Matrix-valued Finite Elements with Applications in Elasticity and Curvature

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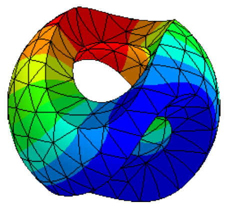

with contributions by

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Vector-valued function spaces

function spaces:

$$
\begin{aligned}
H(\mathrm{curl}) &= \{u \in [L_2]^3 : \mathrm{curl}\, u \in [L_2]^3\} \\
H(\mathrm{div}) &= \{u \in [L_2]^3 : \mathrm{div}\, u \in L_2\}
\end{aligned}
$$

- tangential / normal boundary traces
- tangential / normal continuous finite element spaces ($\mathcal{N}^k, \mathcal{BDM}^k$)
- exact de Rham sequence:
$$H^1 \xrightarrow{\mathrm{grad}} H(\mathrm{curl}) \xrightarrow{\mathrm{curl}} H(\mathrm{div}) \xrightarrow{\mathrm{div}} L_2$$
$\mathrm{range}\, A_i = \ker A_{i+1}$
- regular decomposition $H(\mathrm{curl}) = [H^1]^3 + \nabla H^1$, $H(\mathrm{div}) = [H^1]^3 + \mathrm{curl}[H^1]^3$

Matrix-valued function spaces

$$
\begin{array}{rcl}
H(dd) := H(\operatorname{div}\operatorname{div}) & := & \{\sigma \in [L_2]^{3\times 3, sym} : \operatorname{div}\operatorname{div}\, \sigma \in H^{-1}\} \\
H(cd) := H(\operatorname{curl}\operatorname{div}) & := & \{\sigma \in [L_2]^{3\times 3} \qquad : \operatorname{curl}\operatorname{div}\sigma \in [H^{-1}]^3\} \\
H(cc) := H(\operatorname{curl}\operatorname{curl}) & := & \{\sigma \in [L_2]^{3\times 3, sym} : \operatorname{curl}^T \operatorname{curl}\sigma \in [H^{-1}]^{3\times 3, sym}\}
\end{array}
$$

finite element spaces for $k \geq 0$, slightly non-conforming

$$
\begin{array}{rcl}
V_{dd}^k & = & \{\sigma \in [L_2]^{3\times 3, sym} : \sigma_{|T} \in P^k, \sigma_{nn} \text{ continuous}\} \\
V_{cd}^k & = & \{\sigma \in [L_2]^{3\times 3} \qquad : \sigma_{|T} \in P^k, \sigma_{nt} \text{ continuous}\} \\
V_{cc}^k & = & \{\sigma \in [L_2]^{3\times 3, sym} : \sigma_{|T} \in P^k, \sigma_{tt} \text{ continuous}\}
\end{array}
$$

- In 2D, V_{dd}^k is the Hellan-Herrmann-Johnson finite element space. V_{cc} is the Regge finite element space [Christiansen '11, Li '18]
- Regular decomposition: $H(cc) = [H^1]^{3\times 3, sym} + \varepsilon([H^1]^3)$, ...
- Shape functions are defined on reference elements, two-sided Piola/covariant transformations preserve normal/tangential components, mapping on manifolds

3-step exact sequence

Smooth subspaces form a sequence:

$$H^1 \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{sym-grad}^T} H(cc) \xrightarrow{\text{curl}} H(cd) \xrightarrow{\text{sym-curl}^T} H(dd) \xrightarrow{\text{div}} H^{-1}(\text{div}) \xrightarrow{\text{div}} H^{-1}$$

- 3-step exact sequence: $\text{range}\, A_i = \ker A_{i+2} A_{i+1}$
- includes the Kröner complex (with $\varepsilon = \text{sym-grad}$ and $\text{inc} = \text{curl}^T \text{curl}$).

$$[H^1]^3 \xrightarrow{\varepsilon} H(cc) \xrightarrow{\text{inc}} H(dd) \xrightarrow{\text{div}} H^{-1}$$

