A-posteriori-steered and adaptive *p*-robust multigrid solvers

A THESIS PRESENTED AT THE
SORBONNE UNIVERSITY
DOCTORAL SCHOOL: MATHEMATICAL SCIENCES OF CENTRAL PARIS (ED 386)

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CONTEXT AND GOALS OF THE THESIS

Numerical simulations of physical phenomenona typically follow the steps:

MODELING → DISCRETIZATION → LINEAR ITERATIVE SOLVER

- MODELING: Partial differential equation (PDE)
- DISCRETIZATION: Finite element method (FEM)
- LINEAR ITERATIVE SOLVER: Multigrid method (MG)

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- MODELING: Partial differential equation (PDE) u
- discretization error $\|\nabla(u - u_J)\|$
- DISCRETIZATION: Finite element method (FEM) u_J
- algebraic error $\|\nabla(u_J - u_J^i)\|$
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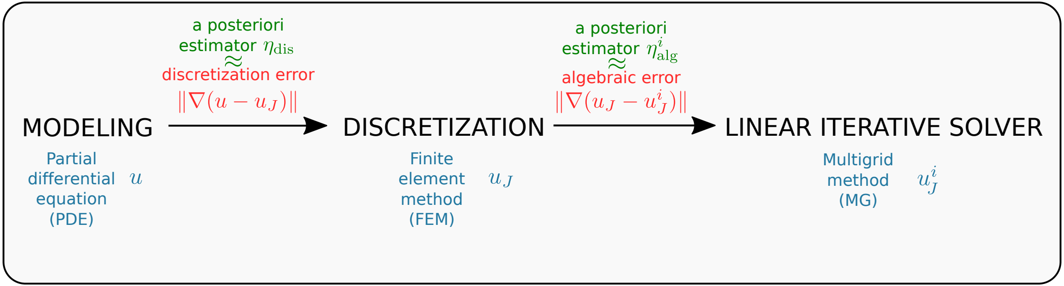

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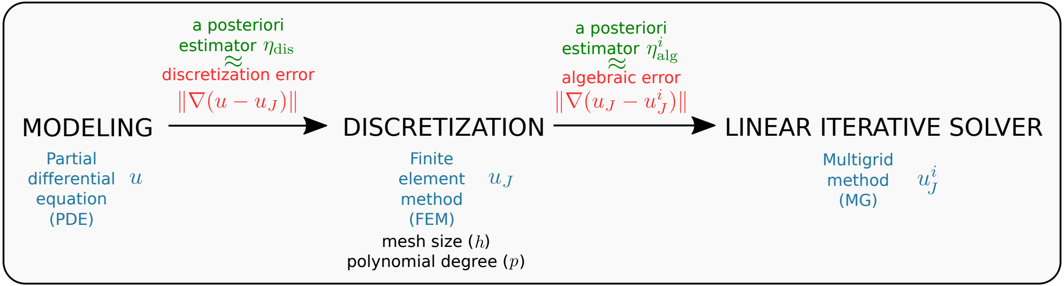

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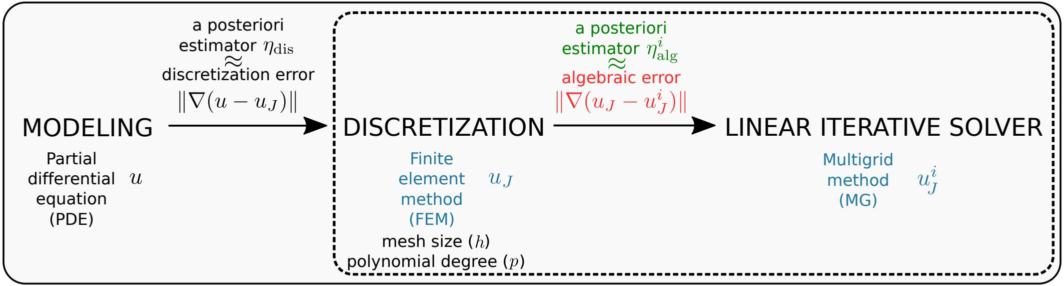

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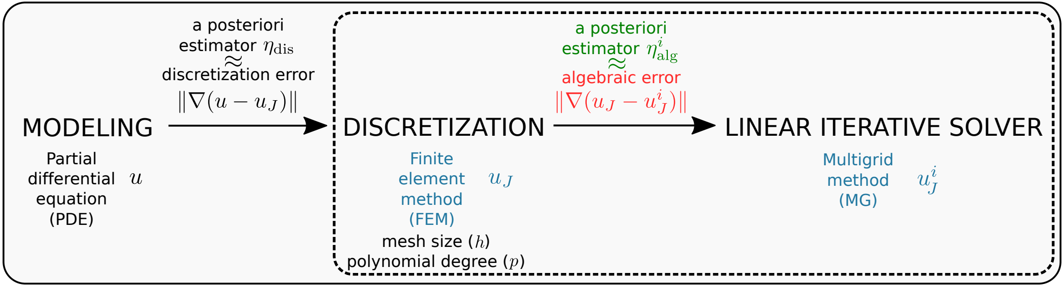

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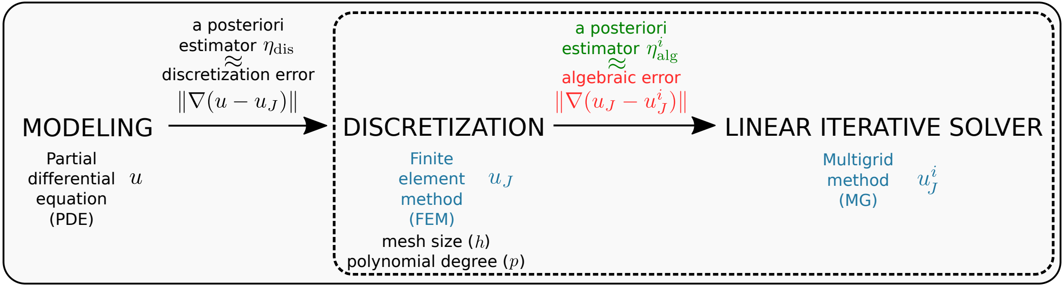

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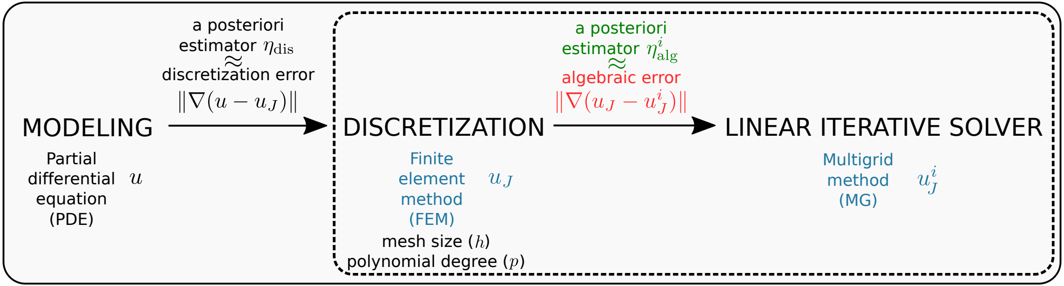

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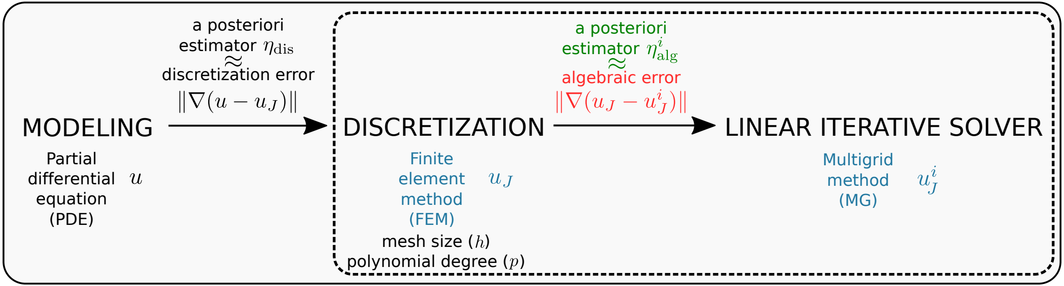

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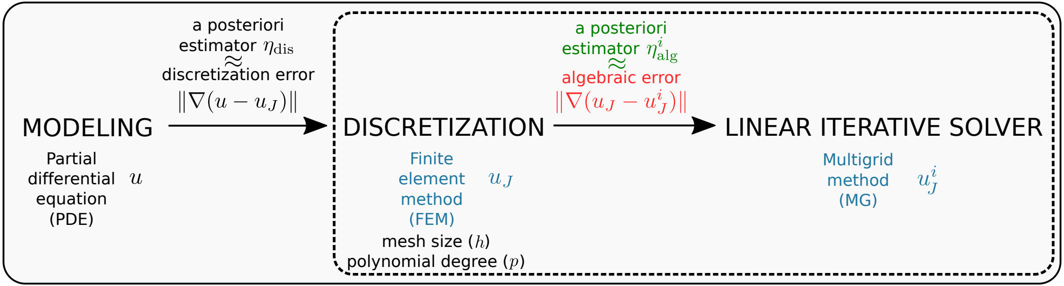

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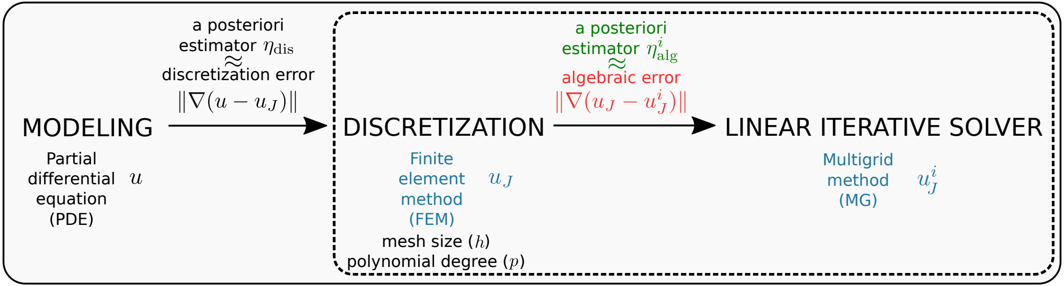

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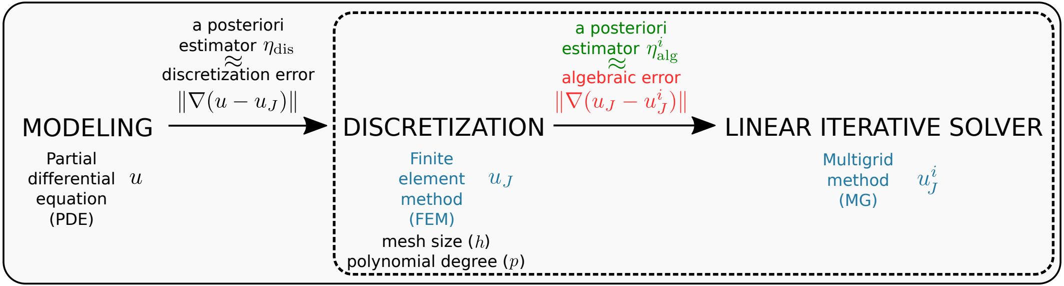

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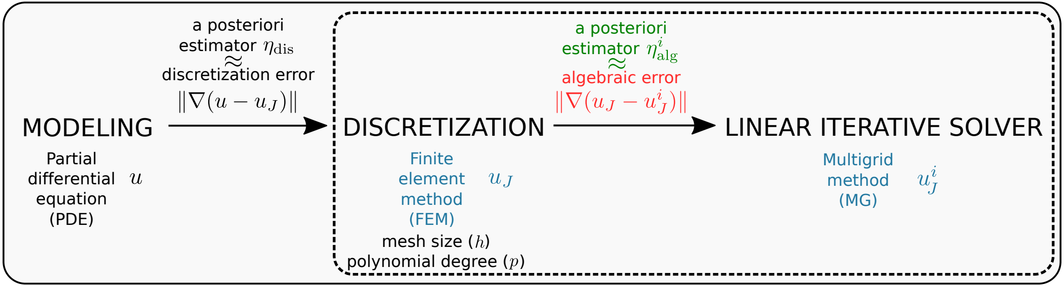

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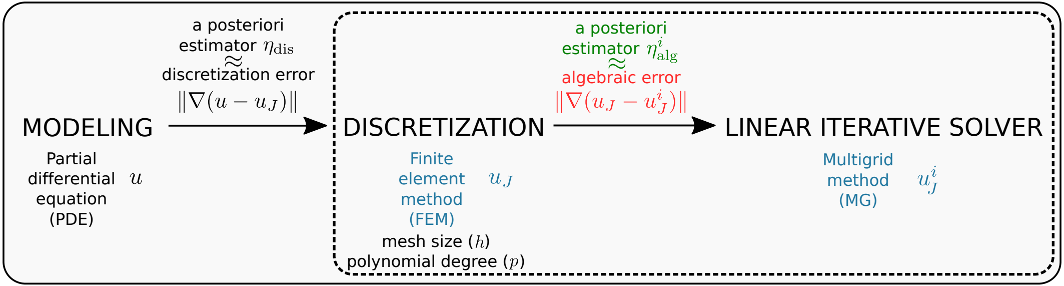

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give us a **multilevel *p*-robust stable decomposition**, *crucial* for our analysis.

Outline and contents of the thesis

- Setting

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Model problem: find $u \in H_0^1(\Omega)$ such that $(\mathbf{K}\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$.

