A multilevel algebraic error estimator and the corresponding iterative solver with *p*-robust behavior*

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ANI MIRAÇI , JAN PAPEŽ, MARTIN VOHRALÍK
Inria Paris & École des Ponts, France

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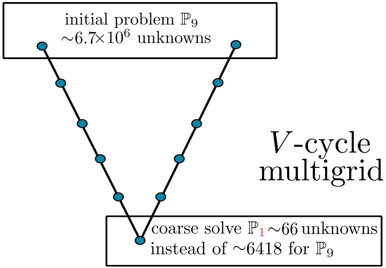

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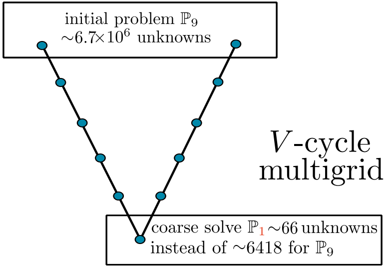

References

- Hackbusch. “Multi-grid methods and applications”. 1985.
- Pavarino. “Additive Schwarz methods for the *p*-version finite element method”. 1994.
- Schöberl et al. “Additive Schwarz preconditioning for *p*-version triangular and tetrahedral finite elements”. 2008.
- Kanschat. “Robust smoothers for high-order discontinuous Galerkin discretizations of advection-diffusion problems”. 2008.
- Antonietti et al. “A uniform additive Schwarz preconditioner for high-order discontinuous Galerkin approximations of elliptic problems”. 2017.
- Antonietti and Pennesi. “V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes”. 2019.
- Sundar, Stadler, and Biros. “Comparison of multigrid algorithms for high-order continuous finite element discretizations”. 2015.

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- Numerical results

Finite element discretization, algebraic system

Setting: $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, an open bounded polytope, $f \in L^2(\Omega)$ a source term.

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$$V_J^p = \mathbb{P}_p(\mathcal{T}_J) \cap H_0^1(\Omega),$$

where $\mathbb{P}_p(\mathcal{T}_J) = \{v_J \in L^2(\Omega), v_J \in \mathbb{P}_p(K)\ \forall K \in \mathcal{T}_J\}$.

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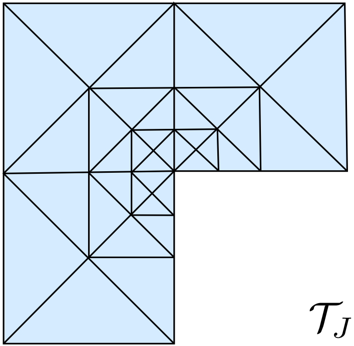

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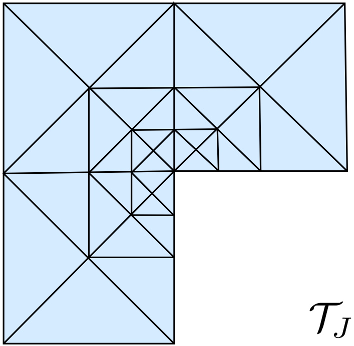

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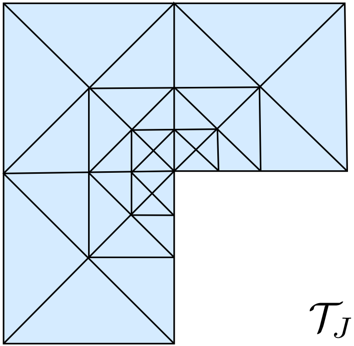

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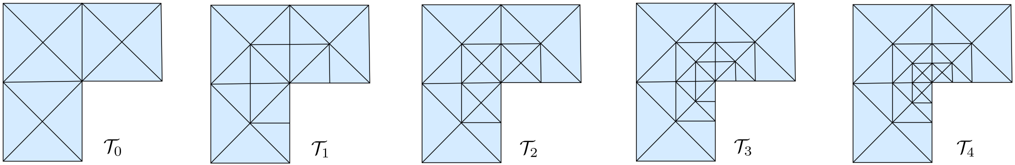
We work with the *basis-independent* functional formulation (FE).

A hierarchy of meshes

Assumptions on $\{\mathcal{T}_j\}_{0\le j\le J}$

- *Shape regularity*: The ratio element diameter over the diameter of the largest ball inscribed in the element is bounded for all elements by $\kappa_{\mathcal{T}} > 0$.
- *Strength of refinement*: For any $j \in \{1, \dots, J\}$, and for all $K \in \mathcal{T}_{j-1}$, $K^* \in \mathcal{T}_j$, such that $K^* \subset K$, h_{K^*} and h_K are comparable.

