Multigrid for Hybridized Discontinuous Galerkin Methods

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General structure

- Degrees of freedom are located on the skeleton of $\mathcal{T}_\ell$
  - Skeletal function space M_ℓ
- Bilinear forms are defined by cellwise Dirichlet-to-Neumann maps
  - for polynomial spaces
  - (in a DG sense)
- ... and numerical trace operators

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How much of the following transfers to HHO?

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How much of the following transfers to HHO? ...and to VEM?

Hybridized mixed method

Matching spaces

$$\nabla \cdot \boldsymbol{W}_T = V_T \qquad M_N = \operatorname{tr}_N \boldsymbol{W}_T \cdot \mathrm{n}.$$

Options: Raviart-Thomas and Brezzi-Douglas-Marini

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Spaces chosen to ensure

- Local problems are solvable
- Trace spaces are sufficiently large

Stabilization needed: add

$$\sum_{T\in\mathcal{T}} \int_{\partial T} \big[\boldsymbol{q}\cdot \mathrm{n} + \tau(u-\lambda)\big] v \,\mathrm{d}\sigma \qquad \forall v \in V_T.$$

Choices for τ: $\mathcal{O}(1)$ or $\mathcal{O}(1/h)$

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Local solvers

Local solver example

$$\int_T \boldsymbol{q}_T \cdot \boldsymbol{p}_T \,\mathrm{d}x - \int_T u_T \nabla\cdot \boldsymbol{p}_T \,\mathrm{d}x = -\int_{\partial T} \lambda \boldsymbol{p}_T \cdot \mathrm{n}\,\mathrm{d}\sigma$$

$$-\int_T \boldsymbol{q}_T \cdot \nabla v_T \,\mathrm{d}x + \int_{\partial T} (\boldsymbol{q}_T \cdot \mathrm{n} + \tau_\ell u_T) v_T \,\mathrm{d}\sigma = \tau_\ell \int_{\partial T} \lambda v_T \,\mathrm{d}\sigma$$

- Defines[1] operators $\mathcal{U}_\ell$, $\boldsymbol{\mathcal{Q}}_\ell$, and $\widehat{\mathcal{Q}}_\ell$ by

$$\mathcal{U}_\ell \lambda_{|T} = u_T, \qquad \boldsymbol{\mathcal{Q}}_\ell \lambda_{|T} = \boldsymbol{q}_T, \qquad \widehat{\mathcal{Q}}_\ell \lambda_{|\partial T} = \boldsymbol{q}_T \cdot \mathrm{n} + \tau_\ell (u_T - \lambda).$$

- Complemented by flux balance

$$a_\ell(\lambda, \mu) \equiv \sum_{T \in \mathcal{T}_\ell} \int_{\partial T} \widehat{\mathcal{Q}}_\ell \lambda \mu \,\mathrm{d}\sigma = \sum_{F \in \mathcal{F}} \int_F [\![\widehat{\mathcal{Q}}_\ell \lambda]\!] \mu \,\mathrm{d}\sigma = 0 \qquad \forall \mu \in M_\ell.$$

[1] **Cockburn, Gopalakrishnan, and Lazarov 2009.**

Useful properties

- The bilinear from $a_\ell(\cdot,\cdot)$ is s.p.d. and

$$a_\ell(\lambda,\mu) = \int_\Omega \boldsymbol{\mathcal{Q}}_\ell\lambda\boldsymbol{\mathcal{Q}}_\ell\mu\,\mathrm{d}x + \sum_{T\in\mathcal{T}_\ell}\int_{\partial T}\tau_\ell(\mathcal{U}_\ell\lambda-\lambda)(\mathcal{U}_\ell\mu-\mu)\,\mathrm{d}\sigma.$$

- λ_ℓ solves

$$a_\ell(\lambda_\ell,\mu) = b_\ell(\mu) \equiv \int_\Omega \mathcal{U}_\ell\mu f\,\mathrm{d}x. \qquad \forall\mu\in M_\ell$$

- Given a solution to $-\Delta u = f$
  - λ_ℓ approximates u on the skeleton
  - $\mathcal{U}_\ell\lambda_\ell$ approximates u in Ω
  - $\boldsymbol{\mathcal{Q}}_\ell\lambda_\ell$ approximates $-\nabla u$ in Ω

