Regular Decompositions: Discrete, Boundary-Aware, and *p*-Stable

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GATIPOR Workshop 2022
Interplay of discretization and algebraic solvers:
A posteriori error estimates and adaptivity

Paris, June 8-10, 2022

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$\Omega \subset \mathbb{R}^3 \triangleq$ bounded Lipschitz domain, *trivial topology*

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$$H(\Lambda^0, \Omega) \xrightarrow{d} H(\Lambda^1, \Omega) \xrightarrow{d} H(\Lambda^2, \Omega) \xrightarrow{d} H(\Lambda^3, \Omega)\,.$$

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nD generalization: Hilbert complexes of differential forms on Ω

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Whitney Forms

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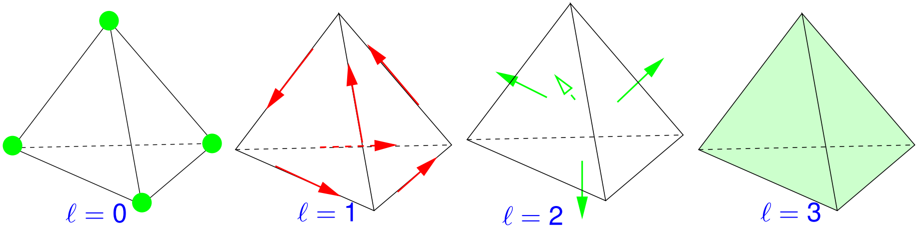

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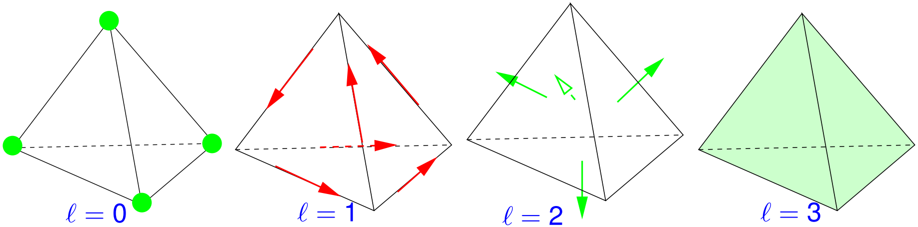

$$\begin{array}{ccccccc}
H^1(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{H}(\textbf{curl}, \Omega) & \xrightarrow{\textbf{curl}} & \boldsymbol{H}(\mathrm{div}, \Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\
\bigcup & & \bigcup & & \bigcup & & \bigcup \\
\mathcal{W}_0^0(\mathcal{T}) & \xrightarrow{\textbf{grad}} & \boldsymbol{\mathcal{W}}_0^1(\mathcal{T}) & \xrightarrow{\textbf{curl}} & \boldsymbol{\mathcal{W}}_0^2(\mathcal{T}) & \xrightarrow{\mathrm{div}} & \mathcal{W}_0^3(\mathcal{T})
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is an exact sequence.

\+ **commuting nodal interpolation operators**:

$$\begin{array}{ccccccc}
C^\infty(\Omega) & \xrightarrow{\textbf{grad}} & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\textbf{curl}} & \boldsymbol{C}^\infty(\Omega) & \xrightarrow{\text{div}} & C^\infty(\Omega) \\
\downarrow \Pi_0^0 & & \downarrow \Pi_0^1 & & \downarrow \Pi_0^2 & & \downarrow \Pi_0^3 \\
\mathcal{W}_0^0(\mathcal{T}) & \xrightarrow{\textbf{grad}} & \boldsymbol{\mathcal{W}}_0^1(\mathcal{T}) & \xrightarrow{\textbf{curl}} & \boldsymbol{\mathcal{W}}_0^2(\mathcal{T}) & \xrightarrow{\text{div}} & \mathcal{W}_0^3(\mathcal{T}) .
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What Next ?