Example: A mesh hierarchy with $J = 4$



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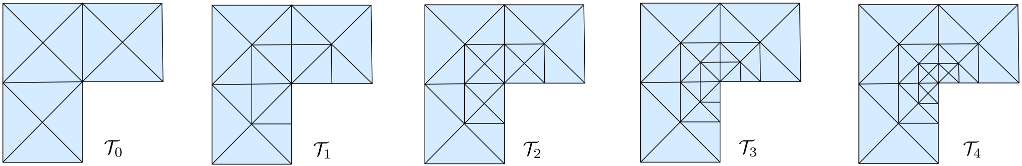
Example: A mesh hierarchy with $J = 4$, associated spaces with $p' \in \{1,\dots,p\}$



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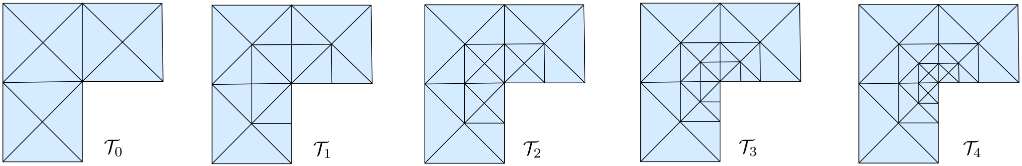

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Note: We can have very general meshes (*highly refined meshes* are also allowed). However, our theoretical results *depend* on the number of refinements J.

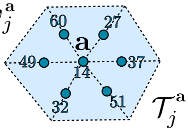
Patches

Let $\mathcal{V}_j$ be the set of vertices of the mesh $\mathcal{T}_j$, $j \in \{1, \ldots, J\}$. Given a vertex $\mathbf{a} \in \mathcal{V}_j$, we denote

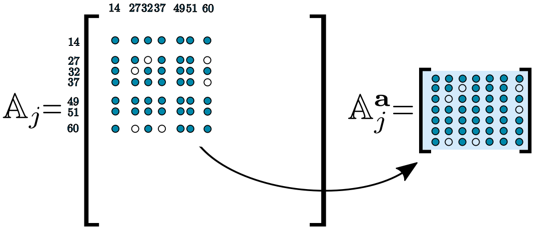
- $\mathcal{T}_j^{\mathbf{a}}$ the patch of elements sharing vertex **a**
- $\omega_j^{\mathbf{a}}$ the corresponding patch subdomain
- $V_j^{\mathbf{a}}$ the associated local space

Example: Geometric (left) and algebraic (right) representation of localizing the problem for $p' = p = 2$, $j \in \{1, \ldots, J-1\}$ and a patch composed of 6 elements:

patch subdomain $\omega_j^{\mathbf{a}}$
for a vertex $\mathbf{a} \in \mathcal{V}_j$

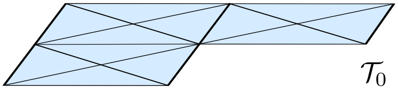


$$V_j^{\mathbf{a}} = \mathbb{P}_{p'}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$$



Multilevel lifting of the algebraic residual: Case $J = 2$

$j = 0:$



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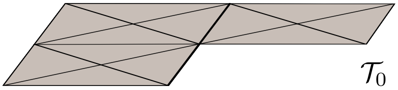
$j = 0 : \rho_0^i \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H_0^1(\Omega)$



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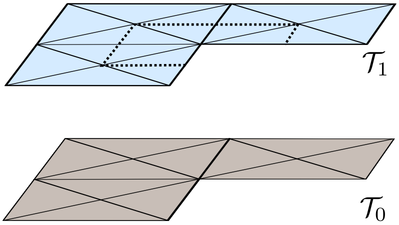
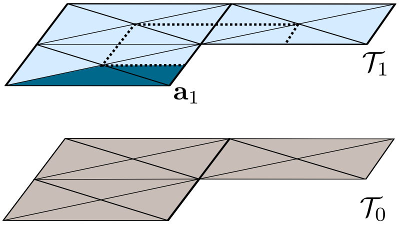


MULTILEVEL LIFTING OF THE ALGEBRAIC RESIDUAL: CASE $J = 2$

$j = 1 : \underbrace{\rho^i_{1,\mathbf{a}_1}}_{\in V_1^{\mathbf{a}_1}}$

$j = 0 : \rho^i_0 \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H^1_0(\Omega)$



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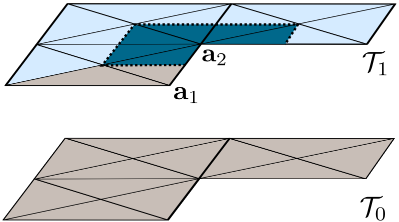

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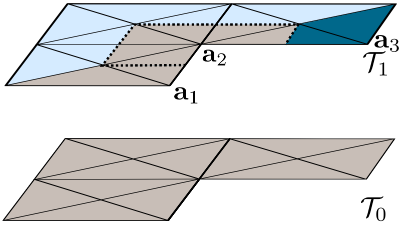

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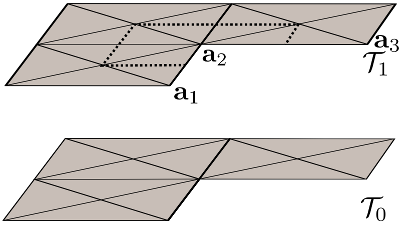