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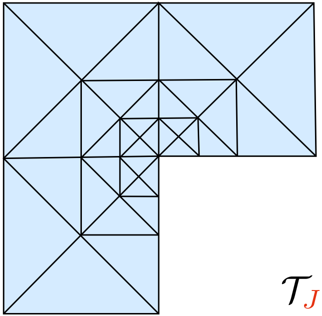

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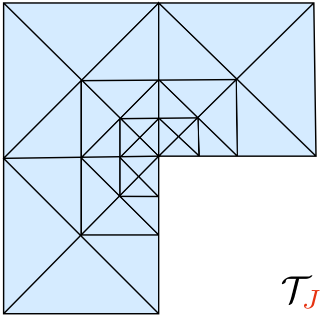

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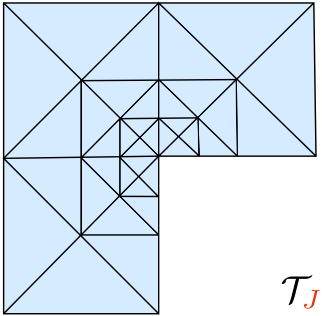

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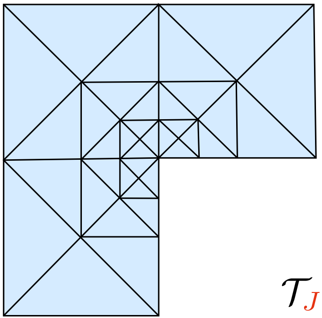

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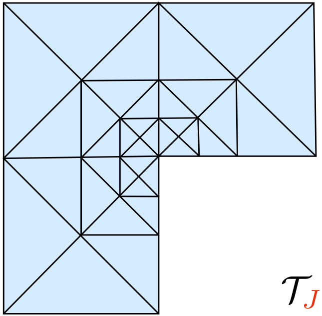

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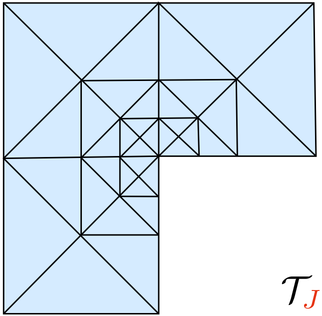

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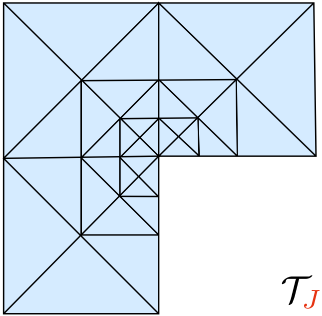

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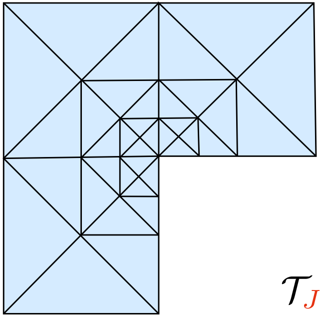

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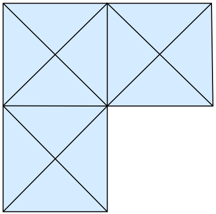
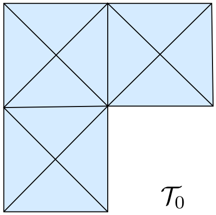
Algebraic residual functional: $v_J \mapsto (f, v_J) - (\mathbf{K}\nabla u_J^i, \nabla v_J) \in \mathbb{R}, \quad v_J \in V_J^p$.

A hierarchy of meshes

Example: Two different hierarchies with $J = 3$ refinements.





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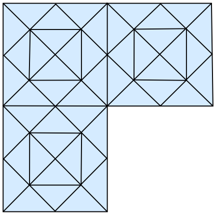

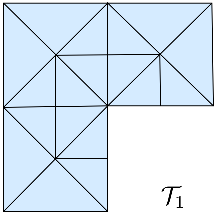

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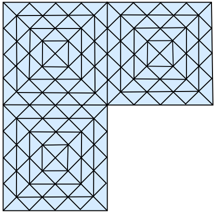

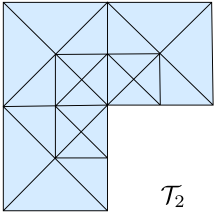

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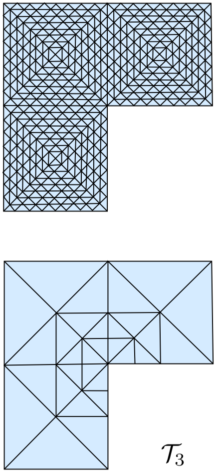

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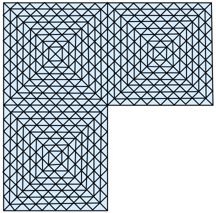

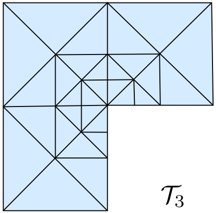

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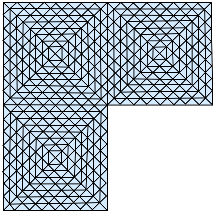

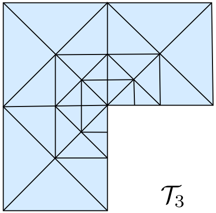

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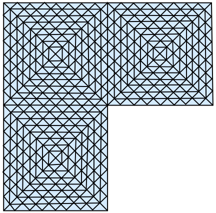
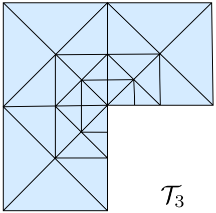
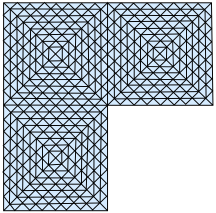




A HIERARCHY OF MESHES AND SPACES



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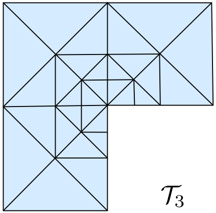

$\mathcal{T}_3$

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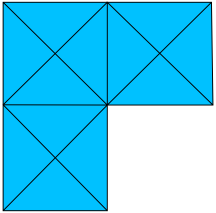

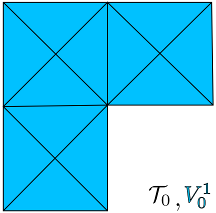

A hierarchy of meshes and spaces

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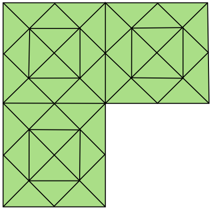

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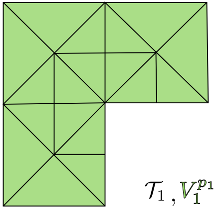

$\mathcal{T}_1$, $V_1^{p_1}$

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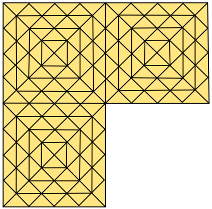

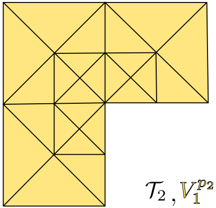

$\mathcal{T}_2, V_1^{p_2}$

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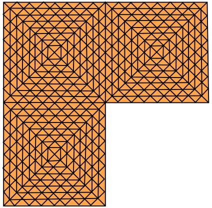

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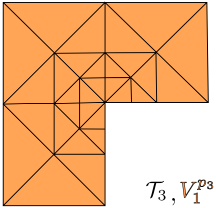

$\mathcal{T}_3$, $V_1^{p_3}$

Patches

Let $\mathcal{V}_j$ be the set of vertices of the mesh $\mathcal{T}_j$, $j \in \{1, \ldots, J\}$. Given a vertex $\mathbf{a} \in \mathcal{V}_j$, we denote

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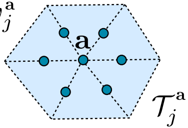
Example: Representation of localizing the problem for $p_j = 2$, $j \in \{1, \ldots, J-1\}$: geometric perspective

patch subdomain $\omega_j^{\mathbf{a}}$ for a vertex $\mathbf{a} \in \mathcal{V}_j$



$$V_j^{\mathbf{a}} = \mathbb{P}_{p_j}(\mathcal{T}_j) \cap H_0^1(\omega_j^{\mathbf{a}})$$

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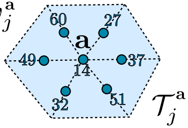
Example: Representation of localizing the problem for $p_j = 2$, $j \in \{1, \ldots, J-1\}$: geometric perspective and algebraic perspective

patch subdomain $\omega_j^{\mathbf{a}}$ for a vertex $\mathbf{a} \in \mathcal{V}_j$



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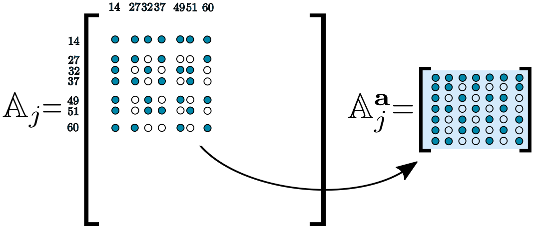

A-POSTERIORI-STEERED MULTIGRID

- V-cycle of geometric multigrid

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- zero pre- and a single post-smoothing step

A-posteriori-steered multigrid

$u_J^i \in V_J^p \qquad u_J^{i+1} = u_J^i + \sum_{j=0}^{J} \lambda_j^i \rho_j^i \in V_J^p \qquad \eta_{\mathrm{alg}}^i = \left(\sum_{j=0}^{J} (\lambda_j^i \|\nabla \rho_j^i\|)^2 \right)^{\frac{1}{2}}$

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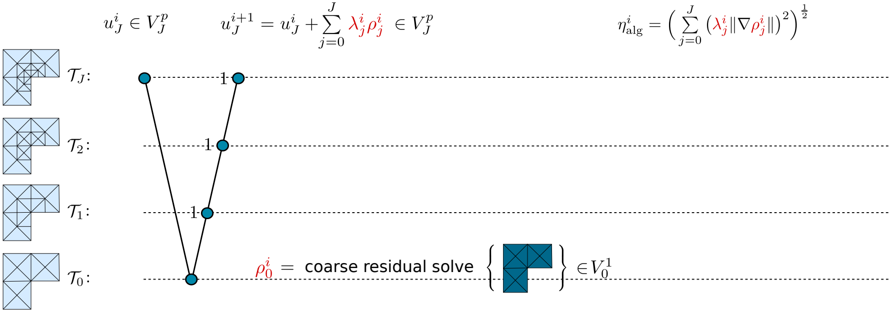

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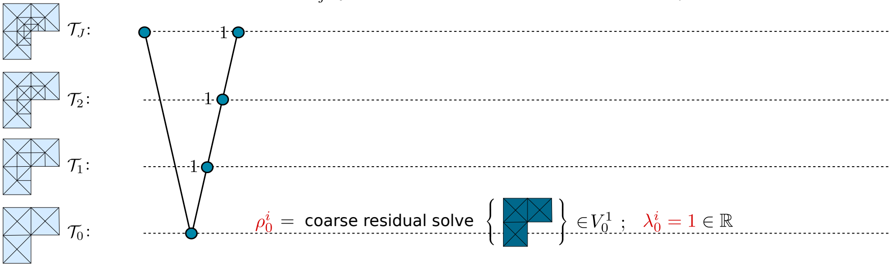

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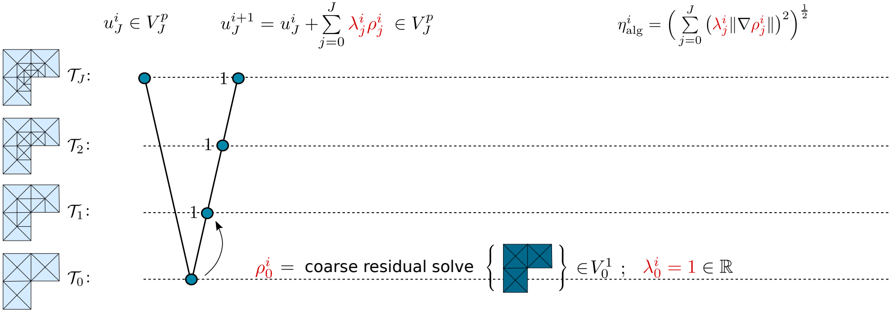
A-POSTERIORI-STEERED MULTIGRID



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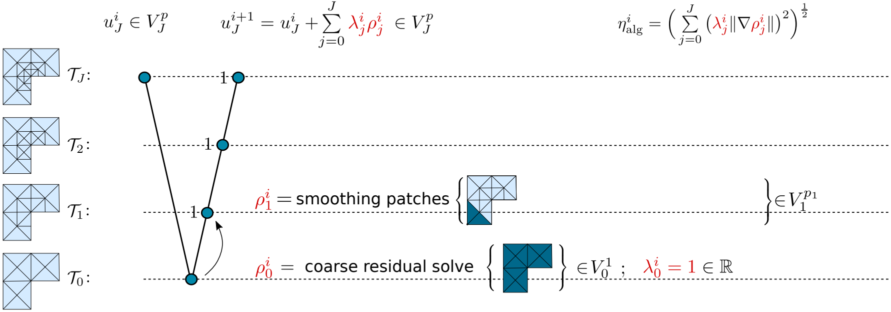

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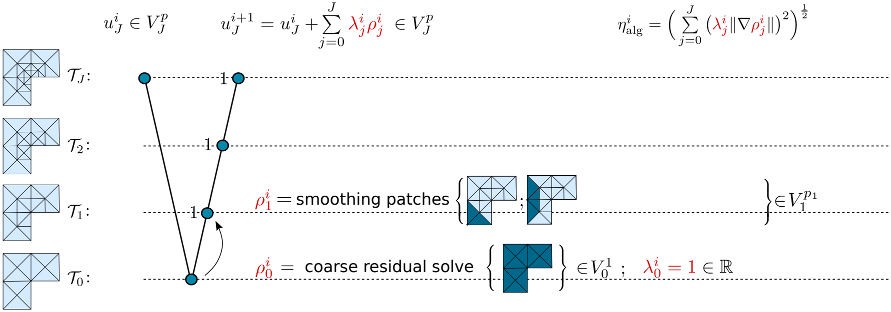

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A-POSTERIORI-STEERED MULTIGRID

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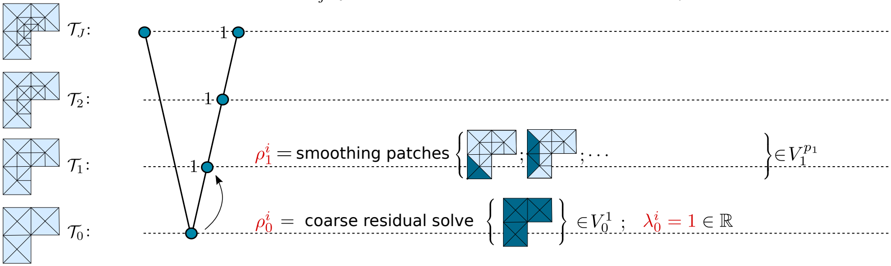

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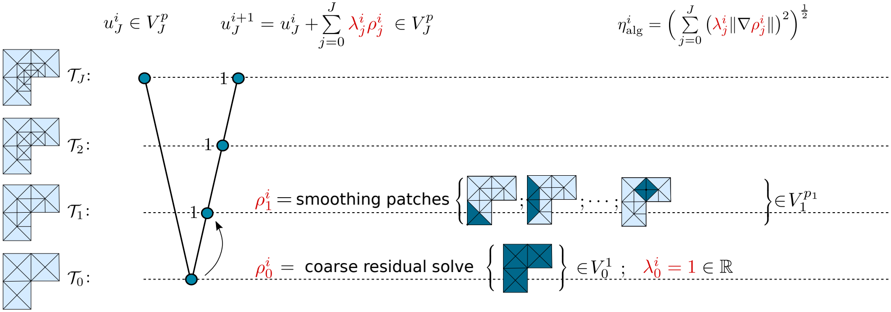

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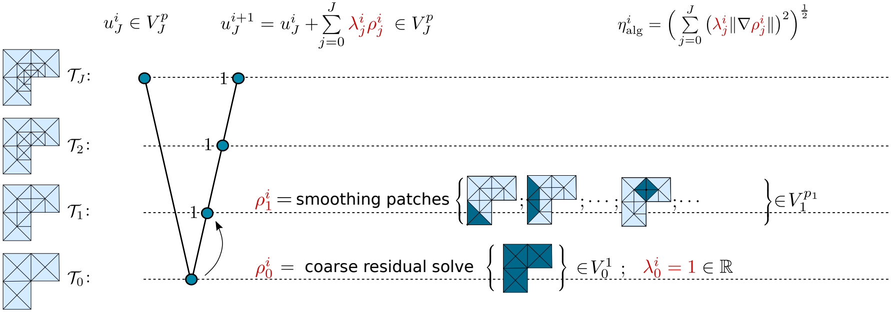
A-posteriori-steered multigrid