- homogeneous boundary conditions on flat boundaries are preserved (t-component for $H(\text{curl})$, tt-component for $H(cc)$, nt-component for $H(cd)$, ...)
- useful for construction of basis functions and designing bubbles not polluting the range of the next operators (elasticity, weakly-symmetric formulation for Stokes)

Distributional derivatives

Let $\sigma \in V_{dd}^k$. Then the distributional divergence $f := \operatorname{div} \sigma$ is

$$\begin{aligned} \langle f, \varphi \rangle &= -\int \sigma : \nabla \varphi = -\sum_T \int_T \sigma : \nabla \varphi = \sum_T \int_T \operatorname{div} \sigma - \int_{\partial T} \sigma_n \varphi \\ &= \sum_T \int_T \operatorname{div} \sigma \varphi - \sum_E \int_E [\sigma_n] \varphi = \sum_T \int_T \underbrace{\operatorname{div} \sigma}_{f_T} \varphi - \sum_E \int_E \underbrace{[\sigma_{nt}]}_{f_E} \varphi_t \end{aligned}$$

$f = \operatorname{div} \sigma$ consists of element-terms and facet-terms:

$$\begin{aligned} f_T &= \operatorname{div}_T \sigma \\ f_E &= [\sigma_{nt}] \qquad \text{vector in tangential space} \end{aligned}$$

It can be applied to $v_h \in \mathcal{N} \subset H(\text{curl})$.

Write duality pairing as

$$\langle \operatorname{div} \sigma, v \rangle \qquad \text{for } \sigma \in V_{dd}^k,\ v \in \mathcal{N}^k$$

Second distributional derivatives

Let f as above, and $g = \operatorname{div} f$. Then

$$\begin{aligned} \langle g, \varphi \rangle &= -\sum_T \int_T f_T \nabla \varphi - \sum_E \int_E f_E \nabla_t \varphi \\ &= \sum_T \int_T \operatorname{div}_T f_T \, \varphi + \sum_E \int_E ([f_{T,n}] + \operatorname{div}_t f_E) \, \varphi + \sum_V \sum_{T:V \in T} (\sigma_{n_1 t_1} - \sigma_{n_2 t_2}) \varphi \end{aligned}$$

$$\begin{aligned} g_T &= \operatorname{div}_T f_T \\ g_E &= [f_{T,n}] + \operatorname{div}_t f_E \\ g_V &= \sum_{T:V \in T} (\sigma_{n_1 t_1} - \sigma_{n_2 t_2}) \end{aligned}$$

g is a measure and can be applied to $v_h \in \mathcal{L}^{k+1} \subset H^1$. Due to the arising point functionals, V_{dd} is slightly non-conforming for $H(dd)$.

$H(dd)$ and $H(cd)$-based methods for Elasticity and Stokes

Find stress $\sigma \in V_{dd}^k$ and displacement $u \in \mathcal{N}^k$ (the TDNNS method: robust for thin structors)

$$\begin{array}{lllll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, u \rangle & = & 0 & \forall\, \tau \in V_{dd} \\ \langle \operatorname{div} \sigma, v \rangle & & & = & f(v) & \forall\, v \in \mathcal{N} \end{array}$$

Astrid Pechstein Phd-thesis and joint work ['11,'12,'18,'21]

Find $\sigma \in V_{cd}^k$, $u \in \mathcal{BDM}^k$, and $p \in P^{k-1}$ (the pressure-robust MCS method for Stokes):

$$\begin{array}{llllll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, u \rangle + (\operatorname{div} u, q) & = & 0 & \forall\, \tau \in V_{cd}, \forall\, q \in P^{k-1} \\ \langle \operatorname{div} \sigma, v \rangle + (\operatorname{div} v, p) & & & = & f(v) & \forall\, v \in \mathcal{BDM}^k \end{array}$$