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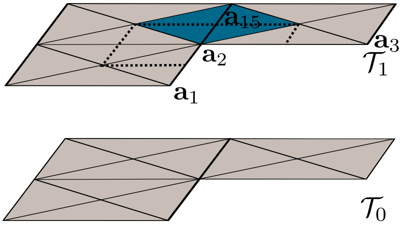

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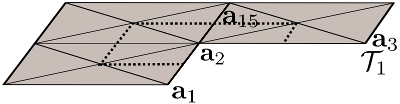

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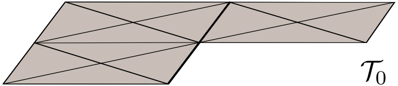

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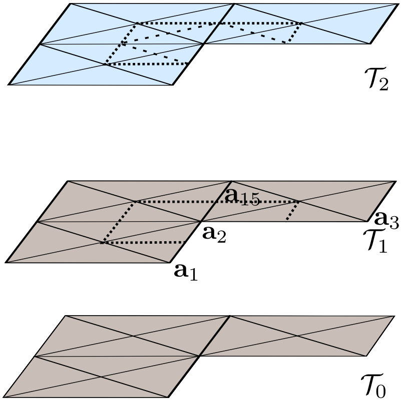

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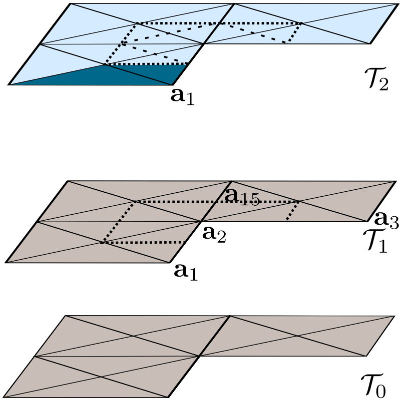

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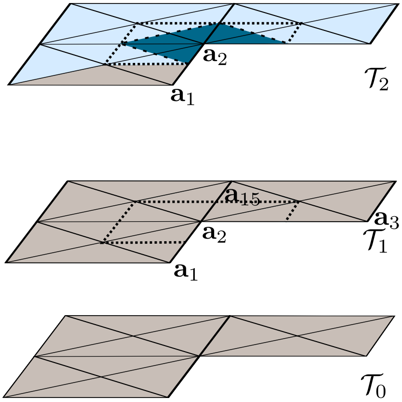

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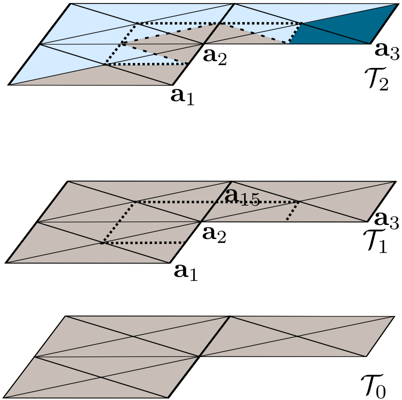

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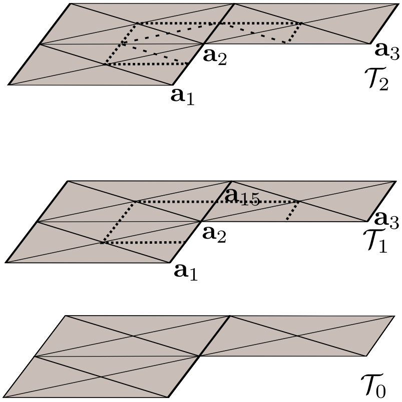

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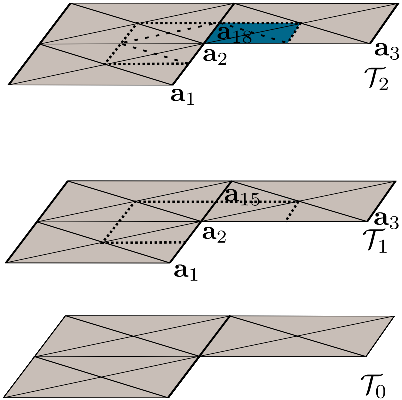
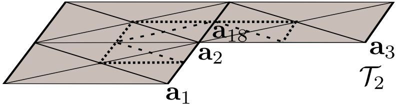
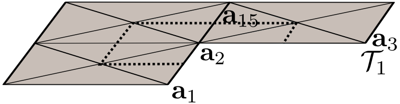


