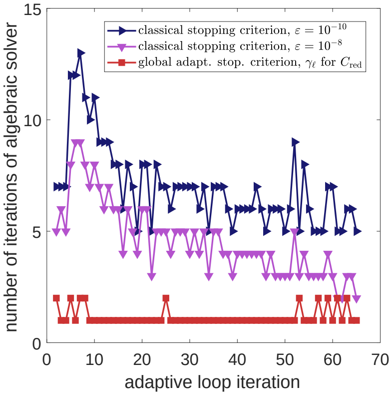

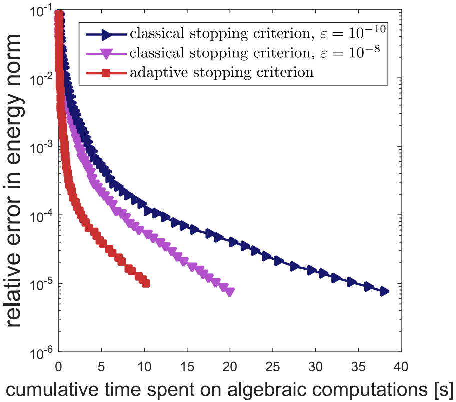

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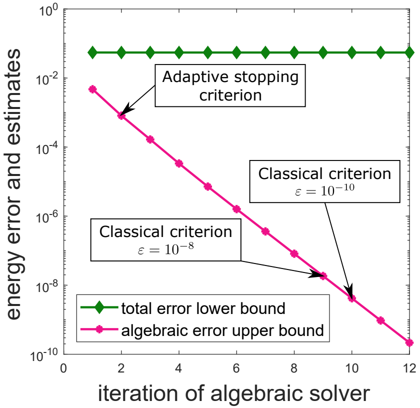
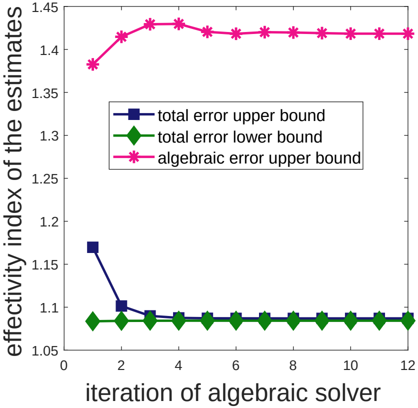
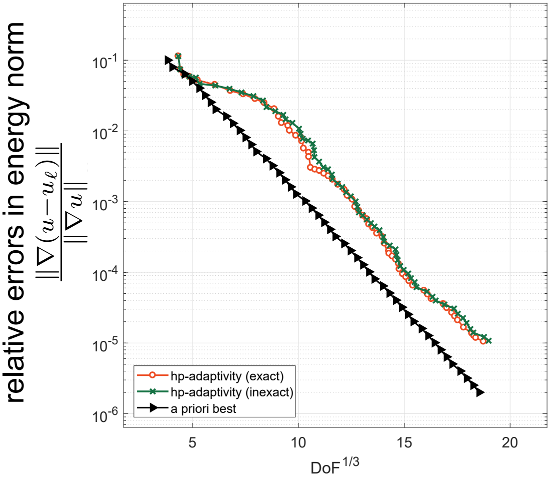




Numerics IV. – exponential convergence retained



✓ Exponential convergence numerically observed also with inexact solvers

Outline

1. An adaptive hp-refinement strategy **with exact solver**
2. An adaptive hp-refinement strategy **with inexact solver**
3. Convergence of adaptive hp-refinement strategies

Theoretical analysis – convergence proofs (exact setting)

Goal

- ensure **error reduction on each step** of the adaptive loop, i.e.