Philip Lederer Phd-thesis and P. Lederer-J. Gopalakrishnan-JS ['20, '20]

$H(dd)$ methods for plates

Hellan-Herrmann-Johnson (HHJ) method for the Kirchhoff plate: ['60s and '70s, I. Comodi '89]

Find bending moments $\sigma \in V_{dd}^k$ and vertical deflection $w \in \mathcal{L}^{k+1}$:

$$\begin{array}{lcccll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, \nabla w \rangle & = & 0 & \forall\, \tau \in V_{dd}^k \\ \langle \operatorname{div} \sigma, \nabla v \rangle & & & = & f(v) & \forall\, v \in \mathcal{L}^{k+1} \end{array}$$

Combination of HHJ and TDNNS for Reissner Mindlin [A. Pechstein-JS '17]:

Find $\sigma \in V_{dd}^k$ and $w \in \mathcal{L}^{k+1}$, $\beta \in \mathcal{N}^k$:

$$\begin{array}{lcccll} \int A\sigma : \tau & + & \langle \operatorname{div} \tau, \beta \rangle & = & 0 & \forall\, \tau \in V_{dd}^k \\ \langle \operatorname{div} \sigma, \delta \rangle & - & \frac{1}{t^2}(\nabla w - \beta, \nabla v - \delta) & = & f(v) & \forall\, v \in \mathcal{L}^{k+1},\ \forall\, \delta \in \mathcal{N}^k, \end{array}$$

Free of locking, and for $t \to 0$ the discrete RM solution converges to the Kirchhoff solution.

The TD-NNS mixed method for elasticity

The elasticity problem is equivalent to the mixed problem: Find $\sigma \in H(\operatorname{div}\operatorname{div})$ and $u \in H(\operatorname{curl})$ such that for tangentially continuous v and normal-normal continuous τ:

$$
\begin{array}{lclcll}
\int A\sigma : \tau & + & \sum_T \left\{ \int_T \operatorname{div}\tau \cdot u - \int_{\partial T} \tau_{n\tau} u_\tau \right\} & = & 0 & \forall\, \tau \\
\sum_T \left\{ \int_T \operatorname{div}\sigma \cdot v - \int_{\partial T} \sigma_{n\tau} v_\tau \right\} & & & = & -\int f \cdot v & \forall\, v
\end{array}
$$

Proof: The second line is equilibrium, plus tangential continuity of the normal stress vector:

$$\sum_T \int_T (\operatorname{div}\sigma + f) v + \sum_E \int_E [\sigma_{n\tau}] v_\tau = 0 \qquad \forall\, v$$

Since the space requires continuity of σ_{nn}, the normal stress vector is continuous.
Element-wise integration by parts in the first line gives

$$\sum_T \int_T (A\sigma - \varepsilon(u)) : \tau + \sum_E \int_E \tau_{nn} [u_n] = 0 \qquad \forall\, \tau$$

This is the constitutive relation, plus normal-continuity of the displacement. Tangential continuity of the displacement is implied by the space $H(\operatorname{curl})$.

Equilibrated residuals for Kirchhoff Plates

Plate problem: Find vertical deflection $w \in H^2_{0,D}$:

$$\int \nabla^2 w : \nabla^2 v = \int f v \qquad \forall\, v \in H^2_{0,D}$$

- Discretize by some method, e.g. C^0-IPDG to compute w_h.
- Local interpolation to some $w_h^* \in H^2$.
- Local postprocessing of $\sigma_h^* \approx \nabla_h^2 w_h$ with $\operatorname{div}\operatorname{div}\sigma_h^* = f$ for $\sigma_h^* \in V_{dd}$.
- Prager-Synge:

$$\|\nabla^2 w_h^* - \nabla^2 w\|_{L_2}^2 + \|\sigma_h^* - \nabla^2 w\|_{L_2}^2 = \|\nabla^2 w_h^* - \sigma_h^*\|_{L_2}^2$$