1. Spaces
2. Regular Decomposition
3. Discrete Regular Decomposition (DRD): *h*-Version
4. Discrete Regular Decomposition: *p*-Version

Helmholtz Decompositions

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✎ spaces: $\Omega \subset \mathbb{R}^3$ bounded Lipschitz domain, $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$

$\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) := \{\mathbf{v} \in \boldsymbol{L}^2(\Omega)\colon\ \mathbf{curl}\,\mathbf{v} \in \boldsymbol{L}^2(\Omega), \mathbf{v} \times \mathbf{n} = 0 \text{ on } \Gamma_D\}$

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Classical $L^2(\Omega)$-orthogonal Helmholtz decompositions:

$$\boldsymbol{L}^2(\Omega) = \begin{cases} \mathbf{grad}\, H^1(\Omega) \oplus \mathbf{curl}\, \boldsymbol{H}_0(\mathbf{curl},\Omega) \oplus \mathcal{H}_D(\Omega)\ , \\ \mathbf{grad}\, H^1_0(\Omega) \oplus \mathbf{curl}\, \boldsymbol{H}(\mathbf{curl},\Omega) \oplus \mathcal{H}_N(\Omega)\ . \end{cases}$$

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➤ Restriction to $\boldsymbol{H}(\mathbf{curl}, \Omega)$, $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)$ (also $\boldsymbol{H}(\mathrm{div}, \Omega)$, $\boldsymbol{H}_{\Gamma_N}(\mathrm{div}, \Omega)$)

$L^2(\Omega)$-orthogonal Helmholtz decompositions of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, $\boldsymbol{H}_0(\mathbf{curl}, \Omega)$:

$$\boldsymbol{H}(\mathbf{curl}, \Omega) = \mathbf{grad}\, H^1(\Omega) \oplus (\boldsymbol{H}_0(\mathrm{div}\, 0, \Omega) \cap \boldsymbol{H}(\mathbf{curl}, \Omega)) \;,$$
$$\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) = \mathbf{grad}\, H_{\Gamma_D}^1(\Omega) \oplus (\boldsymbol{H}_{\Gamma_N}(\mathrm{div}\, 0, \Omega) \cap \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega)) \;.$$

Regular Decompositions

 Ω convex/smooth

$\boldsymbol{H}(\mathrm{div}\,0,\Omega) \cap \boldsymbol{H}_0(\mathbf{curl},\Omega) \subset (H^1(\Omega))^3$

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Ω convex/smooth	$\leftrightarrow$	Ω reentrant edges
$\boldsymbol{H}(\mathrm{div}\,0,\Omega) \cap \boldsymbol{H}_0(\mathbf{curl},\Omega) \subset (H^1(\Omega))^3$		$\boldsymbol{H}(\mathrm{div}\,0,\Omega) \cap \boldsymbol{H}_0(\mathbf{curl},\Omega) \not\subset (H^1(\Omega))^3$

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Regular decomposition theorem: $\exists$ stable splitting

$$\boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega) = \mathbf{V} \oplus \mathbf{grad}\, H^1_{\Gamma_D}(\Omega) \quad , \quad \mathbf{V} \subset (H^1_{\Gamma_D}(\Omega))^3 \; .$$

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$$\mathrm{R} : \begin{cases} \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega) & \rightarrow \quad (H^1_{\Gamma_D}(\Omega))^3 \\ \boldsymbol{L}^2(\Omega) & \rightarrow \quad \boldsymbol{L}^2(\Omega) \end{cases}$$

$$\mathrm{R} + \mathbf{grad} \circ \mathrm{N} = Id \quad \text{on} \quad \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)\ .$$

$\mathrm{N} : \boldsymbol{L}^2(\Omega) \rightarrow H^1_{\Gamma_D}(\Omega)$,

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"Gradients fill gap between $\boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega)$ and $(H^1_{\Gamma_D}(\Omega))^3$"

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h-DRD for Whitney 1-Forms: Goal

[Focus on Whitney forms]