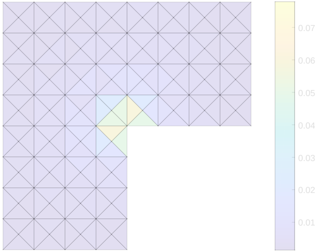
$$\|\nabla(u-u_{\ell+1}^{\mathrm{ex}})\| \leq C_{\ell,\mathrm{red}} \|\nabla(u-u_\ell^{\mathrm{ex}})\| \text{ and } 0 \leq C_{\ell,\mathrm{red}} \leq C_{\theta,d,\kappa_{\mathcal{T}},p_{\max}} < 1$$

→ **convergence** of the hp-adaptive algorithm

$$\lim_{\ell\to\infty} \|\nabla(u-u_\ell^{\mathrm{ex}})\| = 0$$

Changes

1. extension of the marked region by **one extra layer** of elements → $\mathcal{V}_\ell^\sharp$



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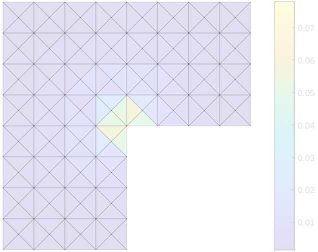

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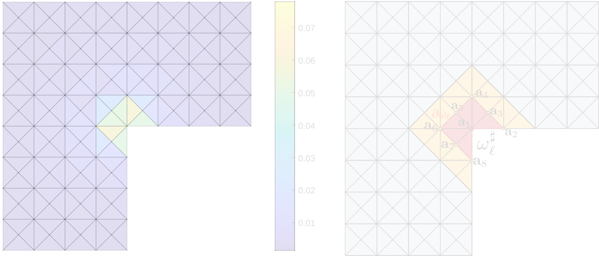

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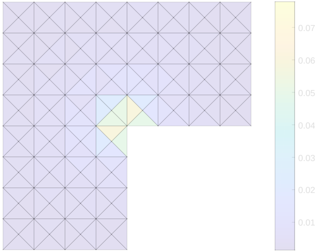

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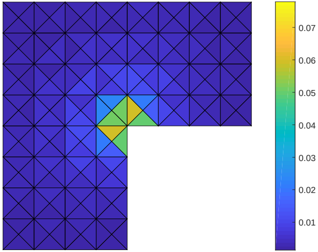

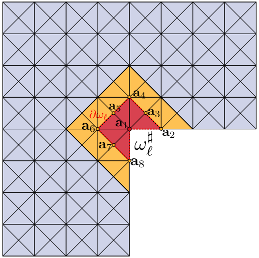

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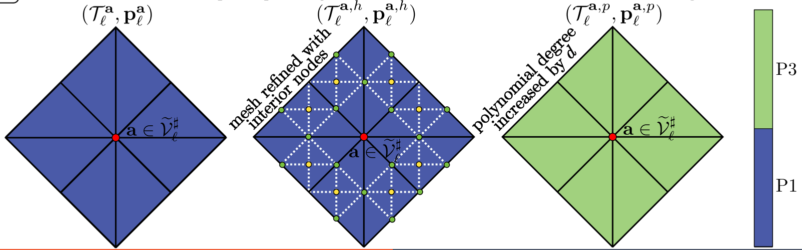
2. $\boxed{\texttt{REFINE}}$: **interior node property** for h-refinement and **stronger** p-refinement



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3. **restriction on maximal polynmial degree**

$$p_{\ell,K} \leq p_{\max}, \ \forall K \in \mathcal{T}_\ell, \ \ell \geq 0$$

Discrete Stability (DS) of the local flux equilibration (exact setting)

Let

- the hat function orthogonality $(f, \psi_\ell^{\mathbf{a}})_{\omega_\ell^{\mathbf{a}}} - (\nabla u_\ell^{\mathrm{ex}}, \nabla \psi_\ell^{\mathbf{a}})_{\omega_\ell^{\mathbf{a}}} = 0 \qquad \forall \mathbf{a} \in \mathcal{V}_\ell^{\mathrm{int}}$
- the local equilibrated flux $\boldsymbol{\sigma}_\ell^{\mathbf{a}}$ be constructed by the local minimization
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Then there holds:
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Bubble function technique

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Main contributions and future perspectives

Refinement strategies

- ✓ sharp bounds on the error reduction factor
- ✓ hp-refinement decision criterion
- ✓ excellent effectivity indices of estimated quantities
- ✓ asymptotic exponential convergence numerically observed
- coarsening, hp-refinement decision taking into account the number of DOFs?
- 3D and more general 2D test cases, anisotropic h- and p-refinements?