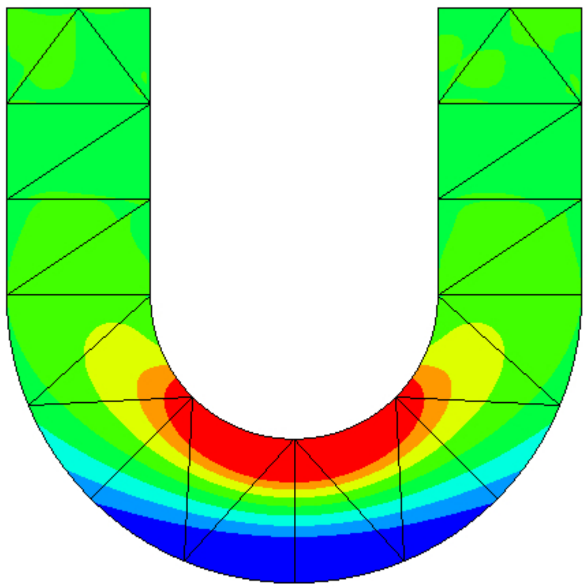
[D. Braess-A. Pechstein-J.S, '20]

Equilibrated residuals for Stokes: P. Lederer + C. Merdon '21

Curved elements

fixed left top, pull right top

Elements of order 5



σ_{xx}

Mapped elements by two-sided Piola:

$$\sigma(x) = \frac{1}{J^2} F \widehat{\sigma}(\hat{x}) F^t$$

Mapping preserves nn-continuity, but not nt-continuity

$\operatorname{div} \sigma$ is not an algebraic transformation of $\widehat{\operatorname{div}} \hat{\sigma}$, but

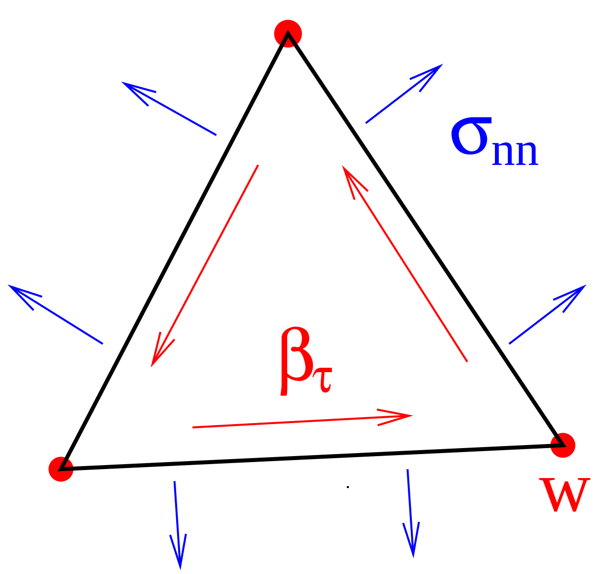
$$\operatorname{div} \sigma = \frac{1}{J} F \, \widehat{\operatorname{div}} \, \hat{\sigma} + something(\nabla F) : \hat{\sigma}$$