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"Continuous" Regular Decomposition Theorem:

$$\forall\, \mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl},\Omega): \quad \exists\ \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3,\ \varphi \in H^1_{\Gamma_D}(\Omega)\ : \quad \mathbf{v} = \mathbf{z} + \mathbf{grad}\,\varphi\ ,$$

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FE space for regular part: $\quad \boldsymbol{\mathcal{V}}_h := (\mathcal{W}^0_{0,\Gamma_D}(\mathcal{T}))^3 \subset (H^1_{\Gamma_D}(\Omega))^3$

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Not possible: $\boldsymbol{\mathcal{V}}_h \not\subset \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\mathcal{T})$

h-DRD of Whitney 1-Forms: Construction & Proof

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Continuous RD of $\mathbf{v}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T})$: $\mathbf{v}_h = \mathbf{z} + \mathbf{grad}\,\varphi$, $\mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

h-DRD of Whitney 1-Forms: Construction & Proof

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! $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$ discrete ➤ $\left\|(Id - \Pi^1_h)\mathbf{z}\right\|_{L^2(\Omega)} \lesssim h\|\mathbf{z}\|_{H^1(\Omega)}$

A special *interpolation error estimate* for edge interpolation

If $\mathbf{v} \in (H^1(\Omega))^3$, $\mathbf{curl}\,\mathbf{v} \in \mathcal{W}^2_0(\mathcal{T})$, then

$$\left\|\mathbf{v} - \Pi^1_h\mathbf{v}\right\|_{0,T} \lesssim h_T\,|\mathbf{v}|_{1,T} \quad \forall T \in \mathcal{T}\,.$$

h-DRD of Whitney 1-Forms: Construction & Proof

Continuous RD of $\mathbf{v}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T})$: $\quad \mathbf{v}_h = \underbrace{\mathbf{z}}_{\notin \mathcal{V}_h} + \mathbf{grad}\, \underbrace{\varphi}_{\notin \mathcal{W}^0_{0,\Gamma_D}}$, $\quad \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

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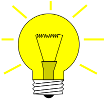

Throw $\mathbf{z} \to \mathcal{V}_h$ by quasi-interpolation $\mathsf{Q}_h : (H^1_{\Gamma_D}(\Omega))^3 \to \mathcal{V}_h$

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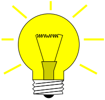
$\notin \boldsymbol{\mathcal{V}}_h \qquad \notin \mathcal{W}^0_{0,\Gamma_D}$

Continuous RD of $\mathbf{v}_h \in \boldsymbol{\mathcal{W}}^1_{0,\Gamma_D}(\mathcal{T})$: $\quad \mathbf{v}_h = \mathbf{z} + \mathbf{grad}\,\varphi$, $\quad \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

! $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$ discrete ➤ $\left\|(Id - \Pi^1_h)\mathbf{z}\right\|_{L^2(\Omega)} \lesssim h\|\mathbf{z}\|_{H^1(\Omega)}$



Throw $\mathbf{z} \to \boldsymbol{\mathcal{V}}_h$ by quasi-interpolation $\mathsf{Q}_h : (H^1_{\Gamma_D}(\Omega))^3 \to \boldsymbol{\mathcal{V}}_h$

Quasi-interpolation operator

= linear, H^1/L^2-stable, boundary-aware $\quad \mathsf{Q}_h : H^1_{\Gamma_D}(\Omega) \to \mathcal{W}^0_{0,\Gamma_D}(\mathcal{T})$

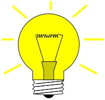
$$\begin{aligned} \|\mathsf{Q}_h\psi\|_{L^2(T)} &\lesssim \|\psi\|_{L^2(\omega_T)}\;, \\ |\mathsf{Q}_h\psi|_{H^1(T)} &\lesssim |\psi|_{H^1(\omega_T)}\;, \qquad \forall T \in \mathcal{T}\;, \\ \|\psi - \mathsf{Q}_h\psi\|_{L^2(T)} &\lesssim h_T|\psi|_{H^1(\omega_T)} \end{aligned}$$