Standard assumptions on local solvers

- The trace of the local reconstruction $\mathcal{U}_\ell\lambda$ approximates the skeletal function λ itself, namely

$$\|\mathcal{U}_\ell\mu - \mu\|_\ell \lesssim h_\ell\|\mathcal{Q}_\ell\mu\|_0. \tag{LS1}$$

- Both $\mathcal{Q}_\ell\lambda$ and $\mathcal{U}_\ell\lambda$ are bounded by the traces:

$$\|\mathcal{Q}_\ell\mu\|_0 \lesssim h_\ell^{-1}\|\mu\|_\ell \quad \text{and} \quad \|\mathcal{U}_\ell\mu\|_0 \lesssim \|\mu\|_\ell. \tag{LS2}$$

- The reconstruction $\mathcal{Q}_\ell\lambda$ approximates the negative gradient of $\mathcal{U}_\ell\lambda$:

$$\|\mathcal{Q}_\ell\mu + \nabla\mathcal{U}_\ell\mu\|_0 \lesssim h_\ell^{-1/2}\|\mathcal{U}_\ell\mu - \mu\|_\ell, \tag{LS3}$$

Norm on skeleton is scaled

$$\|\lambda\|_\ell^2 = \sum h_E\|\lambda\|_E^2$$

Additional assumptions

- Consistency with linear FEM: for $w \in \overline{V}_\ell^c$ and $\mu = \operatorname{tr}_\ell w$ holds
$$\mathcal{Q}_\ell \mu = -\nabla w \quad \text{and} \quad \mathcal{U}_\ell \mu = w. \tag{LS4}$$
- Convergence of the Lagrange multipliers to the projected traces of the analytical solution.
$$\|\Pi_{M_\ell} u - \lambda\|_\ell \lesssim h_\ell^2 |u|_2. \tag{LS5}$$
- The standard spectral properties of the condensed stiffness matrix hold in the sense that
$$C_1 \|\mu\|_\ell^2 \le a_\ell(\mu,\mu) \le C_2 h_\ell^{-2} \|\mu\|_\ell^2. \tag{LS6}$$

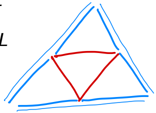
Hierarchies (geometric multigrid)

1. Hierarchy of meshes

$$\mathcal{T}_0 \sqsubset \mathcal{T}_1 \sqsubset \cdots \sqsubset \mathcal{T}_\ell \sqsubset \cdots \sqsubset \mathcal{T}_L$$

2. Hierarchy of spaces

$$\begin{aligned} \text{bulk:} \quad & V_0 \subset \quad \cdots \subset V_L \\ \text{skeletal:} \quad & M_0 \not\subset \quad \cdots \not\subset M_L \end{aligned}$$



3. Injection operator

$$\begin{aligned} \text{bulk:} \quad & I_\ell : V_{\ell-1} \hookrightarrow V_\ell \\ \text{skeletal:} \quad & I_\ell : M_{\ell-1} \to M_\ell \end{aligned}$$

4. Bilinear forms and operators

$$a_\ell(\lambda, \mu) = \langle A_\ell \lambda, \mu \rangle_\ell \qquad \forall \lambda, \mu \in M_\ell$$

The V-cycle

$$B_\ell\colon M_\ell \to M_\ell, \qquad B_0 = A_0^{-1}$$

The V-cycle Algorithm

1. Pre-smoothing: $x^0 = 0$ and for $i = 1, \dots, m$:
$$x^i = x^{i-1} + R_\ell^i(\mu - A_\ell x^{i-1}). \tag{1}$$
2. Set $y^0 = x^m + I_\ell q$, where $q \in M_{\ell-1}$ is defined as
$$q = B_{\ell-1} I_\ell^t(\mu - A_\ell x^m). \tag{2}$$
3. Post-smoothing: for $i = 1, \dots, m$
$$y^i = y^{i-1} + R_\ell^{i+m}(\mu - A_\ell y^{i-1}). \tag{3}$$

Let $B_\ell \mu = y^m$.