Reissner Mindlin Plates and Thin 3D Elements

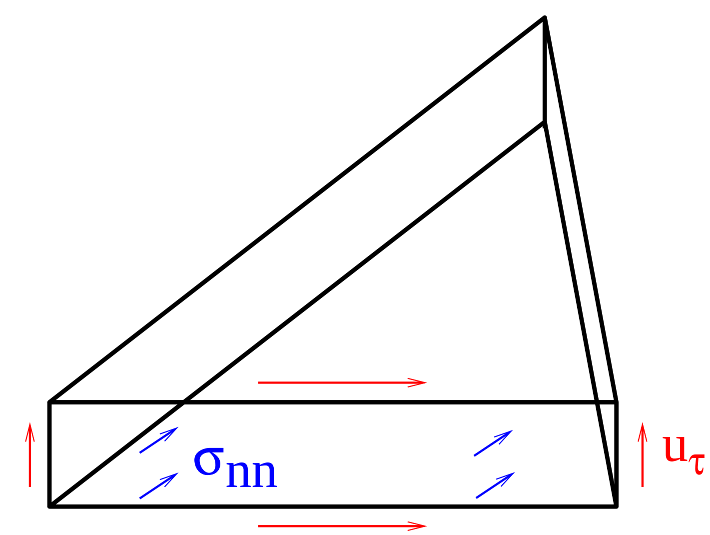
Mixed method with $\sigma = A^{-1}\varepsilon(\beta) \in H(\operatorname{div}\operatorname{div})$, $\beta \in H(\operatorname{curl})$, and $w \in H^1$:

$$L(\sigma; \beta, w) = \|\sigma\|_A^2 + \langle \operatorname{div} \sigma, \beta \rangle - t^{-2} \|\nabla w - \beta\|^2$$

Reissner Mindlin element:



3D prism element:



Hierarchical modeling: 3D discretization contains 2D reduced model

Geometric nonlinear Elasticity

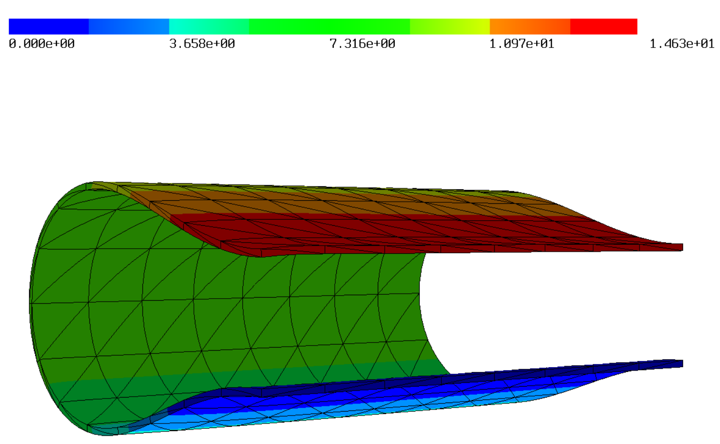
[M. Neunteufel + A. Pechstein + J.S to appear in CMAME, 2021, Phd-thesis M. Neunteufel 2021]