$\omega_T \mathrel{\hat=}$ mesh neighborhood of T.

h-DRD of Whitney 1-Forms: Construction & Proof

Continuous RD of $\mathbf{v}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T})$: $\mathbf{v}_h = \underbrace{\mathbf{z}}_{\notin \mathcal{V}_h} + \mathbf{grad}\,\underbrace{\varphi}_{\notin \mathcal{W}^0_{0,\Gamma_D}}$, $\mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

! $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$ discrete ➤ $\left\|(Id - \Pi^1_h)\mathbf{z}\right\|_{L^2(\Omega)} \lesssim h\|\mathbf{z}\|_{H^1(\Omega)}$



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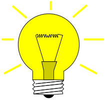
$$\mathbf{z} = \underbrace{\boldsymbol{\Pi}_h(\mathbf{z} - \mathsf{Q}_h\mathbf{z})}_{=:\ \widetilde{\mathbf{v}}_h} + \boldsymbol{\Pi}_h\,\underbrace{(\mathsf{Q}_h\mathbf{z})}_{=:\ \mathbf{z}_h} + \underbrace{(Id - \boldsymbol{\Pi}_h)\mathbf{z}}_{\mathbf{curl}\,=\,0\ !}$$

h-DRD of Whitney 1-Forms: Construction & Proof

Continuous RD of $\mathbf{v}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T})$: $\mathbf{v}_h = \mathbf{z} + \mathbf{grad}\,\varphi$, $\mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

($\mathbf{z} \notin \mathcal{V}_h$, $\varphi \notin \mathcal{W}^0_{0,\Gamma_D}$)

! $\mathbf{curl}\,\mathbf{z} = \mathbf{curl}\,\mathbf{v}_h$ discrete ➤ $\|(Id - \Pi^1_h)\mathbf{z}\|_{L^2(\Omega)} \lesssim h\|\mathbf{z}\|_{H^1(\Omega)}$



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Commuting diagram property

h-DRD of Whitney 1-Forms: Construction & Proof

$$\notin \mathcal{V}_h \qquad \notin \mathcal{W}^0_{0,\Gamma_D}$$

Continuous RD of $\mathbf{v}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T})$: $\quad \mathbf{v}_h = \mathbf{z} + \mathbf{grad}\,\varphi$, $\quad \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

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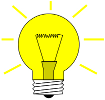

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φ_h as discrete potential: $\quad \underbrace{(Id - \boldsymbol{\Pi}_h)\mathbf{z} + \mathbf{grad}\,\varphi}_{\in \mathcal{W}^1_0(\mathcal{T}),\ \mathbf{curl}\,=\,0} = \mathbf{grad}\,\varphi_h$. ✔

h-DRD of Whitney 1-Forms: Construction & Proof

Continuous RD of $\mathbf{v}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T})$: $\quad \mathbf{v}_h = \underbrace{\mathbf{z}}_{\notin \mathcal{V}_h} + \mathbf{grad}\,\underbrace{\varphi}_{\notin \mathcal{W}^0_{0,\Gamma_D}}$, $\quad \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3$, $\varphi \in H^1_{\Gamma_D}(\Omega)$

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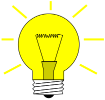

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h-DRD of Whitney 1-Forms: Construction & Proof

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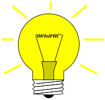

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$$\left\|\widetilde{\mathbf{v}}_h\right\|_{L^2(\Omega)} \leq \|(Id - \boldsymbol{\Pi}_h)(Id - \mathsf{Q}_h)\mathbf{z}\|_{L^2(\Omega)} + \|(Id - \mathsf{Q}_h)\mathbf{z}\|_{L^2(\Omega)} \lesssim h\|\mathbf{z}\|_{H^1(\Omega)}.$$