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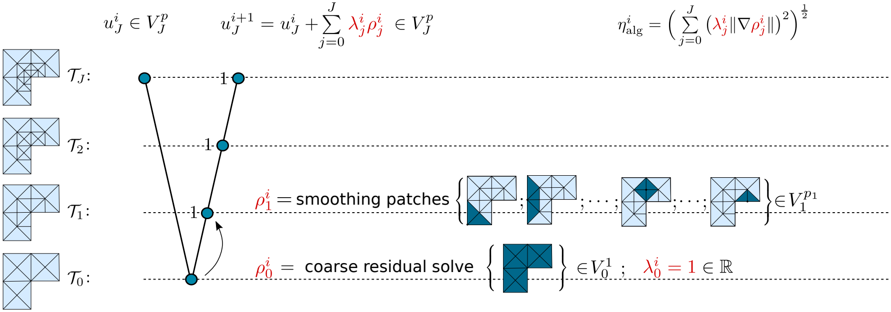

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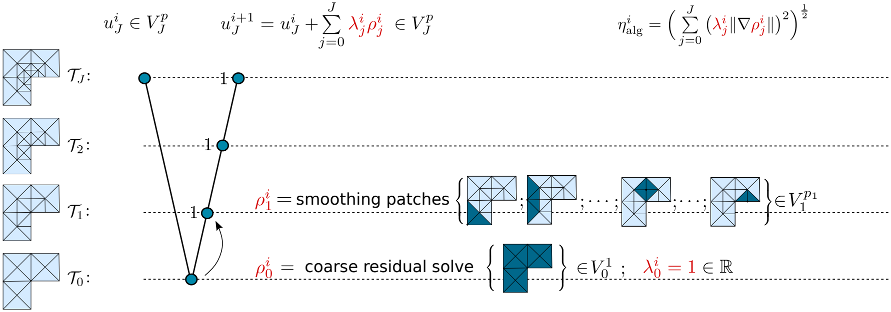

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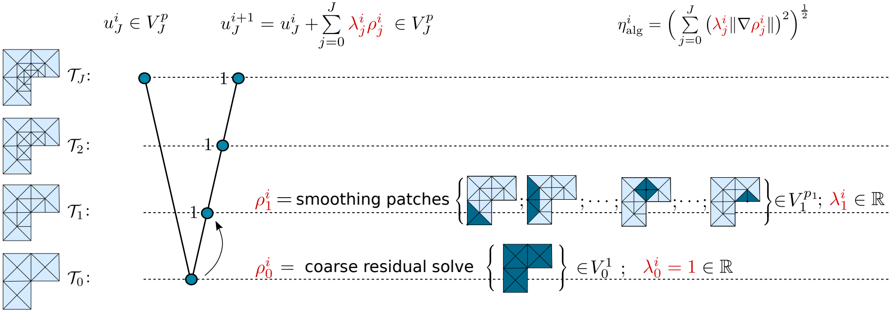

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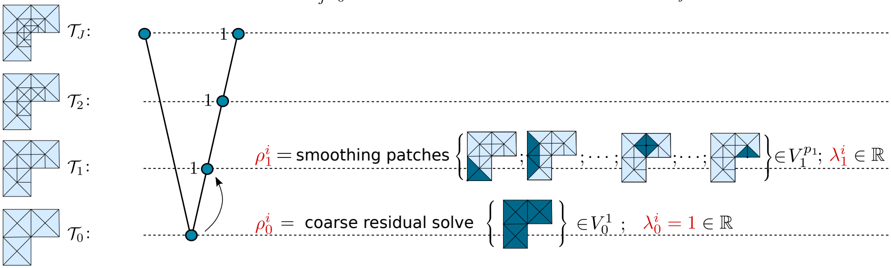

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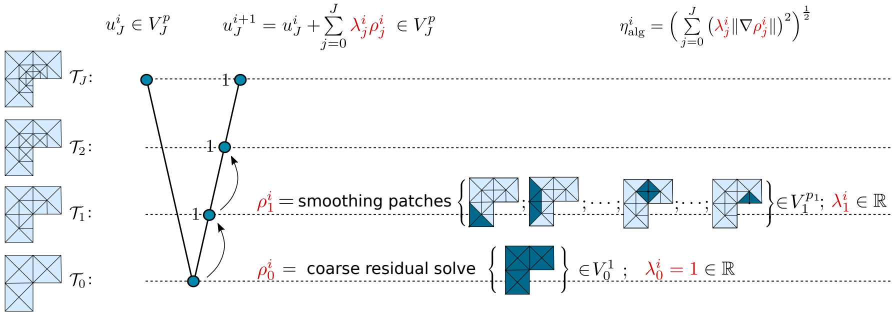

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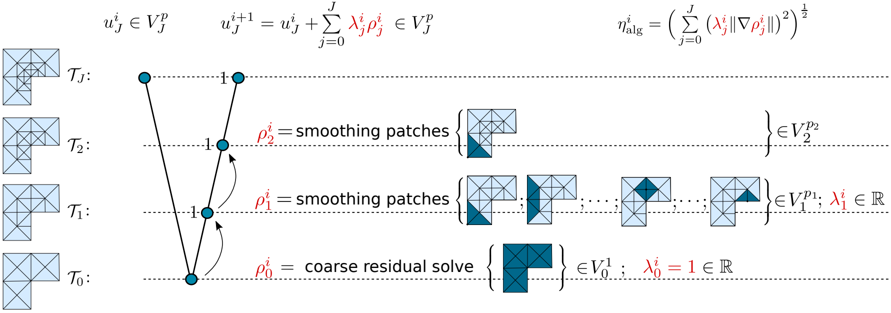

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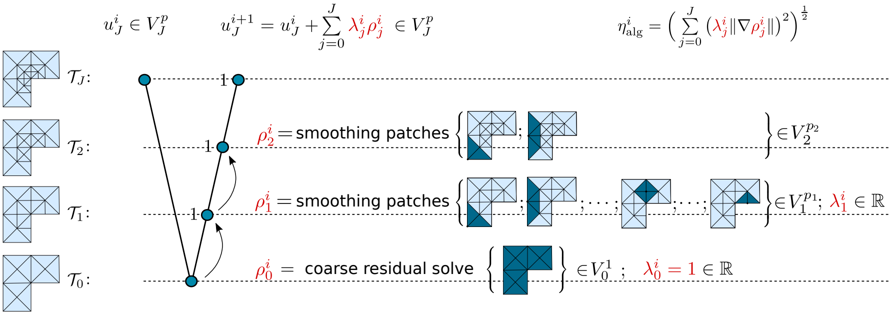

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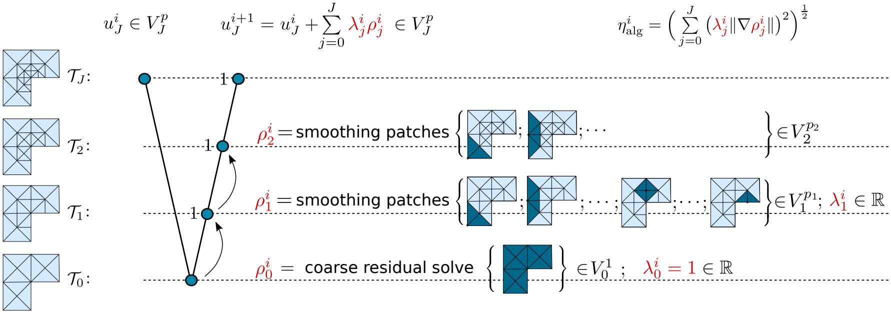

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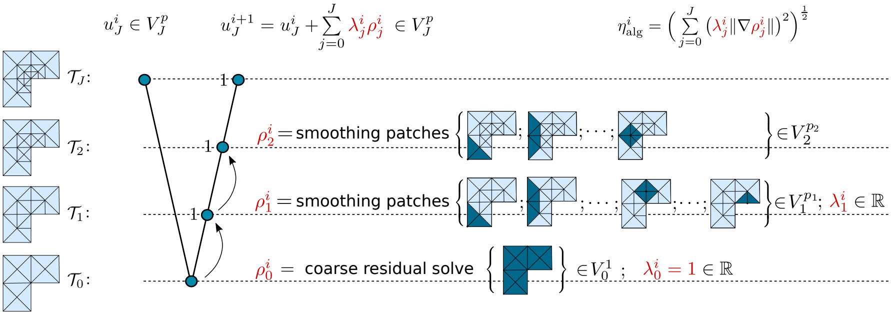
A-posteriori-steered multigrid



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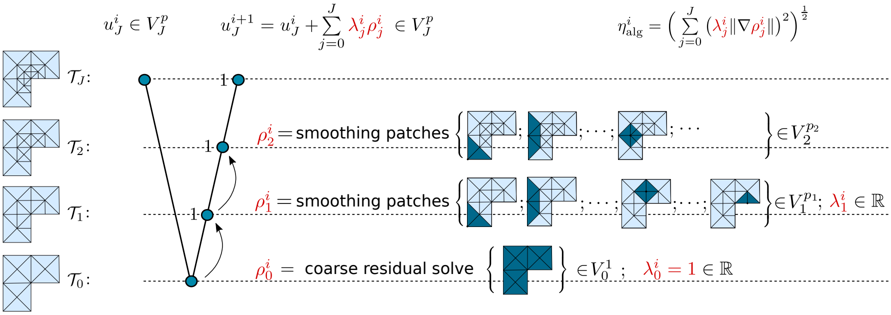

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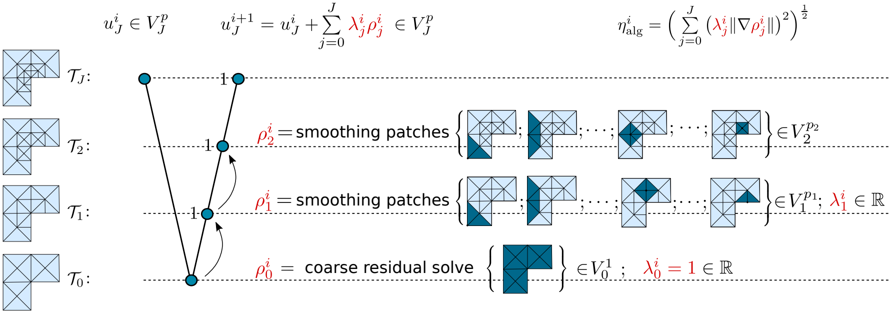

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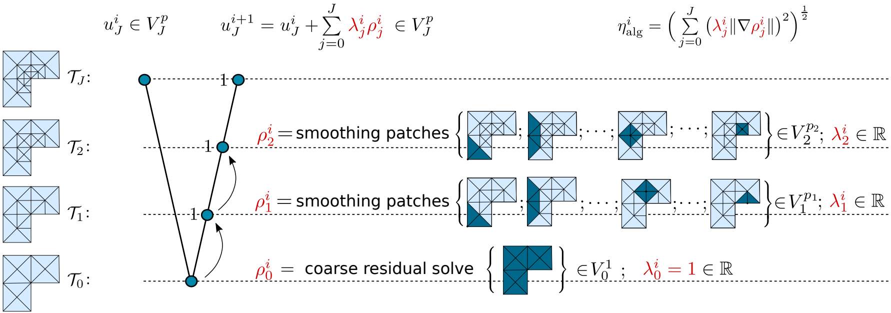

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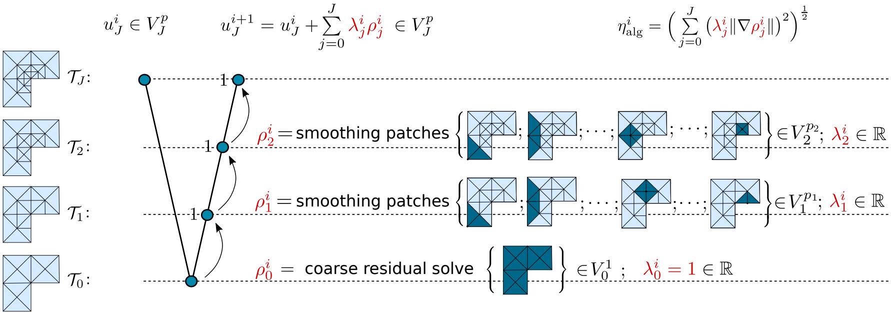

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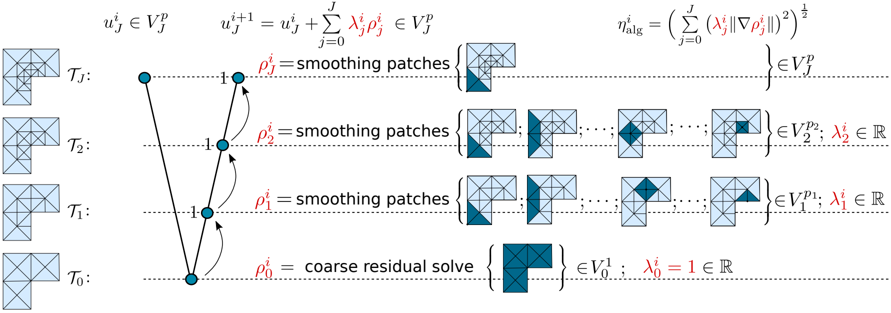

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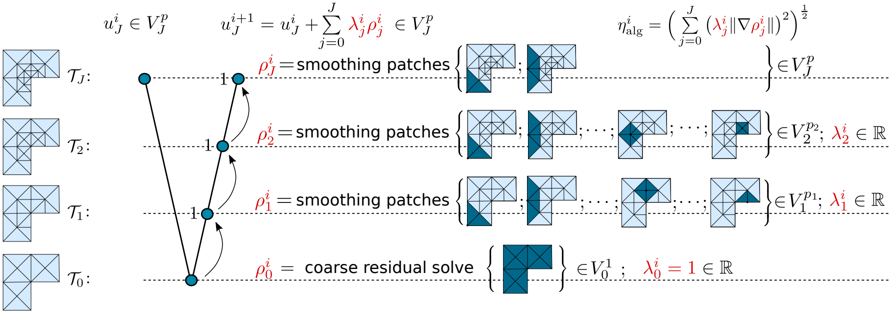

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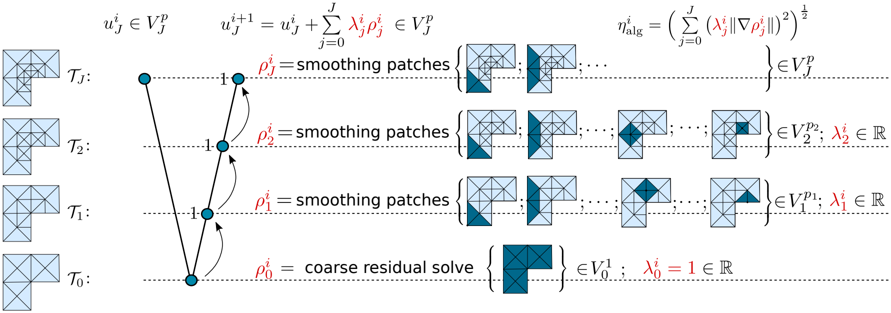