Hu-Washizu three-field mixed formulation

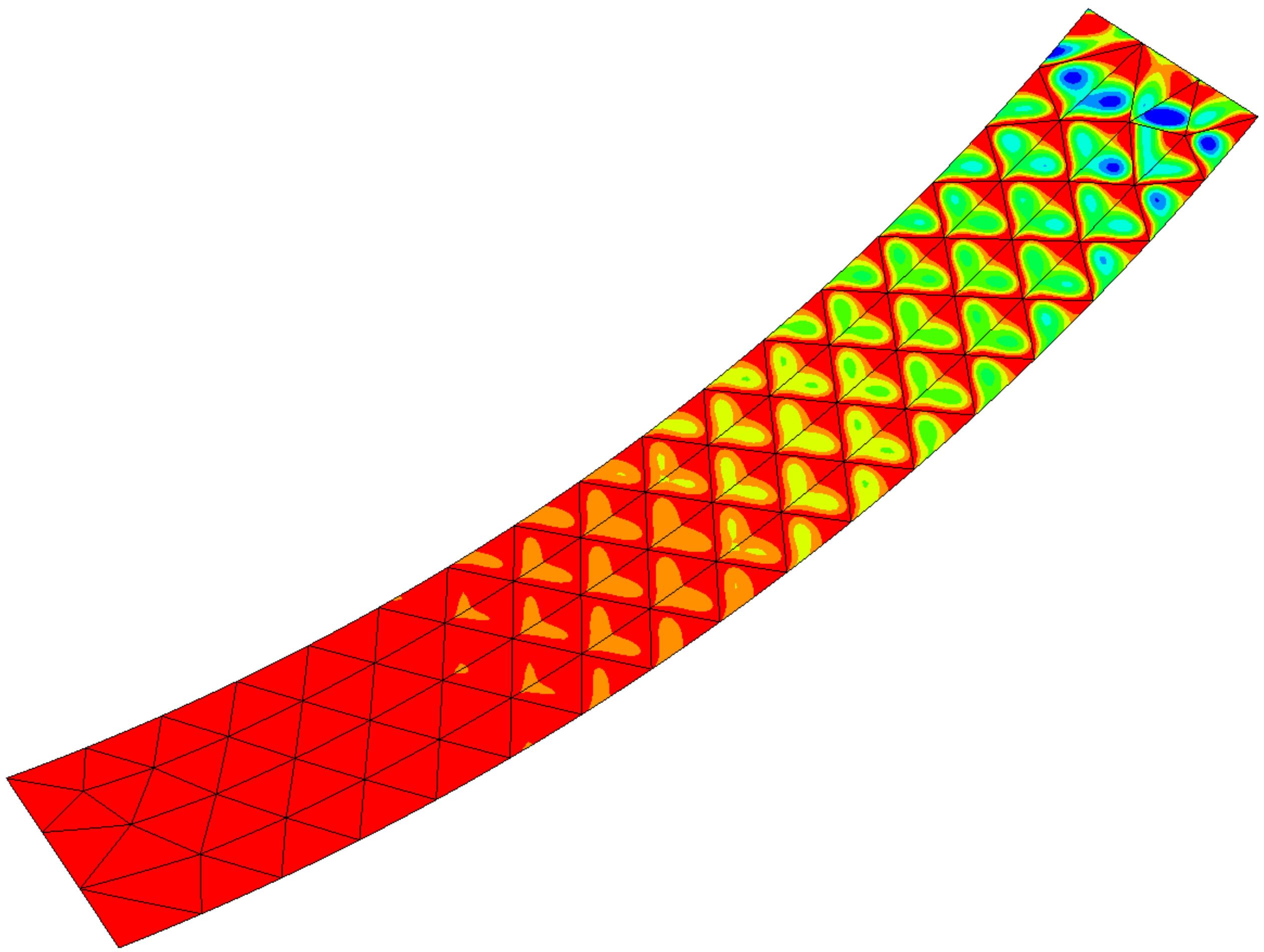
$$\min_{\substack{u,C \\ <C(u)-C,\Sigma>=0}} \int_\Omega W(C)\,dx - \int_\Omega f u\,dx$$

with

- $u \in H(\mathrm{curl})$
- $\Sigma \in H(\operatorname{div}\operatorname{div})$ ... 2^{nd} Piola-Kirchhoff
- $C \in L_2(\mathbb{R}^{d\times d,sym})$ ... Cauchy-Green streain
- $W(.,)$ ... hyperelastic energy functional
- pressure-robust nearly incompressible $(\det F = 1)$



Checkerboarding for Valentine's day



Riemann curvature and Incompatibility

The Kröner complex [Kröner 85, Int. J. Solid Structures]:

linear elasticity:

$$[H^1]^3 \xrightarrow{\varepsilon(\cdot)} H(cc) \xrightarrow{\text{inc}} H(dd)$$

nonlinear elasticity: Cauchy-Green strain and Riemann curvature:

$$[H^1]^3 \xrightarrow{C(\cdot)} H(cc) \xrightarrow{R(\cdot)} H(dd)$$

with

$$\begin{aligned} C(\varphi) &= \nabla\varphi^T \nabla\varphi \\ R_{qijk}(g) &= \partial_j \Gamma_{ikq} - \partial_k \Gamma_{ijq} + \Gamma_{ij}^p \Gamma_{kqp} - \Gamma_{ik}^p \Gamma_{jqp} \end{aligned}$$

with Christoffel symbols Γ.