Summary: h-DRD for Whitney 1-Forms

Summary: *h*-DRD for Whitney 1-Forms

"Continuous" Regular Decomposition Theorem:

$$\forall \mathbf{v} \in \boldsymbol{H}_{\Gamma_D}(\mathbf{curl}, \Omega): \quad \exists \ \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3, \ \varphi \in H^1_{\Gamma_D}(\Omega) \ : \quad \mathbf{v} = \mathbf{z} + \mathbf{grad}\,\varphi \ ,$$

$$\|\mathbf{z}\|_{L^2(\Omega)} \lesssim \|\mathbf{v}\|_{L^2(\Omega)} \ , \quad |\mathbf{z}|_{H^1(\Omega)} \lesssim \|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \ , \quad \|\varphi\|_{L^2(\Omega)} \lesssim \|\mathbf{v}\|_{L^2(\Omega)} \ .$$

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$$\exists\ \varphi_h \in \mathcal{W}^0_{0,\Gamma_D}(\mathcal{T}),\ \mathbf{z}_h \in \boldsymbol{\mathcal{V}}_h,\ \widetilde{\mathbf{v}}_h \in \mathcal{W}^1_{0,\Gamma_D}(\mathcal{T}): \quad \mathbf{v}_h = \Pi^1_h \mathbf{z}_h + \widetilde{\mathbf{v}}_h + \mathbf{grad}\,\varphi_h\ ,$$

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constants *depending only* on Ω, Γ_D, and shape regularity.

Summary: h-DRD for Whitney 1-Forms

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Summary: h-DRD for Whitney 1-Forms

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Summary: *h*-DRD for Whitney 1-Forms

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Summary: h-DRD for Whitney 1-Forms

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Summary: h-DRD for Whitney 1-Forms

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Summary: *h*-DRD for Whitney 1-Forms

"Continuous" Regular Decomposition Theorem:

Applications:

Multigrid theory DD theory	Auxiliary space preconditioners
Error estimators	Edge BEM analysis

constants *depending only* on Ω, Γ_D, and shape regularity.

What Next ?

1. Spaces
2. Regular Decomposition
3. Discrete Regular Decomposition (DRD): *h*-Version
4. Discrete Regular Decomposition: *p*-Version

Higher-Order Discrete Differential Forms

Higher-Order Discrete Differential Forms

- Ω $\hat{=}$ bounded (curvilinear) Lipschitz polyhedron
- $\mathcal{T}$ $\hat{=}$ **tetrahedral mesh** of Ω
- $p \in \mathbb{N}_0$ $\hat{=}$ (uniform) polynomial degree

Higher-Order Discrete Differential Forms

First family of discrete differential forms ($\mathcal{P}_p \hat{=}$ polynomials)

- $\ell = 0$: Lagrangian finite element spaces

$$\mathcal{W}_p^0(\mathcal{T}) := \{v \in H^1(\Omega) : \; v|_T \in \mathcal{P}_{p+1}, \; T \in \mathcal{T}\} \; .$$

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- $\ell = 1$: Nédélec (edge) finite elements

$$\mathcal{W}_p^1(\mathcal{T}) := \left\{ v \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \begin{array}{l} v|_T(\boldsymbol{x}) = \mathbf{p}(\boldsymbol{x}) + \mathbf{q}(\mathbf{x}) \times \boldsymbol{x}, \\ \mathbf{p}, \mathbf{q} \in (\mathcal{P}_p(T))^3, \end{array} \; T \in \mathcal{T} \right\} .$$

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- $\ell = 2$: Raviart-Thomas (face) elements

$$\mathcal{W}^2_p(\mathcal{T}) := \left\{ v \in \boldsymbol{H}(\mathrm{div}, \Omega) : \; \begin{array}{l} v|_T(\boldsymbol{x}) = \mathbf{p}(\boldsymbol{x}) + q(\mathbf{x})\boldsymbol{x}, \\ \mathbf{p} \in (\mathcal{P}_p(T))^3, \; q \in \mathcal{P}_p(T) \end{array} \; T \in \mathcal{T} \right\} \; .$$