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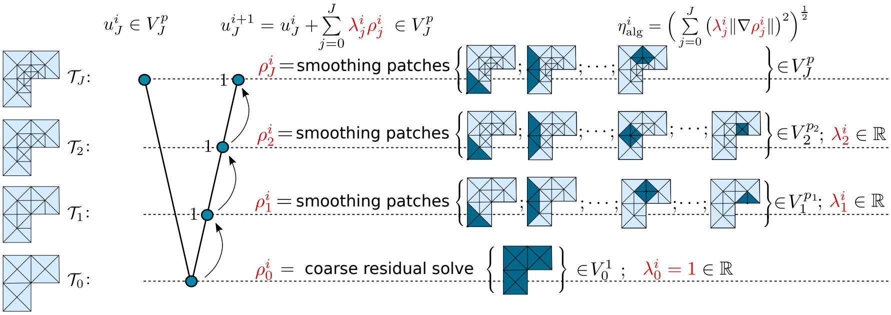

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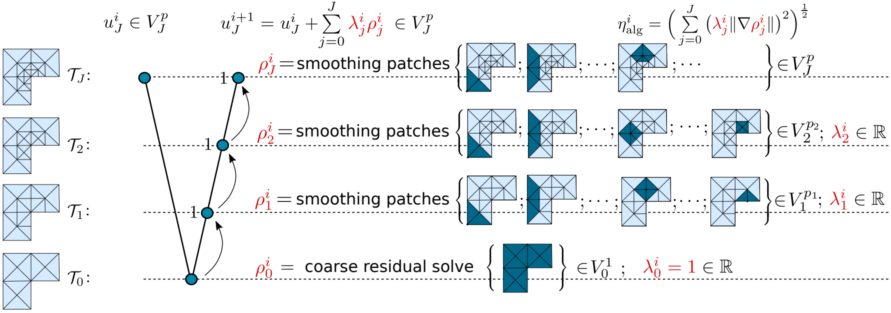

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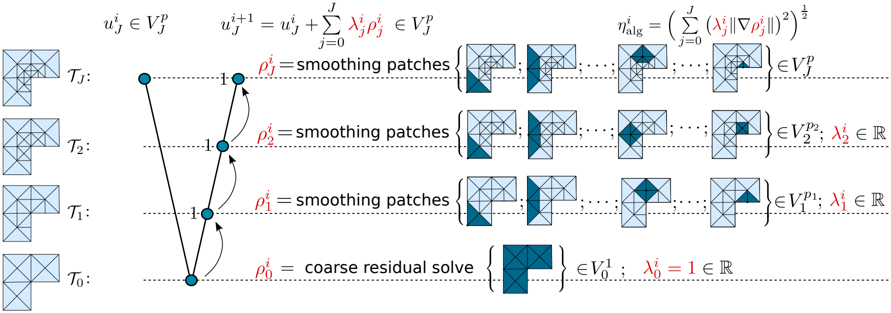

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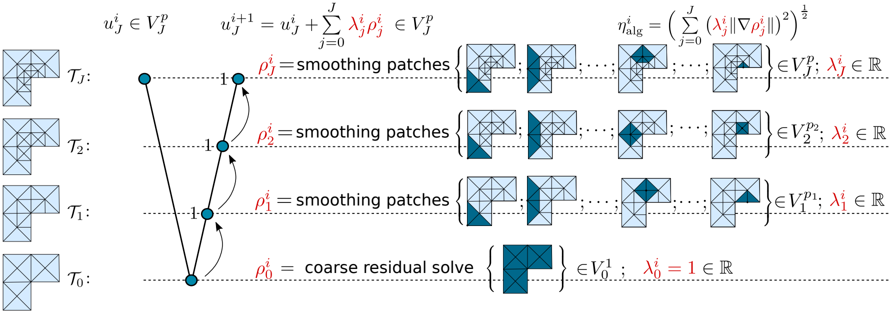

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Multilevel lifting of the algebraic residual

Let $u_J^i \in V_J^p$ be *arbitrary*. We construct its associated *level-wise algebraic residual liftings* $\{\rho_j^i\}_{j=0}^J$ and *level-wise step-sizes* $\{\lambda_j^i\}_{j=0}^J$ as follows:

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Coarse solve: Define $\rho_0^i \in V_0$ by: $(\mathbf{K}\nabla \rho_0^i, \nabla v_0) = (f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0), \quad \forall v_0 \in V_0$ and set $\lambda_0^i := 1$.

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Level-wise local solves: For $j = 1 : J,$ for all $\mathbf{a} \in \mathcal{V}_j,$ define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by :

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A POSTERIORI ESTIMATOR AND SOLVER

Definition 1 (A posteriori estimator of the algebraic error)

Let $u_J^i \in V_J^p$ be *arbitrary*. Let $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ be constructed as above. Define the a posteriori estimator of the algebraic error associated to u_J^i as

$$\eta_{\mathrm{alg}}^i := \left(\sum_{j=0}^{J} \left(\lambda_j^i \|\mathbf{K}^{\frac{1}{2}} \nabla \rho_j^i\|\right)^2 \right)^{\frac{1}{2}}.$$

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Definition 2 (A posteriori-steered solver)

Initialize $u_J^0 = 0$ and let $i = 0$. Perform the following steps:

1. Construct $\{\rho_j^i\}_{j=0}^J$ and $\{\lambda_j^i\}_{j=0}^J$ as detailed above.
2. Update the current approximation $u_J^{i+1} := u_J^i + \sum_{j=0}^J \lambda_j^i \rho_j^i$.
3. If $u_J^{i+1} = u_J^i$, then stop the solver; otherwise increase $i := i + 1$ and go to step 1.

Proposition (Pythagorean error representation of one solver step)

For $u_J^i \in V_J^p$, let $u_J^{i+1} \in V_J^p$ be the next iterate constructed from u_J^i by our solver. Then

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\right\|^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\|^2 - \sum_{j=0}^{J}\left(\lambda_j^i\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|\right)^2.$$

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Corollary (Guaranteed lower bound on the algebraic error)

There holds:

$$\eta_{\mathrm{alg}}^i \le \|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|.$$

MAIN RESULTS[2]

Theorem 1 (p-robust reliable and efficient bound on the algebraic error)

Let $u_J^i \in V_J^p$ be *arbitrary*. Let η_{alg}^i be the associated a posteriori estimator on the algebraic error. Then, in addition to $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\| \geq \eta_{\text{alg}}^i$, there holds:

$$\eta_{\text{alg}}^i \geq \beta\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad 0 < \beta(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1.$$

[2] Chapter 2

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For $u_J^i \in \mathbf{V}_J^p$, let $u_J^{i+1} \in \mathbf{V}_J^p$ be constructed from u_J^i using one step of the solver. There holds:

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+1})\| \leq \alpha\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|, \qquad \alpha = \sqrt{1 - \beta^2}.$$

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Remark:
- The dependence on J is at most *linear* under minimal H^1-regularity.
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[2]Chapter 2

ADDITIONAL RESULTS

Corollary (Equivalence of the two main results)

Proving the *efficiency* of the a posteriori estimator η^i_{alg} is equivalent to proving the solver *contraction*.

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Corollary (Equivalence of error–global estimator–local estimators)

Let the assumptions of Theorem 2 hold. Then

$$
\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\right\|^2 \approx (\eta^i_{\mathrm{alg}})^2 = \sum_{j=0}^{J}\left(\lambda^i_j\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_j\right\|\right)^2 = \left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_0\right\|^2 + \sum_{j=1}^{J}\lambda^i_j\sum_{\mathbf{a}\in\mathcal{V}_j}\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_{j,\mathbf{a}}\right\|^2_{\omega^{\mathbf{a}}_j}.
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- **patches:** larger subdomains can be chosen in the definition of the local spaces $V_j^{\mathbf{a}}$,
- **smoothing:** by modifying the assembly of the local contributions $\rho_{j,\mathbf{a}}^i$
  - *damped additive Schwarz* (dAS) smoothing,
  - *weighted restricted additive Schwarz* (wRAS) smoothing[4],

$$\rho_0^i + \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \rho_{j,\mathbf{a}}^i \quad \text{(AS)}, \qquad \rho_0^i + w \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \rho_{j,\mathbf{a}}^i \quad \text{(dAS)}, \qquad \rho_0^i + \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \mathcal{I}_j^{p_j}(\psi_j^{\mathbf{a}} \rho_{j,\mathbf{a}}^i) \quad \text{(wRAS)},$$

  - for vertex $\mathbf{a} \in \mathcal{V}_j$, we denote the associated hat function by $\psi_j^{\mathbf{a}}$,
  - $\mathcal{I}_j^{p_j}$ is the $\mathbb{P}^{p_j}$ Lagrange interpolation operator on mesh level j.
- **combination of level-wise algebraic residual liftings:** only one global optimal step-size in the finest level J can used.

[3] Chapter 1

[4] Efstathiou and Gander. "Why restricted additive Schwarz converges faster than additive Schwarz". *BIT.* 2003.

ALTERNATIVES[3]

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- **combination of level-wise algebraic residual liftings:** only one global optimal step-size in the finest level J can used.

Some of these variants are *parallelizable* also level-wise.

[3] Chapter 1

[4] Efstathiou and Gander. “Why restricted additive Schwarz converges faster than additive Schwarz”. *BIT.* 2003.

NUMERICAL RESULTS

Consider the test cases:

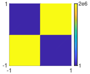
Sine: $u(x,y) = \sin(2\pi x)\sin(2\pi y),\ \Omega := (-1,1)^2,$

Peak: $u(x,y) = x(x-1)y(y-1)e^{-100((x-0.5)^2-(y-0.117)^2)};\ \Omega := (0,1)^2,$

L-shape: $u(r,\theta) = r^{2/3}\sin(2\theta/3);\quad \Omega = (-1,1)^2 \setminus ([0,1]\times[-1,0]),$

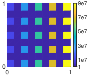
Checkerboard[5]**:** $u(r,\varphi) = r^{\gamma}\mu(\varphi);\quad \Omega := (-1,1)^2$

with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^6)$ or no jump,



Skyscraper: unknown analytic solution; $\Omega := (0,1)^2$

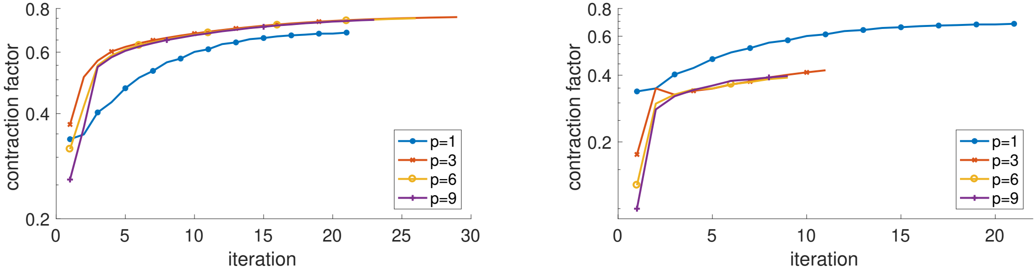
with jump in the diffusion coefficient $\mathcal{J}(\mathbf{K}) = O(10^7)$ or $\mathcal{J}(\mathbf{K}) = O(1)$.



[5] Kellogg. “On the Poisson equation with intersecting interfaces”. *Appl. Anal.* 1975.

Numerical confirmation of p-robustness: contraction factors

L-shape problem, $J = 3$, and mesh hierarchy $p_j = 1$ (left) and $p_j = p$ (right), $j \in \{1, \dots, J-1\}$



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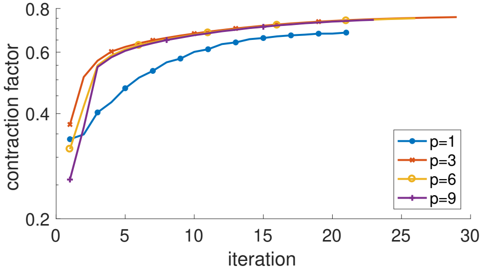
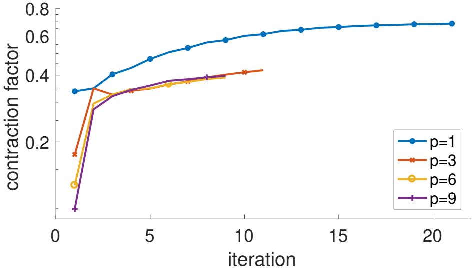


$1 \to 1, p$



$1, p \to p$

Numerical confirmation of p-robustness: iteration numbers

Stopping criterion:

$$\frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{i_s}\|}{\|\mathsf{F}_J\|} \le 10^{-5} \frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{0}\|}{\|\mathsf{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

NUMERICAL CONFIRMATION OF p-ROBUSTNESS: ITERATION NUMBERS

Stopping criterion:

$$\frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{i_s}\|}{\|\mathsf{F}_J\|} \leq 10^{-5} \frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{0}\|}{\|\mathsf{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

H^2-regular (Sine, Peak)

			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Numerical confirmation of p-robustness: iteration numbers

Stopping criterion:

$$\frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{i_s}\|}{\|\mathsf{F}_J\|} \leq 10^{-5} \frac{\|\mathsf{F}_J - \mathbb{A}_J \mathsf{U}_J^{0}\|}{\|\mathsf{F}_J\|}.$$

The mesh hierarchies here are obtained from J *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			H^2-regular				H^1-regular									
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

NUMERICAL CONFIRMATION OF p-ROBUSTNESS: ITERATION NUMBERS

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			H^2-regular				H^1-regular									
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
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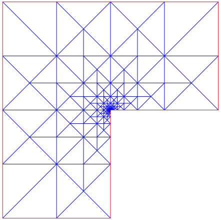
			$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$	$1\to 1,p$	$1,p\to p$
J	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1e^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6e^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	32	14	31	15	23	9	22	9	23	9	25	9	25	9

Numerical **K**- and J-robustness is observed even in low-regularity cases.

Numerical tests for graded meshes

L-shape, $\mathbf{K} = I$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Numerical tests for graded meshes

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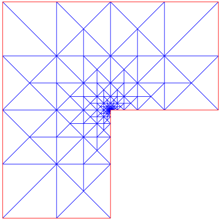
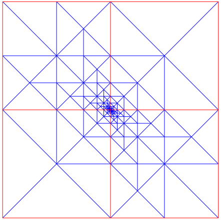
J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

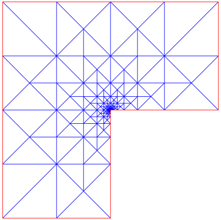
J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



Numerical tests for graded meshes

L-shape, $\mathbf{K} = I$, $1, p \to p$

J	p	i_s	J	p	i_s	J	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



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J	p	i_s	J	p	i_s	J	p	i_s
5	1	33	10	1	57	15	1	97
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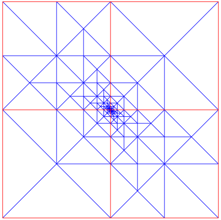
	6	12		6	15		6	20
	9	11		9	12		9	15



These H^1-regular test cases indicate the possibility of *J-dependence*, in accordance with the theoretical results.