Convergence proofs

- ✓ exact and inexact solvers
- ✓ h-robust
- ✓ theoretically justified & practically reasonable stopping criteria
- p-robust version of the proofs avoiding bubble functions?
- computational (quasi)-optimality?

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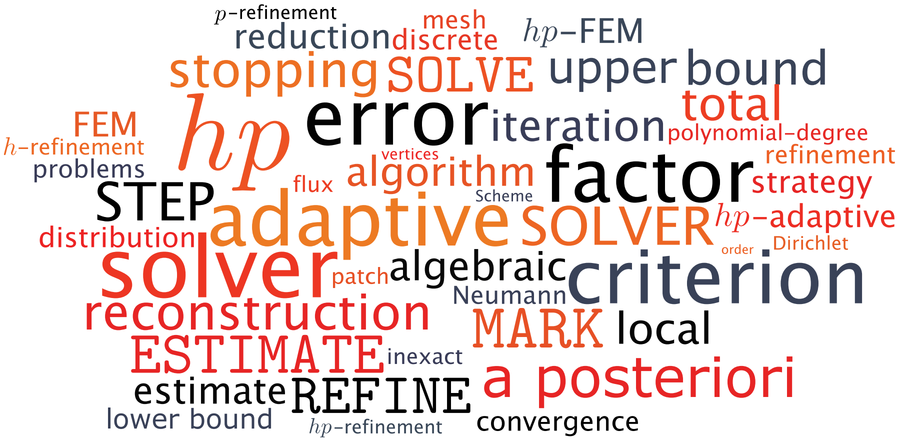
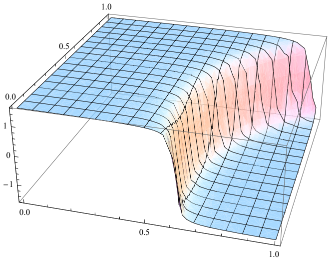


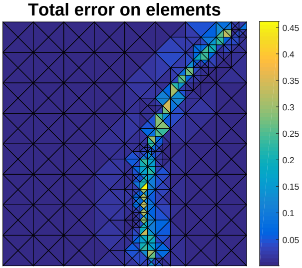
Thank you for your attention!

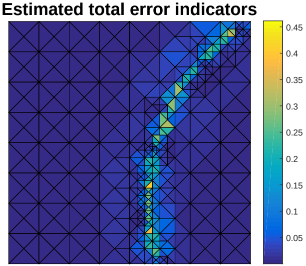
Asymmetric wave front problem: $\Omega := (0,1) \times (0,1)$

Exact solution: $u(r,\varphi) = \arctan\left(100(r - r_0)\right)$

- $r_0 = 0.92$
- $r = \sqrt{(x - x_c)^2 + (y - y_c)^2}$, $x_c = 1.5$, $y_c = 0.25$





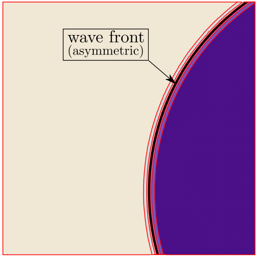


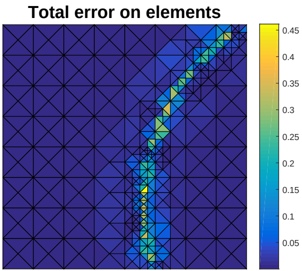
Additional numerical experiments (**inexact setting**)

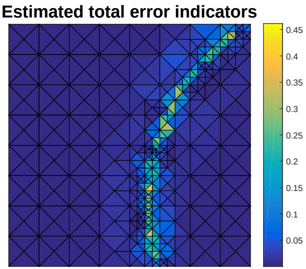
Asymmetric wave front problem: $\Omega := (0,1) \times (0,1)$

Exact solution: $u(r,\varphi) = \arctan\left(100(r - r_0)\right)$

- $r_0 = 0.92$
- $r = \sqrt{(x-x_c)^2 + (y-y_c)^2}$, $x_c = 1.5$, $y_c = 0.25$







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