Higher-Order Discrete Differential Forms

First family of discrete differential forms ($\mathcal{P}_p \hat{=}$ polynomials)

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- $\ell = 3$: Discontinuous piecewise polynomials

$$\mathcal{W}_p^3(\mathcal{T}) := \{v \in L^2(\Omega) : \ v|_T \in \mathcal{P}_p, \ T \in \mathcal{T}\} .$$

Higher-Order Discrete Differential Forms

First family of discrete differential forms ($\mathcal{P}_p \hat{=}$ polynomials)

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$$\mathcal{W}^0_p(\mathcal{T}) := \{v \in H^1(\Omega) : \; v|_T \in \mathcal{P}_{p+1}, \; T \in \mathcal{T}\} \,.$$

- $\ell = 1$: Nédélec (edge) finite elements

$$\mathcal{W}^1_p(\mathcal{T}) := \left\{ v \in \boldsymbol{H}(\mathbf{curl}, \Omega) : \begin{array}{l} v|_T(\boldsymbol{x}) = \mathbf{p}(\boldsymbol{x}) + \mathbf{q}(\mathbf{x}) \times \boldsymbol{x}, \\ \mathbf{p}, \mathbf{q} \in (\mathcal{P}_p(T))^3, \end{array} \; T \in \mathcal{T} \right\} \,.$$

- $\ell = 2$: Raviart-Thomas (face) elements

$$\mathcal{W}^2_p(\mathcal{T}) := \left\{ v \in \boldsymbol{H}(\mathrm{div}, \Omega) : \begin{array}{l} v|_T(\boldsymbol{x}) = \mathbf{p}(\boldsymbol{x}) + q(\mathbf{x})\boldsymbol{x}, \\ \mathbf{p} \in (\mathcal{P}_p(T))^3, \; q \in \mathcal{P}_p(T) \end{array} \; T \in \mathcal{T} \right\} \,.$$

- $\ell = 3$: Discontinuous piecewise polynomials

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▷ FE spaces with locally vanishing $\partial\Omega$-trace: $\mathcal{W}^\ell_{p,\Gamma_D}(\mathcal{T}) \subset H_{\Gamma_D}(\Lambda^\ell, \Omega)$

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\+ B **=** polynomial preserving

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Thm.: (Spectral interpolation error estimate for $\Pi^{0}_{T,p}$)

$$\left|(\mathrm{Id} - \Pi^{0}_{T,p})v\right|_{H^1(T)} \lesssim \frac{h_T}{p+1} |v|_{H^2(T)} \quad \forall v \in H^2(T) \, ,$$

with constants depending only on shape regularity of T.

Boundary-Aware p-Uniform Quasi-Interpolation

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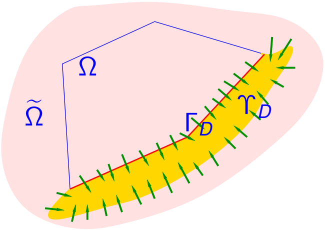

Bulge-contracting deformation T

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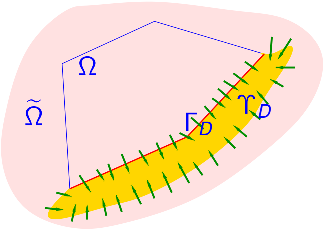

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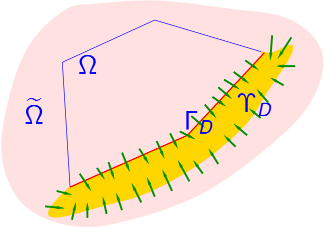

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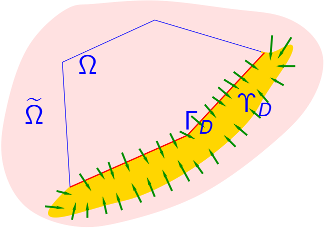