Numerical tests in three space dimensions

Test cases: exact solution u when available; $\mathbf{K} = I$ except where explicitly specified, uniform mesh refinement, $p_j = 1$, $j \in \{1, \ldots, J\}$, and $J = 4$.

Cube: $\Omega := (0,1)^3$,
$u(x,y,z) = x(x-1)y(y-1)z(z-1)$.

Nested cubes: $\Omega := (-1,1)^3$,
unknown analytic solution,
$\mathbf{K} = 10^5 * I$ in $(-0.5, 0.5)^3$.

Checkers cubes: $\Omega := (0,1)^3$,
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$\mathbf{K} = 10^6 * I$ in $(0, 0.5)^3 \cup (0.5, 1)^3$.

NUMERICAL TESTS IN THREE SPACE DIMENSIONS

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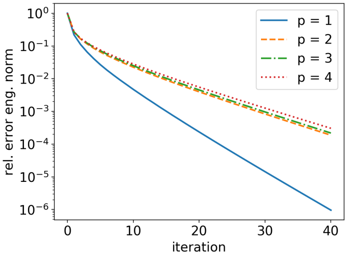

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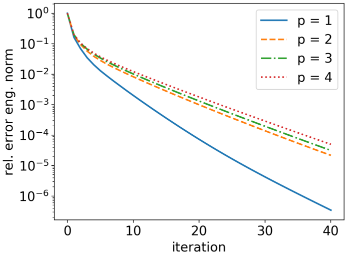

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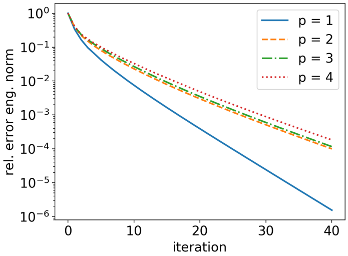

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Cube: $\Omega := (0,1)^3$, $u(x,y,z) = x(x-1)y(y-1)z(z-1)$.

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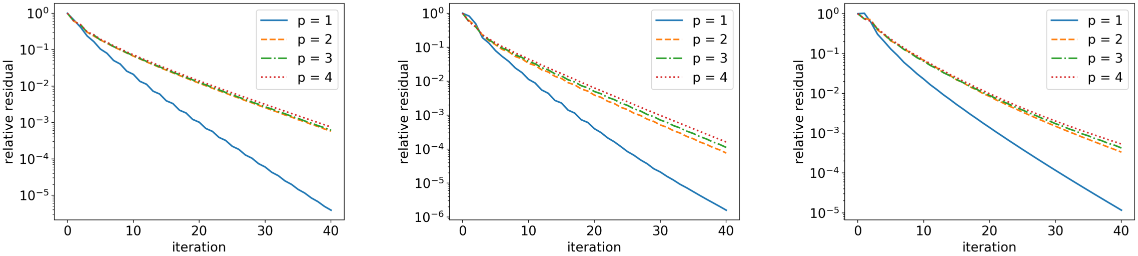

Numerical advantages of optimal step-sizes

Level-wise optimal step-sizes determined by line search:

- *analytically*: **Pythagorean formula** for the algebraic error

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NUMERICAL ADVANTAGES OF OPTIMAL STEP-SIZES

Level-wise optimal step-sizes determined by line search:

- *analytically*: **Pythagorean formula** for the algebraic error
- *numerically*: advantages of using even a single global step-size on level J

		Sine		Peak		L-shape	
J	p	AS	MG(0,1)-J	AS	MG(0,1)-J	AS	MG(0,1)-J
3	1	21	-	19	68	17	44
4	1	23	-	20	-	18	-
5	1	22	-	20	-	17	-

For $p = 1$: **AS** and **MG(0,1)-J** only differ by the use of the global optimal step-size.

NUMERICAL ADVANTAGES OF OPTIMAL STEP-SIZES

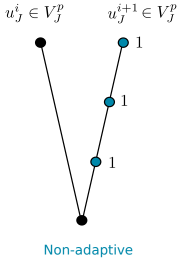
Level-wise optimal step-sizes determined by line search:

- *analytically*: **Pythagorean formula** for the algebraic error
- *numerically*: advantages of using even a single global step-size on level J

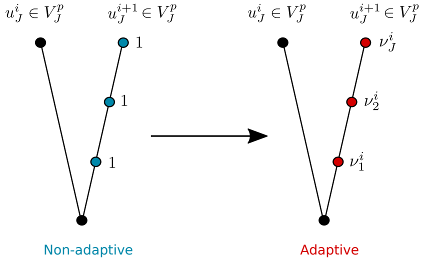
		Sine		Peak		L-shape	
J	p	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J
3	1	21	-	19	68	17	44
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	10	-
4	1	23	-	20	-	18	-
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	9	-
5	1	22	-	20	-	17	-
	3	15	-	15	-	12	-
	6	13	-	14	-	9	-
	9	13	-	13	-	8	-

For $p = 1$: **wRAS** and **MG(0,1)-J** only differ by the use of the global optimal step-size.

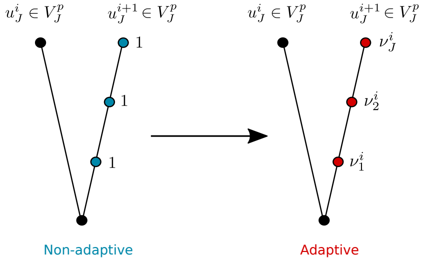
Adaptive number of smoothing steps



ADAPTIVE NUMBER OF SMOOTHING STEPS



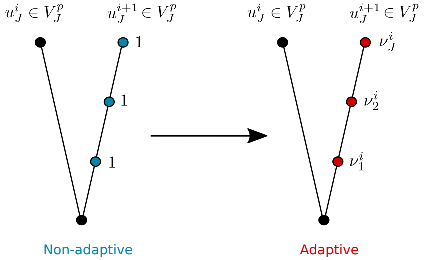
Adaptive number of smoothing steps



Variable number of smoothing steps

- Bramble and Pasciak. "New convergence estimates for multigrid algorithms". *Math. Comp.* 1987.

Adaptive number of smoothing steps



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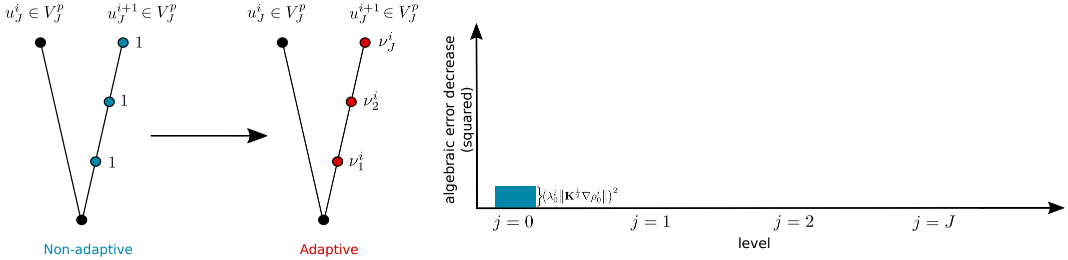
ADAPTIVE NUMBER OF SMOOTHING STEPS



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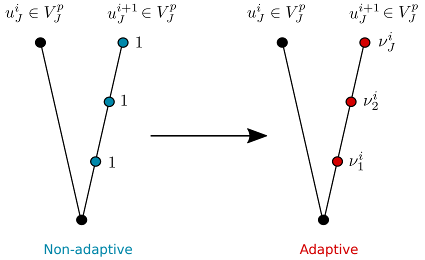

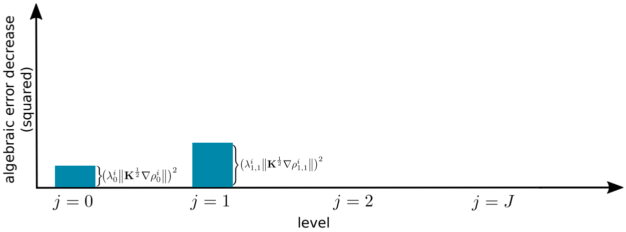

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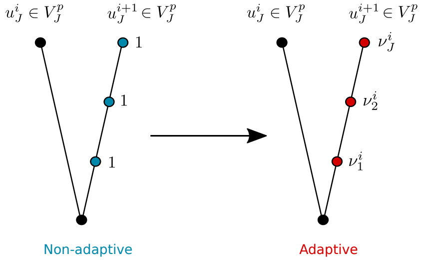

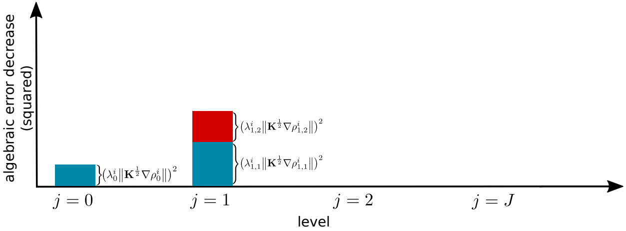

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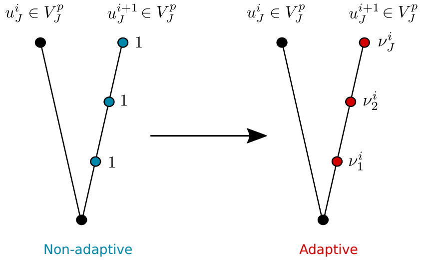

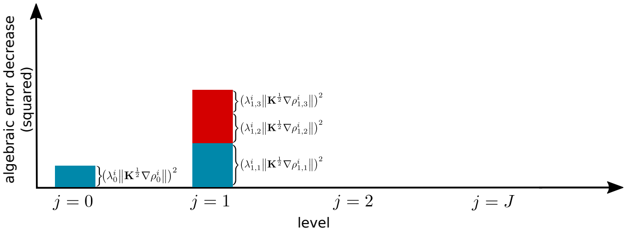

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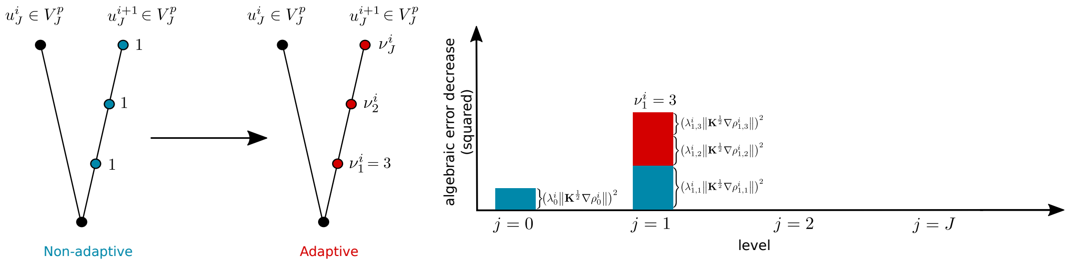

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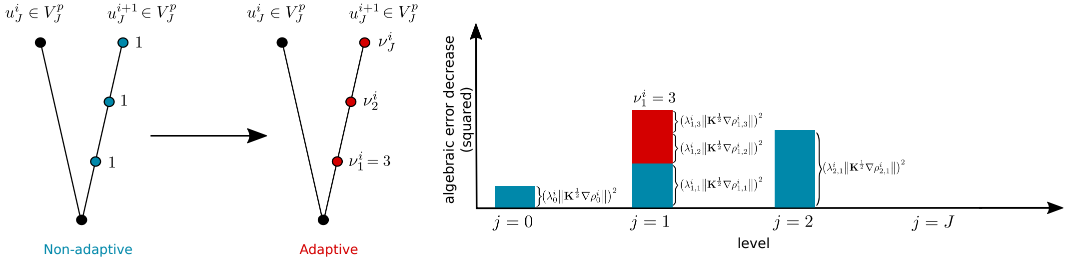

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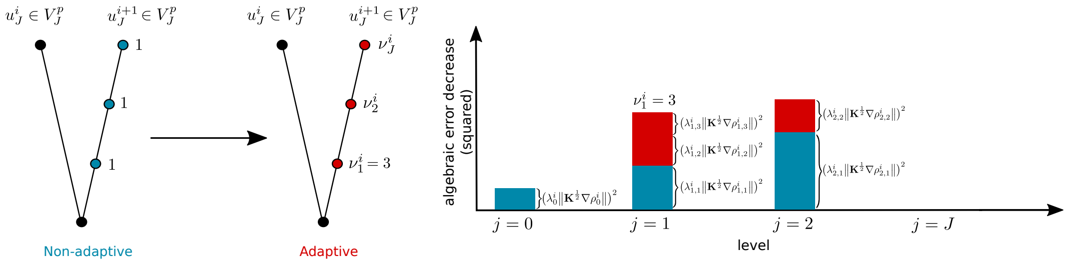

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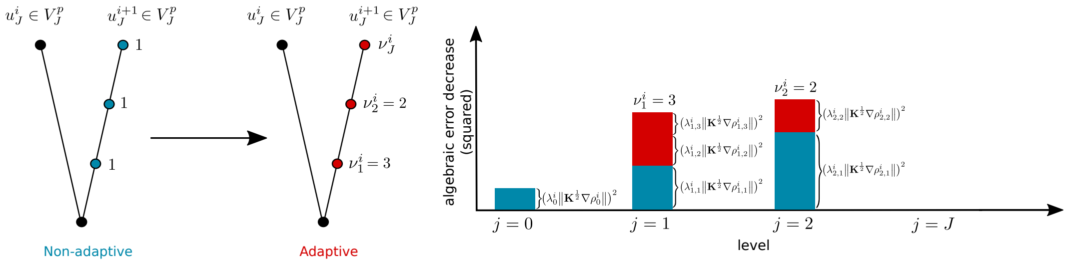

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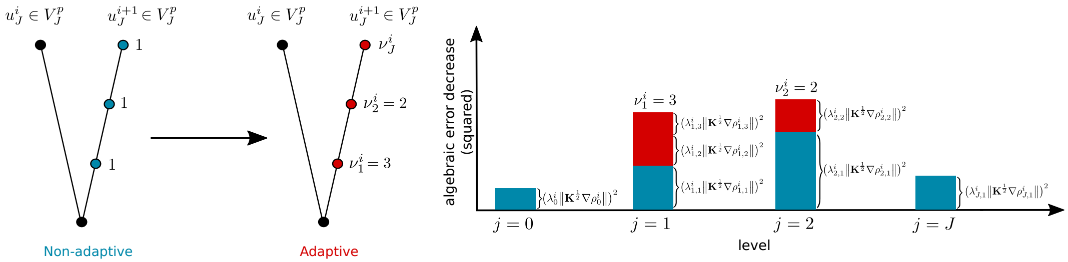