Multilevel lifting of the algebraic residual: Case $J = 2$

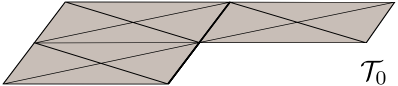
$$j = 2 : \frac{\rho^i_{2,\mathbf{a}_1} + \rho^i_{2,\mathbf{a}_2} + \rho^i_{2,\mathbf{a}_3} + \ldots + \rho^i_{2,\mathbf{a}_{18}}}{J(d+1)} \in V_2^p$$



$$j = 1 : \frac{\rho^i_{1,\mathbf{a}_1} + \rho^i_{1,\mathbf{a}_2} + \rho^i_{1,\mathbf{a}_3} + \ldots + \rho^i_{1,\mathbf{a}_{15}}}{J(d+1)} \in V_1^{p'}$$



$$j = 0 : \rho^i_0 \in V_0 = \mathbb{P}_1(\mathcal{T}_0) \cap H^1_0(\Omega)$$



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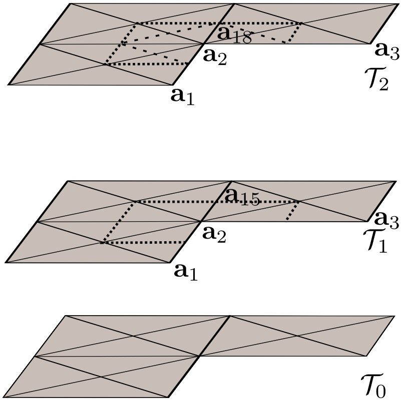

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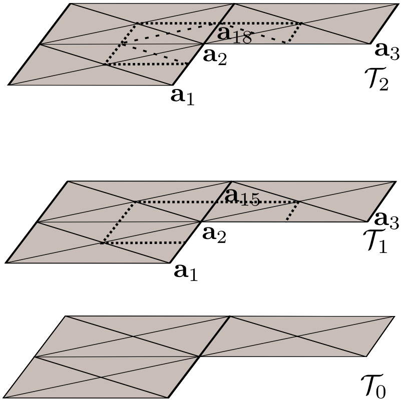

Multilevel lifting of the algebraic residual[1]

Let $u_J^i \in V_J^p$ be *arbitrary*. We define its associated *algebraic residual lifting*.

[1] Papež et al. "Sharp algebraic and total a posteriori error bounds for h and p finite elements via a multilevel approach". HAL preprint 01662944, 2017.

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Let $u_J^i \in V_J^p$ be *arbitrary*. We define its associated *algebraic residual lifting*.

Coarse solve: Define $\rho_0^i \in V_0$ by: $\quad (\nabla \rho_0^i, \nabla v_0) = (f, v_0) - (\nabla u_J^i, \nabla v_0), \quad \forall v_0 \in V_0.$

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where for all $j = \{1, \ldots, J\}$, $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$:

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Remark: $\rho_{J,\mathrm{alg}}^i$ approximates the algebraic error $u_J - u_J^i$ by

- a V-cycle MG(0,1) with piecewise affine coarse solve
- the smoother is *damped additive Schwarz* / *block Jacobi* associated to the patches

[1] Papež et al. "Sharp algebraic and total a posteriori error bounds for *h* and *p* finite elements via a multilevel approach". HAL preprint 01662944, 2017.

Definition 1 (Multilevel a posteriori estimator)

Let $u_J^i \in V_J^p$ be **arbitrary**, and let $\rho_{J,\mathrm{alg}}^i$ be the associated algebraic residual lifting.

Set $\eta_{\mathrm{alg}}^i := \dfrac{(f, \rho_{J,\mathrm{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\mathrm{alg}}^i)}{\|\nabla \rho_{J,\mathrm{alg}}^i\|}$, or else $\eta_{\mathrm{alg}}^i := 0$ if $\rho_{J,\mathrm{alg}}^i = 0$.

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Definition 2 (Multilevel solver)

1. Initialize $u_J^0 \in V_0$ as the solution of $(\nabla u_J^0, \nabla v_0) = (f, v_0), \quad \forall v_0 \in V_0$.
2. Let $i \geq 0$. Set $u_J^{i+1} := u_J^i + \dfrac{(f, \rho_{J,\mathrm{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\mathrm{alg}}^i)}{\|\nabla \rho_{J,\mathrm{alg}}^i\|^2} \rho_{J,\mathrm{alg}}^i$, or else $u_J^{i+1} := u_J^i$ if $\rho_{J,\mathrm{alg}}^i = 0$.

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- without it, the solver would become to MG(0,1) with block Jacobi smoothing

Main results

Theorem 1 (p-robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be **arbitrary**, let η_{alg}^i be the associated a posteriori estimator. There holds

- reliability: $\|\nabla(u_J - u_J^i)\| \geq \eta_{\mathrm{alg}}^i$
- efficiency: $\eta_{\mathrm{alg}}^i \geq \beta(\kappa_{\mathcal{T}}, d, J)\|\nabla(u_J - u_J^i)\|, \qquad 0 < \beta(\kappa_{\mathcal{T}}, d, J) < 1 \qquad \text{(E)}$

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Theorem 2 (p-robust error contraction of the multilevel solver)

Let $u_J^i \in V_J^p$ be **arbitrary**, let u_J^{i+1} be constructed from u_J^i using one step of the multilevel solver. Then there holds

$$\|\nabla(u_J - u_J^{i+1})\| \leq \alpha(\kappa_{\mathcal{T}}, d, J)\|\nabla(u_J - u_J^i)\|, \qquad 0 < \alpha(\kappa_{\mathcal{T}}, d, J) < 1 \qquad \text{(C)}$$

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Corollary 1 (Equivalence of the two main results)

Under the assumptions of Theorems 1 and 2, (E) holds if and only if (C) holds.