Smoothing operator

This talk is not about smoothers

$$x^{i-1} + R_\ell^i(\mu - A_\ell x^{i-1})$$

- Any simple relaxation method: Richardson, Jacobi, Gauß-Seidel, ...
- block versions grouping degrees of freedom on a face
- Anything fitting Bramble and Pasciak 1992

Heterogeneous multigrid methods

- Fine level is HDG
- Second level is a bulk formulation on the same level
- Then do multigrid for the bulk formulation

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Examples:

- Crouzeix-Raviart
  - Equivalence with lowest-order RT-H: Marini 1985
  - Multigrid by Brenner 2004
- conforming P_1
  - Bounded injection: Cockburn, Dubois, Gopalakrishnan, and S. Tan 2013
  - Standard FEM multigrid (Braess/Hackbusch, Bramble/Pasciak/Xu)

Homogeneous multigrid methods

- HDG on all levels preserves discretization properties
  - mass conservation
  - incompressibility
  - well-posedness (vs. Crouzeix-Raviart with elasticity)
- Construction of energy stable injection operators challenging
  - As $V_{\ell-1} \subset V_\ell$, interpreting $\mathcal{U}_{\ell-1}\lambda_{\ell-1}$ as function in V_ℓ is most natural, but not stable.
  - Overview: S. Tan 2009

The "continuous" injection operator

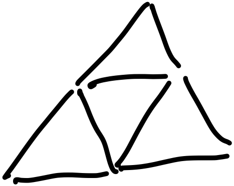
Construction steps: Chen, Lu, and Xu 2014:

1. Given $\lambda_{\ell-1} \in M_{\ell-1}$, compute $u_{\ell-1} = \mathcal{U}_{\ell-1}\lambda_{\ell-1}$
2. Construct $u^c \in V_\ell^c = V_\ell \cap H_0^1(\Omega)$ by averaging operator: given degrees of freedom $\mathcal{N}_{T,i}$ and basis functions $\phi_{T,i}$, compute

$$u^c_{|T} = \sum_i \mathcal{N}_{T,i}\big(\{\{u_{\ell-1}\}\}\big)\phi_{T,i}$$

   where the average is taken over all cells sharing the degree of freedom.
3. Use the embedding $V_{\ell-1}^c \subset V_\ell^c$.
4. Use traces from $V_\ell^c \to M_\ell$.

Sketch of the stencil of the injection operator



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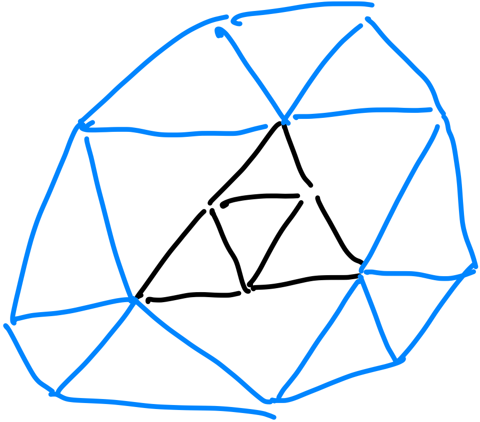

Multigrid analysis

Using Duan, Gao, R. Tan, and Zhang 2007

Multigrid assumptions

1. Regularity approximation assumption:

$$|a_\ell(\lambda - I_\ell P_{\ell-1}\lambda, \lambda)| \le C_1 h_\ell^2 \|A_\ell \lambda\|_\ell^2 \qquad \forall \lambda \in M_\ell. \tag{A1}$$

2. Stability of the "Ritz quasi-projection" $P_{\ell-1}$ and injection I_ℓ :

$$\|\lambda - I_\ell P_{\ell-1}\lambda\|_{a_\ell} \le C_2 \|\lambda\|_{a_\ell} \qquad \forall \lambda \in M_\ell. \tag{A2}$$

3. Smoothing property: Bramble/Pasciak 92

$$a_{\ell-1}(P_{\ell-1}\lambda, \mu) = a_\ell(\lambda, I_\ell \mu) \qquad \forall \mu \in M_{\ell-1},$$