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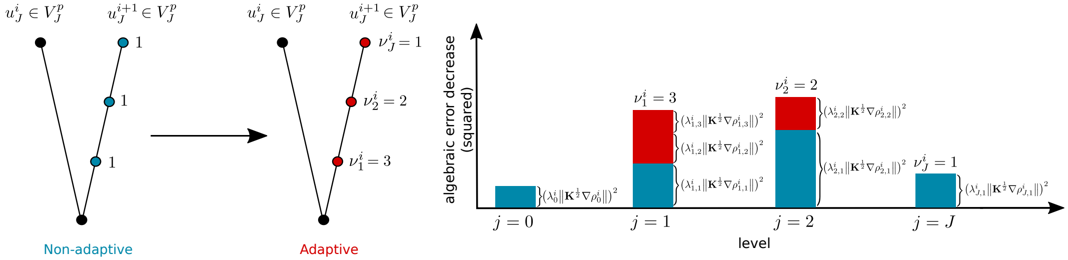

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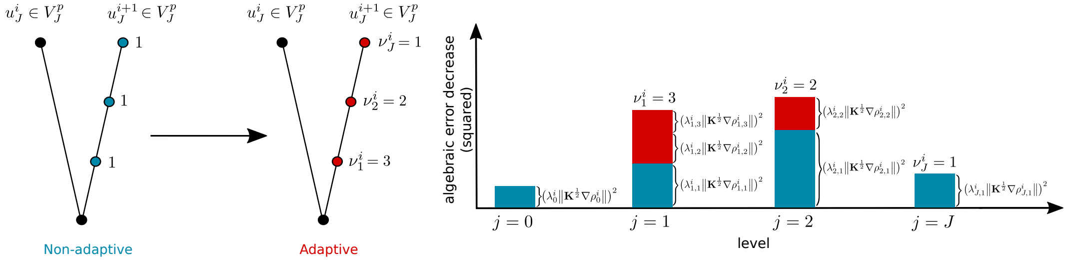

$$\left(\lambda_{j,\nu}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|\right)^2 \geq \theta^2 \left(\sum_{k=0}^{j-1} \sum_{\ell=1}^{\nu_k^i} \left(\lambda_{k,\ell}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|\right)^2 + \sum_{\ell=1}^{\nu-1} \left(\lambda_{j,\ell}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|\right)^2 \right)$$

Variable number of smoothing steps/multigrid cycles:

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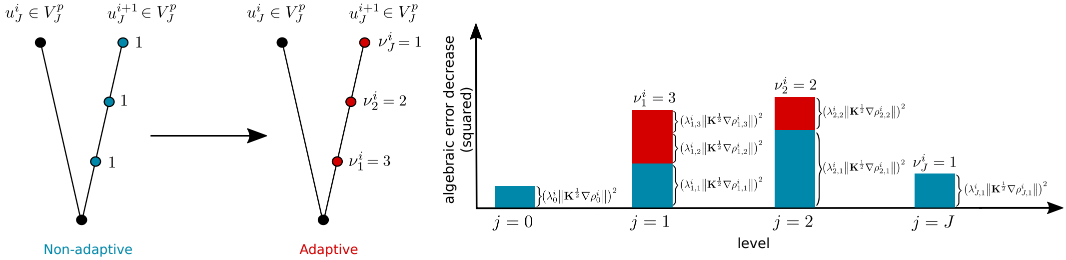
ADAPTIVE NUMBER OF SMOOTHING STEPS



$$\underbrace{\left(\lambda_{j,\nu}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\nu}^i\|\right)^2}_{\text{current smoothing}} \geq \theta^2 \left(\sum_{k=0}^{j-1} \sum_{\ell=1}^{\nu_k^i} \left(\lambda_{k,\ell}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{k,\ell}^i\|\right)^2 + \sum_{\ell=1}^{\nu-1} \left(\lambda_{j,\ell}^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\ell}^i\|\right)^2 \right)$$

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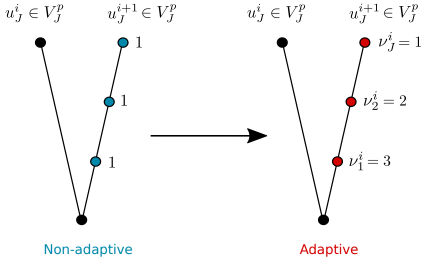
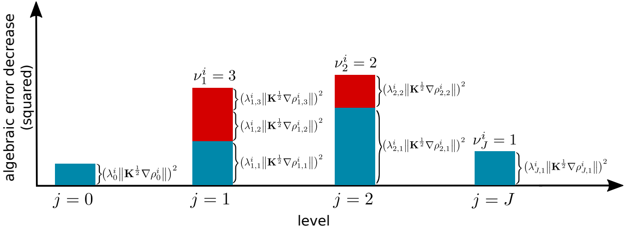
ADAPTIVE NUMBER OF SMOOTHING STEPS





$$\underbrace{\left(\lambda^i_{j,\nu}\|\mathbf{K}^{\frac12}\nabla\rho^i_{j,\nu}\|\right)^2}_{\text{current smoothing}} \geq \theta^2\left(\underbrace{\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}\left(\lambda^i_{k,\ell}\|\mathbf{K}^{\frac12}\nabla\rho^i_{k,\ell}\|\right)^2}_{\text{all smoothings on previous levels}} + \sum_{\ell=1}^{\nu-1}\left(\lambda^i_{j,\ell}\|\mathbf{K}^{\frac12}\nabla\rho^i_{j,\ell}\|\right)^2\right)$$

Variable number of smoothing steps/multigrid cycles:

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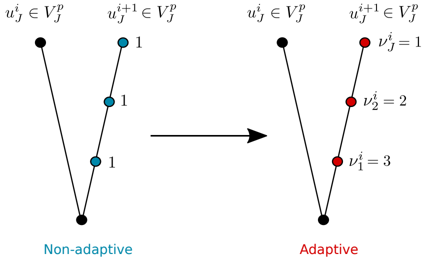
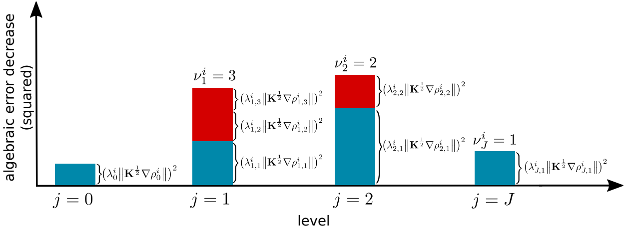
Adaptive number of smoothing steps





$$\underbrace{\left(\lambda^i_{j,\nu}\|\mathbf{K}^{\frac12}\nabla\rho^i_{j,\nu}\|\right)^2}_{\text{current smoothing}} \ge \theta^2\left(\underbrace{\sum_{k=0}^{j-1}\sum_{\ell=1}^{\nu_k^i}\left(\lambda^i_{k,\ell}\|\mathbf{K}^{\frac12}\nabla\rho^i_{k,\ell}\|\right)^2}_{\text{all smoothings on previous levels}} + \underbrace{\sum_{\ell=1}^{\nu-1}\left(\lambda^i_{j,\ell}\|\mathbf{K}^{\frac12}\nabla\rho^i_{j,\ell}\|\right)^2}_{\text{previous smoothings on current level}}\right)$$

Variable number of smoothing steps/multigrid cycles:

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- Thekale, Gradl, Klamroth, and Rüde. “Optimizing the number of multigrid cycles in the full multigrid algorithm.” *Numer. Linear Algebra Appl.* 2010.

Numerical tests

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and mesh hierarchy $p_j = p$, $j \in \{1, \ldots, J-1\}$.

	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

Numerical tests

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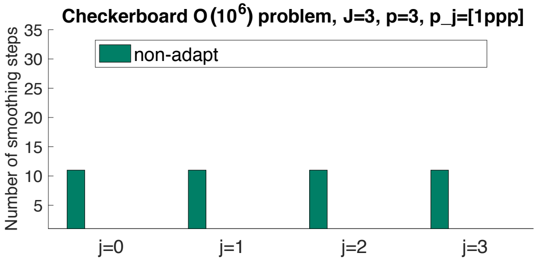
	$p_j = p$, non-adapt										
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1



NUMERICAL TESTS

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and mesh hierarchy $p_j = p$, $j \in \{1, \ldots, J-1\}$.

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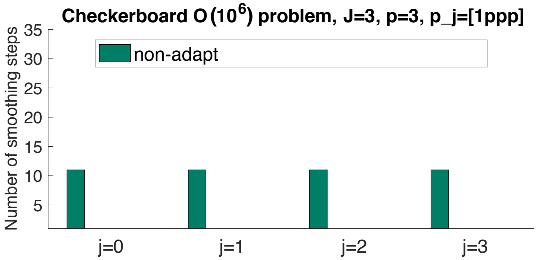
	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



Numerical tests

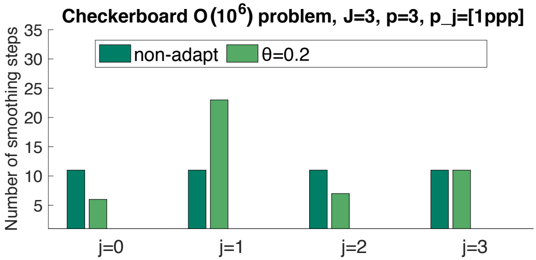
Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and mesh hierarchy $p_j = p$, $j \in \{1, \ldots, J-1\}$.

$p_j = p$, non-adapt

	it=1	it=2	it=3	it=4	it=5	it=6	it=7	it=8	it=9	it=10	it=11
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

$p_j = p$, $\theta = 0.2$

	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1



Numerical tests

Checkerboard case, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $p = 3$, $J = 3$, and mesh hierarchy $p_j = p$, $j \in \{1, \ldots, J-1\}$.

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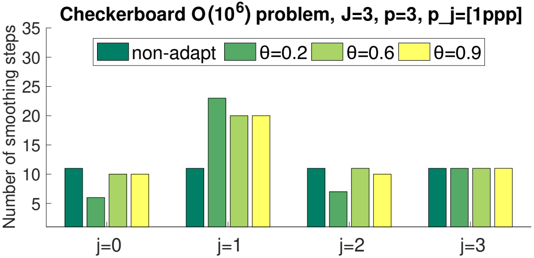
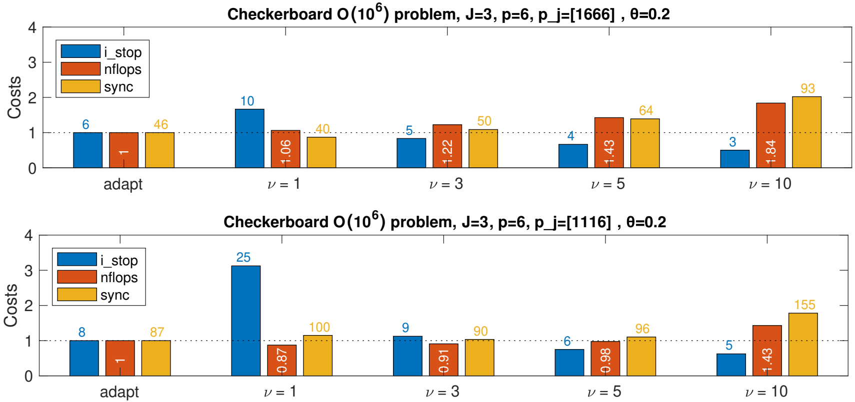
level 0	1	1	1	1	1	1	1	1	1	1	1
level 1	1	1	1	1	1	1	1	1	1	1	1
level 2	1	1	1	1	1	1	1	1	1	1	1
level 3	1	1	1	1	1	1	1	1	1	1	1

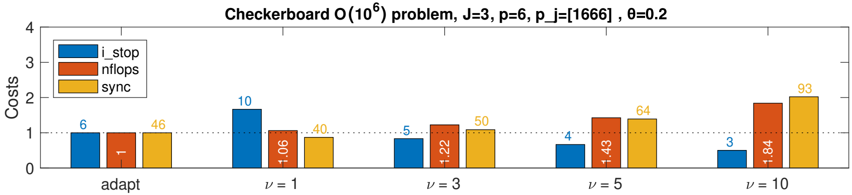
	$p_j = p$, $\theta = 0.2$					
	it=1	it=2	it=3	it=4	it=5	it=6
level 0	1	1	1	1	1	1
level 1	3	4	4	4	4	4
level 2	2	1	1	1	1	1
level 3	2	2	2	2	2	1

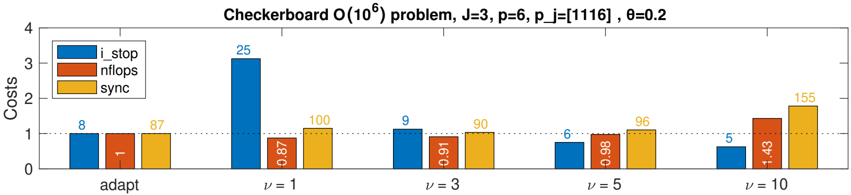


Number of post-smoothing steps: adaptive vs fixed



Number of post-smoothing steps: adaptive vs fixed





$$\textbf{nflops} := \frac{|\mathcal{V}_0|^3}{3} + \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \frac{\mathrm{ndof}(V_j^{\mathbf{a}})^3}{3} + \sum_{i=1}^{i_s}\Big[2|\mathcal{V}_0|^2 + \sum_{j=1}^{J} \nu_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} 2\mathrm{ndof}(V_j^{\mathbf{a}})^2\Big] + \sum_{i=1}^{i_s}\sum_{j=1}^{J}\Big[2\,\mathrm{nnz}(\mathcal{I}_{j-1}^{j}) + 2\,\mathrm{nnz}(\mathcal{I}_{j}^{j-1}) + 2\nu_j^i\,\mathrm{nnz}(\mathbb{A}_j) + 3\nu_j^i(2\,\mathrm{size}(\mathbb{A}_j))\Big];$$

$$\textbf{sync} := i_s + \sum_{i=1}^{i_s}\sum_{j=1}^{J} \nu_j^i.$$

Comparison with other multilevel solvers

We compare our methods with [6,7,8] in terms of the number of iterations (and CPU times[9]).

[6] Antonietti et al. *J. Sci. Comput.* 2017.
[7] Botti et al. *J. Comput. Phys.* 2017.
[8] Schöberl. "C++11 Implementation of Finite Elements in NGSolve". *Tech. report.* 2014.
[9] The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée "NEF" computation cluster, however, in a sequential Matlab script.

COMPARISON WITH OTHER MULTILEVEL SOLVERS

We compare our methods with [6,7,8] in terms of the number of iterations (and CPU times[9]).

		∼MG(0,1) -bJ $1, p \to p$		∼MG(0,adapt) -bJ (wRAS) $1 \nearrow p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$		MG(3,3)-GS $1 \nearrow p$	
J	p	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m

[6] Antonietti et al. *J. Sci. Comput.* 2017.

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		∼MG(0,1) -bJ $1, p \to p$		∼MG(0,adapt) -bJ (wRAS) $1 \nearrow p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)-PCG(iChol) $1 \nearrow p$		MG(0,1)-bGS $1 \to 1, p$		MG(3,3)-GS $1 \nearrow p$	
J	p	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m
								not p-robust				not p-robust	

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[9] The experiments were run on one Dell C6220 dual-Xeon E5-2650 node of Inria Sophia Antipolis - Méditerranée “NEF” computation cluster, however, in a sequential Matlab script.