SKETCH OF THE PROOF OF THEOREM 1 : $\eta^i_{\text{alg}} \geq \beta \|\nabla(u_J - u^i_J)\|$

- Due to the definition of η^i_{alg}

$$\text{if } \rho^i_{J,\text{alg}} \neq 0 : \qquad \eta^i_{\text{alg}} = \frac{(f, \rho^i_{J,\text{alg}}) - (\nabla u^i_J, \nabla \rho^i_{J,\text{alg}})}{\|\nabla \rho^i_{J,\text{alg}}\|}$$

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Sketch of the proof of Theorem 1 : $\eta^i_{\mathrm{alg}} \ge \beta \|\nabla(u_J - u^i_J)\|$

- Due to the definition of η^i_{alg} , it is enough to show:

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Our approach consists in giving a:

1. lower bound on $(f, \rho_{J,\mathrm{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\mathrm{alg}}^i)$

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[2] Schöberl et al. "Additive Schwarz preconditioning for *p*-version triangular and tetrahedral finite elements". 2008.

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Leading to:

$$(\eta^i_{\mathrm{alg}})^2 \overset{❶❷}{\gtrsim} \|\nabla \rho^i_0\|^2 + \sum_{j=1}^{J} \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho^i_{j,\mathbf{a}}\|^2_{\omega^{\mathbf{a}}_j} \overset{❸}{\gtrsim} \|\nabla(u_J - u^i_J)\|^2$$

[2] Schöberl et al. “Additive Schwarz preconditioning for *p*-version triangular and tetrahedral finite elements”. 2008.

Sketch of the proof of Theorem 1 : $\eta^i_{\mathrm{alg}} \geq \beta \|\nabla(u_J - u^i_J)\|$

- Due to the definition of η^i_{alg}, it is enough to show:

$$\text{if } \rho^i_{J,\mathrm{alg}} \neq 0: \qquad \eta^i_{\mathrm{alg}} = \frac{(f, \rho^i_{J,\mathrm{alg}}) - (\nabla u^i_J, \nabla \rho^i_{J,\mathrm{alg}})}{\|\nabla \rho^i_{J,\mathrm{alg}}\|} \geq \beta \|\nabla(u_J - u^i_J)\|,$$

$$\text{if } \rho^i_{J,\mathrm{alg}} = 0: \qquad \eta^i_{\mathrm{alg}} = 0 = \|\nabla(u_J - u^i_J)\|.$$

  Our approach consists in giving a:

  1. lower bound on $(f, \rho^i_{J,\mathrm{alg}}) - (\nabla u^i_J, \nabla \rho^i_{J,\mathrm{alg}})$: *the damping proves to be crucial*
  2. upper bound on $\|\nabla \rho^i_{J,\mathrm{alg}}\|^2$: *rather straightforward*
  3. upper bound on $\|\nabla(u_J - u^i_J)\|^2$: *more delicate* [2]

  by the **splitting** $\|\nabla \rho^i_0\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho^i_{j,\mathbf{a}}\|^2_{\omega^{\mathbf{a}}_j}$.

Leading to:

$$\|\nabla(u_J - u^i_J)\|^2 \geq (\eta^i_{\mathrm{alg}})^2 \overset{❶❷}{\gtrsim} \|\nabla \rho^i_0\|^2 + \sum_{j=1}^J \sum_{\mathbf{a} \in \mathcal{V}_j} \|\nabla \rho^i_{j,\mathbf{a}}\|^2_{\omega^{\mathbf{a}}_j} \overset{❸}{\gtrsim} \|\nabla(u_J - u^i_J)\|^2$$

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Sketch of the proof of Theorem 1 : $\eta_{\mathrm{alg}}^i \geq \beta \|\nabla(u_J - u_J^i)\|$

- Due to the definition of η_{alg}^i , it is enough to show:

$$\text{if } \rho_{J,\mathrm{alg}}^i \neq 0: \qquad \eta_{\mathrm{alg}}^i = \frac{(f, \rho_{J,\mathrm{alg}}^i) - (\nabla u_J^i, \nabla \rho_{J,\mathrm{alg}}^i)}{\|\nabla \rho_{J,\mathrm{alg}}^i\|} \geq \beta \|\nabla(u_J - u_J^i)\|,$$

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Leading to:

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SKETCH OF THE PROOF OF THEOREM 1 : $\eta^i_{\mathrm{alg}} \geq \beta \|\nabla(u_J - u^i_J)\|$