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Boundary-Aware p-Uniform Quasi-Interpolation

Smoothed resolution function $\epsilon : \widetilde{\Omega} \to \mathbb{R}^+ \in C^\infty$, $\epsilon \approx h_T/p+1$ on $T \in \mathcal{T}$

Mollification ($\rho \in C^\infty$, $\mathrm{supp}(\rho) \subset B_1(0)$, six vanishing moments)

$$(\mathsf{E}v)(\boldsymbol{x}) = \int_{\mathbb{R}^3} v(\boldsymbol{y})\rho_{\delta\epsilon(\boldsymbol{x})}(\boldsymbol{x}-\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,, \quad \boldsymbol{x} \in \widetilde{\Omega},\ v \in L^1(\widetilde{\Omega})\,.$$

► $$\mathsf{Q}_p := \mathsf{I}_p \circ \mathsf{E} \circ \mathsf{T}^* : L^1(\widetilde{\Omega}) \to \mathcal{W}_p^0(\mathcal{T})$$

The **quasi-interpolation operators** $\mathsf{Q}_p : L^1(\widetilde{\Omega}) \to \mathcal{W}_p^0(\mathcal{T}) \subset H^1(\Omega)$ satisfy

(Q$_1$) $\mathsf{Q}_p v \in \mathcal{W}^0_{p,\Gamma_D}(\mathcal{T}) \subset H^1_{\Gamma_D}(\Omega)$, if $v|_{\Upsilon_D} = 0$ (boundary-aware),

(Q$_2$) $\|\mathsf{Q}_p u\|_{L^2(\Omega)} \lesssim \|u\|_{L^2(\widetilde{\Omega})}$ for all $u \in L^2(\widetilde{\Omega})$,

(Q$_3$) $|\mathsf{Q}_p u|_{H^1(\Omega)} \lesssim |u|_{H^1(\widetilde{\Omega})}$ for all $u \in H^1(\widetilde{\Omega})$,

(Q$_4$) $\left\|\epsilon^{-1}(\mathsf{Id} - \mathsf{Q}_p)u\right\|_{L^2(\Omega)} \lesssim |u|_{H^1(\widetilde{\Omega})}$ for all $u \in H^1(\widetilde{\Omega})$,

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"Continuous" Regular Decomposition Theorem:

$$\forall \mathbf{v} \in \boldsymbol{H}(\mathbf{curl},\Gamma_D) \quad \exists \; \mathbf{z} \in (H^1_{\Gamma_D}(\Omega))^3,\; \varphi \in H^1_{\Gamma_D}(\Omega) \; : \quad \mathbf{v} = \mathbf{z} + \mathbf{grad}\,\varphi \;,$$

$$\|\mathbf{z}\|_{L^2(\Omega)} \lesssim \|\mathbf{v}\|_{L^2(\Omega)} \;, \quad |\mathbf{z}|_{H^1(\Omega)} \lesssim \|\mathbf{v}\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \;, \quad \|\varphi\|_{L^2(\Omega)} \lesssim \|\mathbf{v}\|_{L^2(\Omega)} \;.$$

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$$\left(\sum_{T\in\mathcal{T}} \left\| \frac{p+1}{h_T} \widetilde{\mathbf{v}}_p \right\|^2_{L^2(T)} \right)^{1/2} \lesssim \|\mathbf{v}_p\|_{\boldsymbol{H}(\mathbf{curl},\Omega)} \ ,$$

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R. HIPTMAIR AND C. PECHSTEIN, *Discrete regular decompositions of tetrahedral discrete 1-forms*, in Maxwell's Equations: Analysis and Numerics, U. Langer, D. Pauly, and S. Repin, eds., vol. 24 of Radon Series on Computational and Applied Mathematics, De Gruyter, Stuttgart, 2019, ch. 7, pp. 199–258.

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THANK YOU