Adaptive local smoothing

Recall: $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|^2 \approx \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}^2.$

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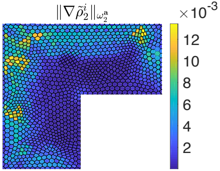

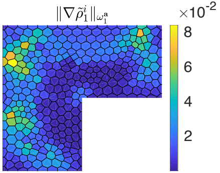

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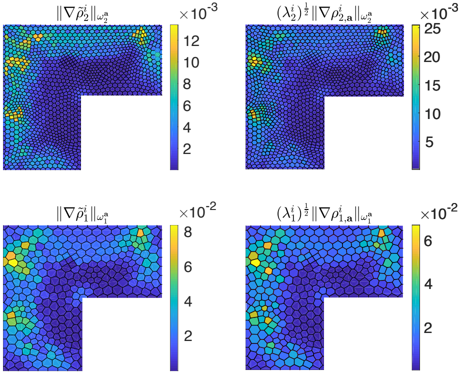

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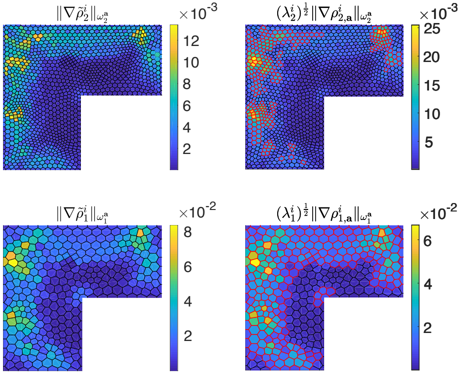

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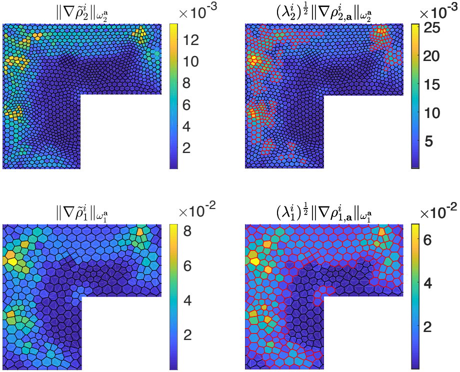

Local smoothing in adaptively-refined meshes

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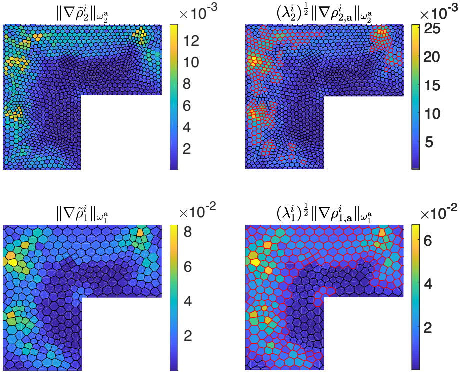

Local smoothing in adaptively-refined meshes v. adaptive local smoothing on a given mesh hierarchy

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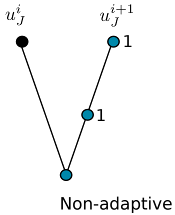

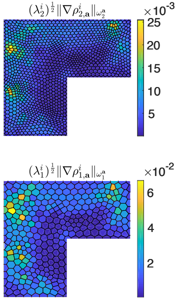

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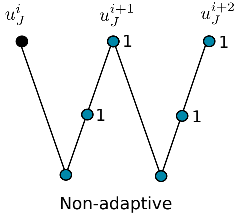

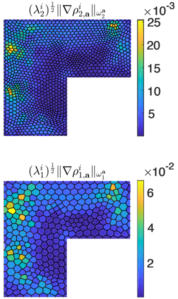

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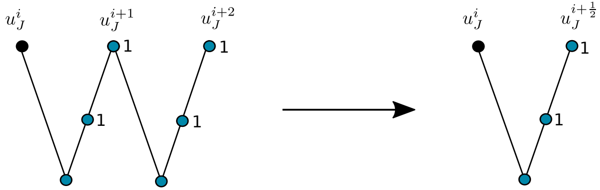
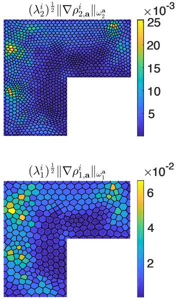


Non-adaptive

Adaptive local smoothing



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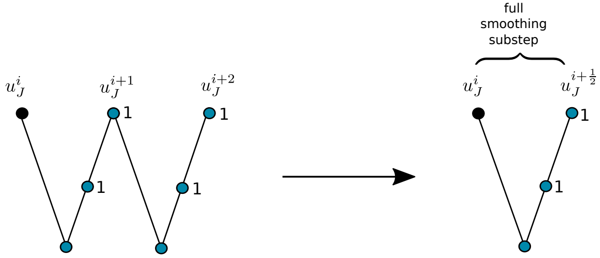

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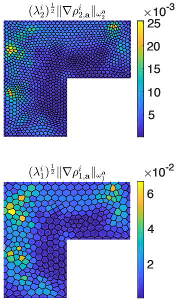

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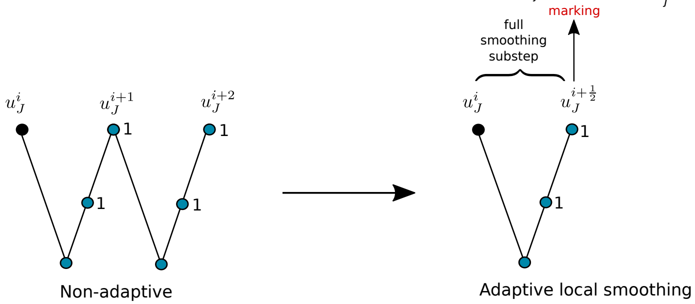

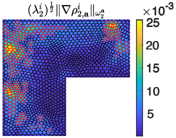

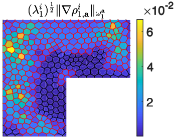

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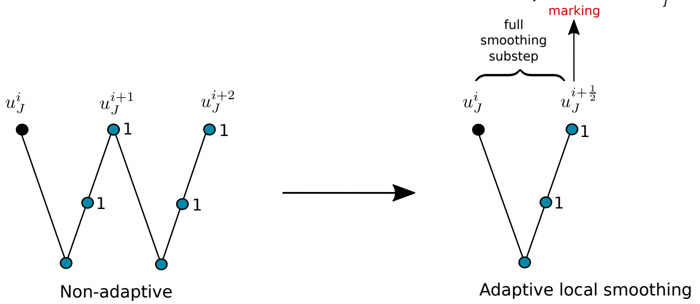

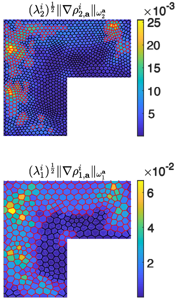

bulk-chasing criterion[11]: $\theta^2\left(\|\mathbf{K}^{\frac12}\nabla\rho_0^i\|^2 + \sum_{j=1}^J \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\mathbf{K}^{\frac12}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}\right) \le \sum_{j\in\mathcal{M}} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j} \|\mathbf{K}^{\frac12}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}$

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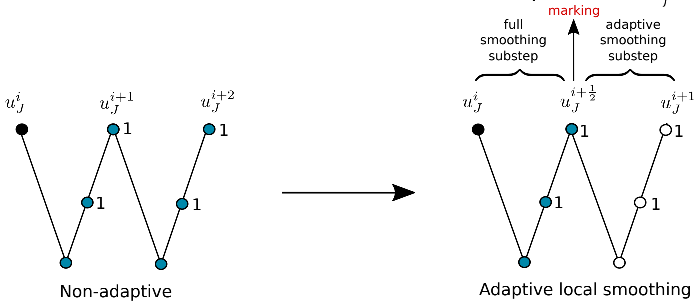

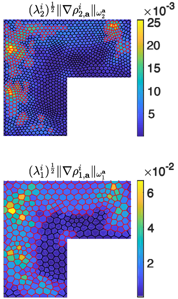

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Algorithm: A-posteriori-steered multigrid with local adaptive smoothing

Input: [p, J, Dörfler's parameter θ, adaptivity parameter γ, tolerance tol]

$i := 0$; $u_J^i := 0$; $\eta_{\text{alg}}^i := 10\text{tol}$;

while $\eta_{\text{alg}}^i \geq tol$ **do**

  $i := i + 1$; $u_J^i := u_J^{i-1}$; $(u_J^i, \eta_{\text{alg}}^i) := \texttt{FULL\_SMOOTHING\_SUBSTEP}(p, J, u_J^i)$;

  if $\eta_{\text{alg}}^i <$ tol **break while loop**;

  $\left(\mathcal{M}, \{\mathbf{a} \in \mathcal{M}_j\}_{j \in \mathcal{M}}\right) := \texttt{DÖRFLER\_MARKING}\ (\theta, \eta_{\text{alg}}^i)$;

  if [$\texttt{TEST\_ADAPT}(\gamma)$] **then**

    $(u_J^i, \eta_{\text{alg}}^i) := \texttt{ADAPTIVE\_SMOOTHING\_SUBSTEP}\ \left(p, J, u_J^i, \mathcal{M}, \{\mathbf{a} \in \mathcal{M}_j\}_{j \in \mathcal{M}}\right)$;

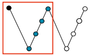
  end

end

$i_{\text{stop}} := i$;

Output: [$u_J^{i_{\text{stop}}}$, $\eta_{\text{alg}}^{i_{\text{stop}}}$]

FULL_SMOOTHING_SUBSTEP (p, J, u_J^i):



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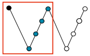
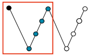


Coarse solve: Define $\rho_0^i \in V_0$ by: $(\mathbf{K}\nabla \rho_0^i, \nabla v_0) = (f, v_0) - (\mathbf{K}\nabla u_J^i, \nabla v_0), \quad \forall v_0 \in V_0$ and set $\lambda_0^i := 1$.

FULL_SMOOTHING_SUBSTEP (p, J, u_J^i):



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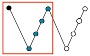
Level-wise local solves: For $j = 1 : J$, for all $\mathbf{a} \in \mathcal{V}_j$, define $\rho_{j,\mathbf{a}}^i \in V_j^{\mathbf{a}}$ by :

$$(\mathbf{K}\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\mathbf{K}\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} \lambda_k^i (\mathbf{K}\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

$\texttt{FULL\_SMOOTHING\_SUBSTEP}\ (p, J, u_J^i)$:



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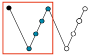
Level-wise contributions: Define $\rho_j^i \in V_j^{p_j}$ by: $\rho_j^i := \sum_{\mathbf{a}\in\mathcal{V}_j} \rho_{j,\mathbf{a}}^i$,

$$\text{and set:} \quad \lambda_j^i := \frac{(f, \rho_j^i) - (\mathbf{K}\nabla(u_J^i + \sum_{k=0}^{j-1} \lambda_k^i \rho_k^i), \nabla\rho_j^i)}{\left\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\right\|^2}.$$

FULL_SMOOTHING_SUBSTEP (p, J, u_J^i):



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$$(\mathbf{K}\nabla\rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\mathbf{K}\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \sum_{k=0}^{j-1} \lambda_k^i (\mathbf{K}\nabla\rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}}, \quad \forall v_{j,\mathbf{a}} \in V_j^{\mathbf{a}}.$$

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Outputs: Define the estimator $\eta_{\mathrm{alg}}^i := \left(\sum_{j=0}^{J} \left(\lambda_j^i \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_j^i\|\right)^2\right)^{\frac{1}{2}}$ and update $u_J^i \longleftarrow u_J^i + \sum_{j=0}^{J} \lambda_j^i \rho_j^i$.

DÖRFLER_MARKING $(\theta, \eta^i_{\mathrm{alg}})$:

Since we have the **error localization:** $\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u^i_J)\|^2 \approx \|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_0\|^2 + \sum_{j=1}^{J}\lambda^i_j\sum_{\mathbf{a}\in\mathcal{V}_j}\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_{j,\mathbf{a}}\|^2_{\omega^{\mathbf{a}}_j}$,

we use a bulk-chasing criterion:

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Outputs: Marked levels $\mathcal{M}$ and marked vertices on marked levels $\{\mathbf{a} \in \mathcal{M}_j\}_{j\in\mathcal{M}}$.

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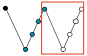

Coarse solve only if $0\in\mathcal{M}$ and **level-wise local solves** only in patches whose vertices are **marked** give us the **level-wise contibutions** $\{\lambda_j^i\}_{j\in\mathcal{M}}$, $\{\rho_j^i\}_{j\in\mathcal{M}}$.

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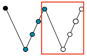

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Outputs: Update the estimator $\eta^i_{\text{alg}} := \left(\sum_{j\in\mathcal{M}}(\lambda^i_j\|\mathbf{K}^{\frac{1}{2}}\nabla\rho^i_j\|)^2\right)^{\frac{1}{2}}$ and update $u^i_J \longleftarrow u^i_J + \sum_{j\in\mathcal{M}}\lambda^i_j\rho^i_j$.

Proposition (Pythagorean error representation per substep)

For $u_J^i \in \mathbf{V}_J^p$, let $u_J^{i+\frac{1}{2}} \in \mathbf{V}_J^p$ and $u_J^{i+1} \in \mathbf{V}_J^p$ be constructed from u_J^i from the full-smoothing and adaptive-smoothing substep, respectively. Then

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Corollary (Guaranteed lower bound on the algebraic error per substep)

There holds:

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i})\right\| \geq \eta_{\mathrm{alg}}^{i},$$

$$\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+\frac{1}{2}})\right\| \geq \eta_{\mathrm{alg}}^{i+\frac{1}{2}}.$$

Main results[11]

Theorem 3 (p-robust error contraction of the multilevel solver)

For $u_J^i \in \mathbf{V}_J^p$, let $u_J^{i+\frac{1}{2}} \in \mathbf{V}_J^p$ and $u_J^{i+1} \in \mathbf{V}_J^p$ be constructed from u_J^i from the full-smoothing and adaptive-smoothing substep when the analysis-driven TEST_ADAPT is satisfied, respectively. Then

$$\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+\frac{1}{2}})\| \leq \alpha\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\| \qquad 0 < \alpha(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1,$$
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[11] Chapter 3

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Theorem 4 (p-robust efficient bound on the algebraic error)

There holds: $\eta_{\mathrm{alg}}^i \geq \beta\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^i)\|$ and $\eta_{\mathrm{alg}}^{i+\frac{1}{2}} \geq \overline{\beta}\left\|\mathbf{K}^{\frac{1}{2}}\nabla(u_J - u_J^{i+\frac{1}{2}})\right\|$, $\beta = \sqrt{1-\alpha^2}$, $\overline{\beta} = \sqrt{1-\overline{\alpha}^2}$.

[11] Chapter 3

Can we predict the distribution of the algebraic error?