- Due to the definition of η^i_{alg}, it is enough to show:

$$\text{if } \rho^i_{J,\mathrm{alg}} \neq 0: \qquad \eta^i_{\mathrm{alg}} = \frac{(f, \rho^i_{J,\mathrm{alg}}) - (\nabla u^i_J, \nabla \rho^i_{J,\mathrm{alg}})}{\|\nabla \rho^i_{J,\mathrm{alg}}\|} \geq \beta \|\nabla(u_J - u^i_J)\|,$$

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Corollary 2 (Equivalence error-splitting)

$$\|\nabla(u_J - u^i_J)\|^2 \approx \|\nabla \rho^i_0\|^2 + \sum_{j=1}^J \sum_{\mathbf{a}\in\mathcal{V}_j} \|\nabla \rho^i_{j,\mathbf{a}}\|^2_{\omega^{\mathbf{a}}_j}$$

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Numerical results

Consider the following problem:

L-shape domain problem: $u(r,\theta) = r^{2/3}\sin(2\theta/3)$; $\quad \Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0])$.

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We focus on testing numerically the *p*-robust behavior of our solver, a common choice for the **stopping criterion** is

$$\frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{i_{\text{stop}}}\|}{\|\mathsf{F}_J\|} \leq 10^{-5} \frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{0}\|}{\|\mathsf{F}_J\|}.$$

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We expect a p-robust solver

- to reach the above stopping criterion in a *similar number of iterations* i_{stop}
- to have *similar error contraction factors* $\|\nabla(u_J - u_J^{i+1})\|/\|\nabla(u_J - u_J^{i})\|$ at all iterations

for different polynomial degrees p, given a fixed J number of mesh levels.

Numerical results: L-shape problem

Comparing the number of iterations i_{stop} to reach the stopping criterion for **multigrid** with *Jacobi* and *Gauss-Seidel* smoothing.

						MG(0,1)	
						Jacobi	GS
J	p	DoF				i_{stop}	i_{stop}
3	1	5057				44	9
	3	46 273				-	49
	6	185 857				-	228
	9	418 753				-	586
4	1	20 481				-	9
	3	185 857				-	42
	6	744 961				-	186
	9	1 677 313				-	454
5	1	82 433				-	8
	3	744 961				-	35
	6	2 982 913				-	147
	9	6 713 857				-	333

Numerical results: L-shape problem

Comparing the number of iterations i_{stop} to reach the stopping criterion for **multigrid** with *Jacobi* and *Gauss-Seidel* smoothing.

			"small" patches			MG(0,1)	
			dAS			Jacobi	GS
J	p	DoF	i_{stop}			i_{stop}	i_{stop}
3	1	5057	76			44	9
	3	46 273	26			-	49
	6	185 857	23			-	228
	9	418 753	21			-	586
4	1	20 481	95			-	9
	3	185 857	29			-	42
	6	744 961	27			-	186
	9	1 677 313	25			-	454
5	1	82 433	112			-	8
	3	744 961	32			-	35
	6	2 982 913	31			-	147
	9	6 713 857	28			-	333

Numerical results: L-shape problem

Comparing the number of iterations i_{stop} to reach the stopping criterion for **multigrid** with *Jacobi* and *Gauss-Seidel* smoothing.

J	p	DoF	"small" patches dAS i_{stop}			MG(0,1) Jacobi i_{stop}	MG(0,1) GS i_{stop}
3	1	5057	76			44	9
	3	46 273	26			-	49
	6	185 857	23			-	228
	9	418 753	21			-	586
4	1	20 481	95			-	9
	3	185 857	29			-	42
	6	744 961	27			-	186
	9	1 677 313	25			-	454
5	1	82 433	112			-	8
	3	744 961	32			-	35
	6	2 982 913	31			-	147
	9	6 713 857	28			-	333

$1 \leq j \leq J$:

$$\rho^i_{J,\mathrm{alg}} = \rho^i_0 + \sum_{j=1}^{J} \frac{\sum_{\mathbf{a}\in\mathcal{V}_j} \rho^i_{j,\mathbf{a}}}{J(d+1)} \qquad \text{(dAS)}$$

$$\rho^i_{J,\mathrm{alg}} = \rho^i_0 + \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \mathcal{I}^p_j(\psi^{\mathbf{a}}_j \rho^i_{j,\mathbf{a}}), \qquad \text{(wRAS)}$$

Numerical results: L-shape problem

Comparing the number of iterations i_{stop} to reach the stopping criterion for **multigrid** with *Jacobi* and *Gauss-Seidel* smoothing.

			"small" patches			MG(0,1)	
			dAS			Jacobi	GS
J	p	DoF	i_{stop}			i_{stop}	i_{stop}
3	1	5057	76			44	9
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- $\mathcal{I}^p_j$ is the $\mathbb{P}^p$ Lagrange interpolation operator on mesh level j
- For vertex $\mathbf{a} \in \mathcal{V}_j$, we denote the associated hat function by $\psi^{\mathbf{a}}_j$

NUMERICAL RESULTS: L-SHAPE PROBLEM

Comparing the number of iterations i_{stop} to reach the stopping criterion for **multigrid** with *Jacobi* and *Gauss-Seidel* smoothing.

			"small" patches			MG(0,1)	
			dAS	wRAS		Jacobi	GS
J	p	DoF	i_{stop}	i_{stop}		i_{stop}	i_{stop}
3	1	5057	76	17		44	9
	3	46 273	26	12		-	49
	6	185 857	23	10		-	228
	9	418 753	21	10		-	586
4	1	20 481	95	18		-	9
	3	185 857	29	12		-	42
	6	744 961	27	10		-	186
	9	1 677 313	25	9		-	454
5	1	82 433	112	17		-	8
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	6	2 982 913	31	9		-	147
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			"small" patches		"big" patches	MG(0,1)	
			dAS	wRAS	wRAS	Jacobi	GS
J	p	DoF	i_{stop}	i_{stop}	i_{stop}	i_{stop}	i_{stop}
3	1	5057	76	17	8	44	9
	3	46 273	26	12	5	-	49
	6	185 857	23	10	5	-	228
	9	418 753	21	10	5	-	586
4	1	20 481	95	18	8	-	9
	3	185 857	29	12	5	-	42
	6	744 961	27	10	5	-	186
	9	1 677 313	25	9	5	-	454
5	1	82 433	112	17	8	-	8
	3	744 961	32	12	5	-	35
	6	2 982 913	31	9	5	-	147
	9	6 713 857	28	8	4	-	333

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COMPARISON WITH OTHER MULTILEVEL SOLVERS

We compare our methods with 4 well-established options (motivated by literature[3][4][5]) in terms of the number of iterations (and CPU times[6]).

[3] Antonietti and Pennesi. “V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes”. 2019.

[4] Botti et al. “h-multigrid agglomeration based solution strategies for discontinuous Galerkin discretizations of incompressible flow problems”. 2017.

[5] Schöberl. “C++11 Implementation of Finite Elements in NGSolve”. 2014.

[6] The experiments were run on one **Dell C6220** dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée “NEF” computation cluster, however, in a sequential Matlab script.

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J	p	DoF	wRAS $1, p \to p$ i_{stop}	wRAS $1, p \to p$ time				
3	1	5057	17	0.0 s				
	3	46 273	12	0.2 s				
	6	185 857	10	1.5 s				
	9	418 753	10	7.2 s				
4	1	20 481	18	0.0 s				
	3	185 857	12	1.0 s				
	6	744 961	10	8.4 s				
	9	1 677 313	9	29.7 s				
5	1	82 433	17	0.2 s				
	3	744 961	12	3.4 s				
	6	2 982 913	9	24.3 s				
	9	6 713 857	8	2.2m				

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J	p	DoF	i_{stop}	time	i_{stop}	time
3	1	5057	17	0.0 s	17	0.0 s
	3	46 273	12	0.2 s	18	0.2 s
	6	185 857	10	1.5 s	15	1.7 s
	9	418 753	10	7.2 s	14	7.7 s
4	1	20 481	18	0.0 s	18	0.0 s
	3	185 857	12	1.0 s	18	1.0 s
	6	744 961	10	8.4 s	15	7.5 s
	9	1 677 313	9	29.7 s	13	36.1 s
5	1	82 433	17	0.2 s	17	0.2 s
	3	744 961	12	3.4 s	17	3.6 s
	6	2 982 913	9	24.3 s	14	26.8 s
	9	6 713 857	8	2.2m	12	2.2m

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J	p	DoF	wRAS $1, p \to p$ i_{stop}	wRAS $1, p \to p$ time	wRAS_1 $1 \to 1, p$ i_{stop}	wRAS_1 $1 \to 1, p$ time	PCG(MG (3,3)-bJ) $p \to p$ i_{stop}	PCG(MG (3,3)-bJ) $p \to p$ time
3	1	5057	17	0.0 s	17	0.0 s	7	0.0 s
	3	46 273	12	0.2 s	18	0.2 s	3	0.2 s
	6	185 857	10	1.5 s	15	1.7 s	2	2.0 s
	9	418 753	10	7.2 s	14	7.7 s	2	10.5 s
4	1	20 481	18	0.0 s	18	0.0 s	8	0.1 s
	3	185 857	12	1.0 s	18	1.0 s	3	0.8 s
	6	744 961	10	8.4 s	15	7.5 s	3	11.4 s
	9	1 677 313	9	29.7 s	13	36.1 s	2	30.3 s
5	1	82 433	17	0.2 s	17	0.2 s	8	0.3 s
	3	744 961	12	3.4 s	17	3.6 s	3	3.6 s
	6	2 982 913	9	24.3 s	14	26.8 s	3	38.9 s
	9	6 713 857	8	2.2m	12	2.2m	2	3.5m