Dörfler's bulk-chasing criterion: $\theta^2 \left(\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_0^i\|^2 + \sum_{j=1}^{J} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{V}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}} \right) \leq \sum_{j\in\mathcal{M}} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j} \|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|^2_{\omega_j^{\mathbf{a}}}.$

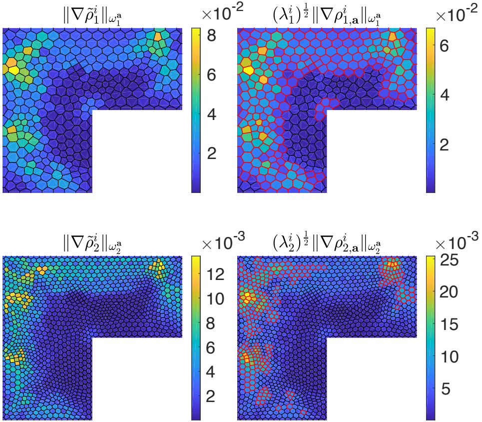
Hierarchy: uniform refinement, $J = 2$, $p_1 = p_2 = 3$.

- local algebraic error indicators $\|\mathbf{K}^{\frac{1}{2}}\nabla\rho_{j,\mathbf{a}}^i\|_{\omega_j^{\mathbf{a}}}$
- local algebraic error distribution $\|\mathbf{K}^{\frac{1}{2}}\nabla\tilde{\rho}_j^i\|_{\omega_j^{\mathbf{a}}}$

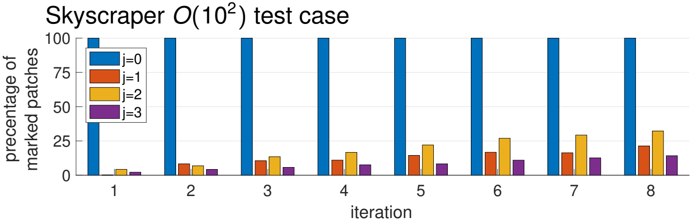
with $\tilde{\rho}_0^i = \rho_0^i$ and $\tilde{\rho}_j^i \in V_j^{p_j}$, for $j \in \{1,\dots,J\}$, given by

$$(\mathbf{K}\nabla\tilde{\rho}_j^i,\nabla v_j) = (f,v_j) - (\mathbf{K}\nabla u_J^i,\nabla v_j) - \sum_{k=0}^{j-1}(\mathbf{K}\nabla\tilde{\rho}_k^i,\nabla v_j) \quad \forall v_j \in V_j^{p_j},$$

so that $\sum_{j=0}^{J}\tilde{\rho}_j^i = u_J - u_J^i$.



Does the adaptivity pay off?



Hierarchy: $J = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$

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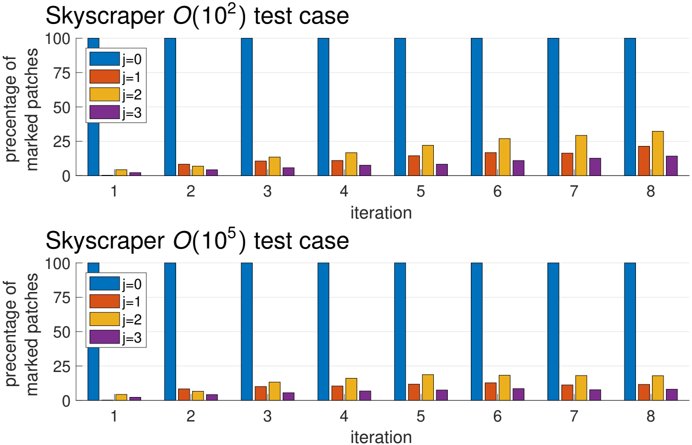

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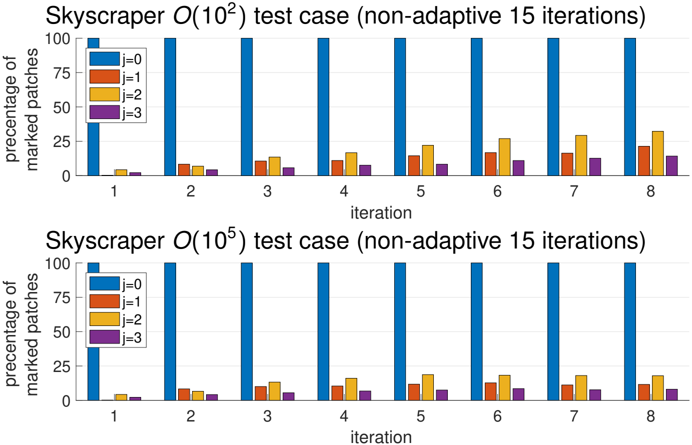

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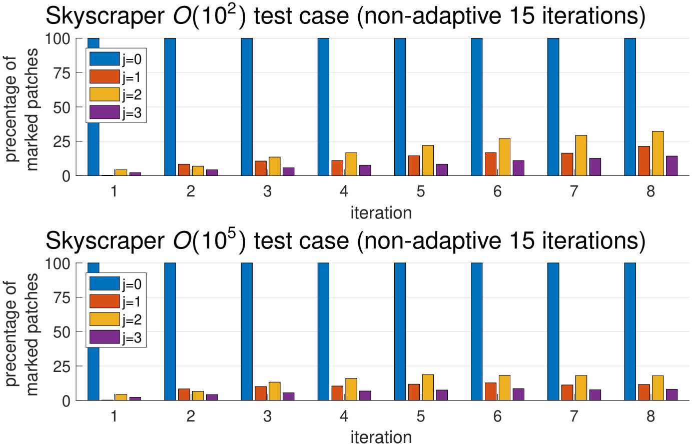

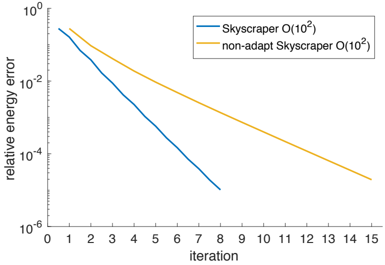

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EXTENSIONS TO MFE: MULTIGRID

Introduce the discrete spaces:

$$\mathbf{V}_J^f \subset \mathbf{V}^f := \{\mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega),\ \nabla\cdot\mathbf{v} = f,\ \mathbf{v}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}$$
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Discrete problem: find $\mathbf{u}_J \in \mathbf{V}_J^f$ so that

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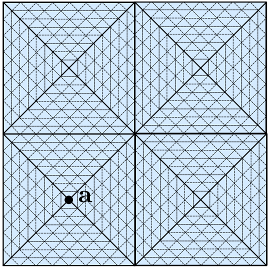

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Main results[12]

Theorem 5 (*p*-robust error contraction of the multilevel solver)

Let $d = 2$. For $\mathbf{u}_J^i \in \mathbf{V}_J^f$, let $\mathbf{u}_J^{i+1} \in \mathbf{V}_J^f$ be constructed from $\mathbf{u}_J^i$ using one step of the solver (**multigrid** or **domain decomposition**). There holds:

$$\left\|\mathbf{K}^{-\frac{1}{2}}(\mathbf{u}_J - \mathbf{u}_J^{i+1})\right\| \leq \alpha \left\|\mathbf{K}^{-\frac{1}{2}}(\mathbf{u}_J - \mathbf{u}_J^{i})\right\|, \qquad 0 < \alpha(\kappa_{\mathcal{T}}, J, d, \mathbf{K}) < 1.$$

[12] Chapter 4

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Theorem 6 (p-robust reliable and efficient bound on the algebraic error)

Let $d = 2$. Let $\mathbf{u}_J^i \in \mathbf{V}_J^f$ be *arbitrary*. Let η_{alg}^i be the associated a posteriori estimator on the algebraic error. Then, in addition to $\left\|\mathbf{K}^{-\frac{1}{2}}(u_J - u_J^i)\right\| \geq \eta_{\text{alg}}^i$, there holds:

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[12] Chapter 4

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The numerical tests in 3D were performed with NGSolve[13].

[13] Schöberl. "C++11 Implementation of Finite Elements in NGsolve". *Tech. report.* 2014.

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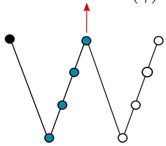
THANK YOU FOR YOUR ATTENTION!

Analysis-driven test for adaptive local smoothing

When the following tests are satisfied:



$$\sum_{j\in\mathcal{M}} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j} \left(\sum_{k=j}^{J} \lambda_k^i \mathbf{K}\nabla\rho_k^i, \nabla\rho_{j,\mathbf{a}}^i \right)_{\omega_{j,0}^{\mathbf{a}}} \le \gamma^2 \sum_{j\in\mathcal{M}} \lambda_j^i \sum_{\mathbf{a}\in\mathcal{M}_j} \left\| \mathbf{K}^{\frac{1}{2}} \nabla\rho_{j,\mathbf{a}}^i \right\|_{\omega_{j,0}^{\mathbf{a}}}^2,$$

$$\lambda_j^i \le 2(d+1) \quad \forall j \in \{0,\ldots,J\},$$

for $\gamma \in (0,1)$ a user-prescribed parameter, proceed to the adaptive-smoothing substep.

J-DEPENDENCE FOR dAS SMOOTHING

Construction:

$$\rho_j^i = \frac{1}{w_1} \sum_{\mathbf{a} \in \mathcal{V}_j} \rho_{j,\mathbf{a}}^i, \quad 1 \le j \le J,$$

$$(\nabla \rho_{j,\mathbf{a}}^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} = (f, v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - (\nabla u_J^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}} - \frac{1}{w_2} \sum_{k=0}^{j-1} (\nabla \rho_k^i, \nabla v_{j,\mathbf{a}})_{\omega_j^{\mathbf{a}}},$$

Compatibility condition: $1 \le w_1 < 6J(d+1)$ and $w_2 \ge \max\left(1, \frac{5J^2(d+1)^2}{w_1(6J(d+1) - w_1)}\right).$

$w_1 = J(d+1)$ and $w_2 = 1$: $\quad \frac{1}{12 C_{\mathrm{SMD}} J^2 \sqrt{2(d+1)^3}} \le \beta,$

$w_1 = d+1$ and $w_2 = J$: $\quad \frac{1}{12 C_{\mathrm{SMD}} J \sqrt{2(d+1)^3}} \le \beta,$

$w_1 = w_2 = \sqrt{J(d+1)}$: $\quad \frac{1}{12\sqrt{2} C_{\mathrm{SMD}} J^{\frac{5}{4}} (d+1)} \le \beta,$

Parallelizable level-wise:

$w_1 = 1$ and $w_2 = \infty$: $\quad \frac{1}{8 C_{\mathrm{SMD}} \sqrt{J}(d+1)} \le \beta,$

$w_1 = 4\sqrt{J}$ and $w_2 = \infty$: $\quad \frac{1}{8 C_{\mathrm{SMD}} \sqrt{J(d+1)}} \le \beta.$

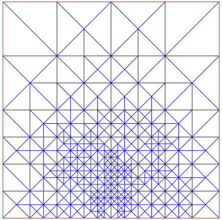
Test with H^2-regular solution on graded meshes

Peak, $1, p \to p$

J	p	i_s
4	1	14
	3	11
	6	9
	9	8

J	p	i_s
8	1	16
	3	9
	6	8
	9	8

J	p	i_s
16	1	16
	3	9
	6	8
	9	9



DEPENDENCE ON THE MARKING PARAMETER

L-shape test case

J	p_j	$\theta = 0.7$ niter	$\theta = 0.7$ nflops	$\theta = 0.9$ niter	$\theta = 0.9$ nflops	$\theta = 0.95$ niter	$\theta = 0.95$ nflops	$\theta = 0.99$ niter	$\theta = 0.99$ nflops
4	1 1 1 1 1	21(0)	7.24×10^{7}	21(0)	7.24×10^{7}	21(0)	7.24×10^{7}	21(0)	7.24×10^{7}
	1 1 2 2 3	9(4)	1.28×10^{9}	8(5)	1.24×10^{9}	8(5)	1.24×10^{9}	6(5)	1.06×10^{9}
	1 2 3 5 6	6(3)	2.97×10^{10}	6(4)	3.03×10^{10}	5(5)	2.92×10^{10}	4(4)	2.70×10^{10}
	1 3 5 7 9	6(6)	2.90×10^{11}	5(5)	2.78×10^{11}	5(5)	2.78×10^{11}	4 (4)	2.68×10^{11}

Skyscraper test case (diff. contrast $O(10^2)$)

J	p_j	$\theta = 0.7$ niter	$\theta = 0.7$ nflops	$\theta = 0.9$ niter	$\theta = 0.9$ nflops	$\theta = 0.95$ niter	$\theta = 0.95$ nflops	$\theta = 0.99$ niter	$\theta = 0.99$ nflops
4	1 1 1 1 1	19(0)	6.31×10^{7}	19(0)	6.31×10^{7}	19(0)	6.31×10^{7}	19(0)	6.31×10^{7}
	1 1 2 2 3	10(4)	1.38×10^{9}	8(7)	1.34×10^{9}	8(7)	1.35×10^{9}	6(6)	1.10×10^{9}
	1 2 3 5 6	8(4)	3.38×10^{10}	6(6)	3.15×10^{10}	6(6)	3.15×10^{10}	5(5)	2.92×10^{10}
	1 3 5 7 9	7(7)	2.99×10^{11}	6(6)	2.88×10^{11}	5(5)	2.77×10^{11}	5(5)	2.77×10^{11}

Skyscraper test case (diff. contrast $O(10^5)$)

J	p_j	$\theta = 0.7$ niter	$\theta = 0.7$ nflops	$\theta = 0.9$ niter	$\theta = 0.9$ nflops	$\theta = 0.95$ niter	$\theta = 0.95$ nflops	$\theta = 0.99$ niter	$\theta = 0.99$ nflops
4	1 1 1 1 1	19(0)	6.31×10^{7}	19(0)	6.31×10^{7}	19(0)	6.31×10^{7}	19(0)	6.31×10^{7}
	1 1 2 2 3	11(5)	1.53×10^{9}	8(7)	1.34×10^{9}	8(7)	1.35×10^{9}	7(7)	1.26×10^{9}
	1 2 3 5 6	8(4)	3.38×10^{10}	6(6)	3.15×10^{10}	6(6)	3.15×10^{10}	5(5)	2.91×10^{10}
	1 3 5 7 9	7(7)	2.99×10^{11}	6(6)	2.88×10^{11}	5(5)	2.77×10^{11}	5(5)	2.77×10^{11}

$$\textbf{nflops} := \frac{|\mathcal{V}_0|^3}{3} + \sum_{j=1}^{J} \sum_{\mathbf{a}\in\mathcal{V}_j} \frac{\mathrm{ndof}(V_j^{\mathbf{a}})^3}{3} + \sum_{i=1}^{i_s} \left[2\delta_0^i |\mathcal{V}_0|^2 + \sum_{j\in\mathcal{M}\setminus\{0\}} \sum_{\mathbf{a}\in\mathcal{M}_j} 2\mathrm{ndof}(V_j^{\mathbf{a}})^2 \right] + \sum_{i=1}^{i_s} \sum_{j=1}^{J} \left[2\,\mathrm{nnz}(\mathcal{I}_{j-1}^{j}) + 2\,\mathrm{nnz}(\mathcal{I}_{j}^{j-1}) + 2\,\mathrm{nnz}(\mathbb{A}_j) + 3(2\,\mathrm{size}(\mathbb{A}_j)) \right]$$