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			wRAS $1, p \to p$		wRAS_1 $1 \to 1, p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		
J	p	DoF	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time	
3	1	5057	17	0.0 s	17	0.0 s	7	0.0 s	4	0.1 s	
	3	46 273	12	0.2 s	18	0.2 s	3	0.2 s	14	0.5 s	
	6	185 857	10	1.5 s	15	1.7 s	2	2.0 s	21	7.6 s	
	9	418 753	10	7.2 s	14	7.7 s	2	10.5 s	63	1.2m	
4	1	20 481	18	0.0 s	18	0.0 s	8	0.1 s	7	0.1 s	
	3	185 857	12	1.0 s	18	1.0 s	3	0.8 s	29	4.1 s	
	6	744 961	10	8.4 s	15	7.5 s	3	11.4 s	49	58.9 s	
	9	1 677 313	9	29.7 s	13	36.1 s	2	30.3 s	167	12.5m	
5	1	82 433	17	0.2 s	17	0.2 s	8	0.3 s	19	0.8 s	
	3	744 961	12	3.4 s	17	3.6 s	3	3.6 s	77	57.7 s	
	6	2 982 913	9	24.3 s	14	26.8 s	3	38.9 s	129	11.6m	
	9	6 713 857	8	2.2m	12	2.2m	2	3.5m	+200	+1.0 h	

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			wRAS $1, p \to p$		wRAS_1 $1 \to 1, p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$	
J	p	DoF	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time
3	1	5057	17	0.0 s	17	0.0 s	7	0.0 s	4	0.1 s	9	0.0 s
	3	46 273	12	0.2 s	18	0.2 s	3	0.2 s	14	0.5 s	8	1.0 s
	6	185 857	10	1.5 s	15	1.7 s	2	2.0 s	21	7.6 s	7	2.4 s
	9	418 753	10	7.2 s	14	7.7 s	2	10.5 s	63	1.2m	6	7.4 s
4	1	20 481	18	0.0 s	18	0.0 s	8	0.1 s	7	0.1 s	9	0.0 s
	3	185 857	12	1.0 s	18	1.0 s	3	0.8 s	29	4.1 s	8	4.3 s
	6	744 961	10	8.4 s	15	7.5 s	3	11.4 s	49	58.9 s	7	11.9 s
	9	1 677 313	9	29.7 s	13	36.1 s	2	30.3 s	167	12.5m	6	29.2 s
5	1	82 433	17	0.2 s	17	0.2 s	8	0.3 s	19	0.8 s	8	0.1 s
	3	744 961	12	3.4 s	17	3.6 s	3	3.6 s	77	57.7 s	8	16.1 s
	6	2 982 913	9	24.3 s	14	26.8 s	3	38.9 s	129	11.6m	7	44.5 s
	9	6 713 857	8	2.2m	12	2.2m	2	3.5m	+200	+1.0 h	6	2.1m

[3] Antonietti and Pennesi. "V-cycle multigrid algorithms for discontinuous Galerkin methods on non-nested polytopic meshes". 2019.

[4] Botti et al. "h-multigrid agglomeration based solution strategies for discontinuous Galerkin discretizations of incompressible flow problems". 2017.

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Comparison with other multilevel solvers

We compare our methods with 4 well-established options (motivated by literature[3][4][5]) in terms of the number of iterations (and CPU times[6]).

			wRAS $1, p \to p$		wRAS$_1$ $1 \to 1, p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$		MG(3,3)-GS $1 \nearrow p$	
J	p	DoF	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time	i_{stop}	time
3	1	5057	17	0.0 s	17	0.0 s	7	0.0 s	4	0.1 s	9	0.0 s	3	0.0 s
	3	46 273	12	0.2 s	18	0.2 s	3	0.2 s	14	0.5 s	8	1.0 s	4	0.1 s
	6	185 857	10	1.5 s	15	1.7 s	2	2.0 s	21	7.6 s	7	2.4 s	9	1.6 s
	9	418 753	10	7.2 s	14	7.7 s	2	10.5 s	63	1.2m	6	7.4 s	9	4.3 s
4	1	20 481	18	0.0 s	18	0.0 s	8	0.1 s	7	0.1 s	9	0.0 s	3	0.0 s
	3	185 857	12	1.0 s	18	1.0 s	3	0.8 s	29	4.1 s	8	4.3 s	4	0.3 s
	6	744 961	10	8.4 s	15	7.5 s	3	11.4 s	49	58.9 s	7	11.9 s	5	2.9 s
	9	1 677 313	9	29.7 s	13	36.1 s	2	30.3 s	167	12.5m	6	29.2 s	8	16.0 s
5	1	82 433	17	0.2 s	17	0.2 s	8	0.3 s	19	0.8 s	8	0.1 s	3	0.1 s
	3	744 961	12	3.4 s	17	3.6 s	3	3.6 s	77	57.7 s	8	16.1 s	4	1.5 s
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- apply our method to more involved problems.

Thank you for your attention!