Stability of injection and quasi-projection

- Bound on the adjoint: Chen, Lu, and Xu 2014

$$
\begin{aligned}
a_\ell(I_\ell\lambda, I_\ell\lambda) &\lesssim a_{\ell-1}(\lambda,\lambda) && \forall\lambda\in M_{\ell-1},\\
a_{\ell-1}(P_{\ell-1}\lambda, P_{\ell-1}\lambda) &\lesssim a_\ell(\lambda,\lambda) && \forall\lambda\in M_\ell.
\end{aligned}
$$

Proof of (A1) (excerpt)

$$
\begin{aligned}
a_\ell(\lambda - I_\ell P_{\ell-1}\lambda, \lambda) &= a_\ell(\lambda, \lambda) - a_{\ell-1}(P_{\ell-1}\lambda, P_{\ell-1}\lambda) \\
&\cong (\boldsymbol{\mathcal{Q}}_\ell\lambda, \boldsymbol{\mathcal{Q}}_\ell\lambda)_0 - (\boldsymbol{\mathcal{Q}}_{\ell-1}P_{\ell-1}\lambda, \boldsymbol{\mathcal{Q}}_{\ell-1}P_{\ell-1}\lambda)_0 \\
&\quad + \frac{\tau_\ell}{h_\ell}\|\mathcal{U}_\ell\lambda - \lambda\|_\ell^2 - \frac{\tau_{\ell-1}}{h_{\ell-1}}\|\mathcal{U}_{\ell-1}P_{\ell-1}\lambda - P_{\ell-1}\lambda\|_{\ell-1}^2.
\end{aligned}
$$

Binomial factorization

$$
(\boldsymbol{q}_\ell, \boldsymbol{q}_\ell)_0 - (\boldsymbol{q}_{\ell-1}, \boldsymbol{q}_{\ell-1})_0 = (\boldsymbol{q}_\ell - \boldsymbol{q}_{\ell-1}, \boldsymbol{q}_\ell + \boldsymbol{q}_{\ell-1})
$$

- Both approximate the gradient of an auxiliary function
- Assume elliptic regularity: $\|\boldsymbol{q}_\ell - \boldsymbol{q}_{\ell-1}\| \lesssim h_\ell$
- h_ℓ^2 is needed

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$$

- Both approximate the gradient of an auxiliary function
- Assume elliptic regularity: $\|\boldsymbol{q}_\ell - \boldsymbol{q}_{\ell-1}\| \lesssim h_\ell$
- h_ℓ^2 is needed
- Quasi-orthogonality!

Convergence result

Convergence Theorem (Lu/Rupp/Kanschat 2021)

The homogeneous HDG multigrid method with the continuous injection operator, a standard smoother and $\tau_\ell \sim h_\ell^{-1}$ has a contraction number bounded uniformly below 1.

IMA J. Numer. Anal. 2021 online (arXiv 2011.14018)

Quasi-orthogonality

For any $\lambda \in M_\ell$ and any function w in the conforming lowest order finite element space, there holds

$$\sum_{T \in \mathcal{T}} \int_T (\mathcal{Q}_\ell \lambda - \mathcal{Q}_{\ell-1} P_{\ell-1} \lambda) \nabla w \, \mathrm{d}x = 0.$$

Properties of the injection operator

1. Stability of the injection operator:

$$\|I_\ell \lambda\|_\ell \lesssim \|\lambda\|_{\ell-1} \qquad \forall \lambda \in M_{\ell-1}.$$

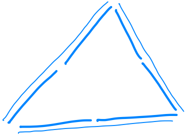
2. Trace identity for conforming linear finite elements:

$$I_\ell \operatorname{tr}_{\ell-1} w = \operatorname{tr}_\ell w \qquad \forall w \in \overline{V}^{\mathsf{c}}_{\ell-1},$$

Note: the second assumption already guarantees quasi-orthogonality

New injection operators

- Injection operators which are the identity on refined faces



New injection operators

- Injection operators which are the identity on refined faces

- Options for "new faces":
  1. Linear interpolation (2D only)
  2. Higher order interpolation (3D)
  3. Use reconstruction $\mathcal{U}_{\ell-1}\lambda_{\ell-1}$

All of them are local!

Convergence Theorem (Lu/Rupp/Kanschat 2022)

If the injection operators are stable, the trace identity holds, the HDG assumptions hold, and $\tau_\ell h_\ell \lesssim 1$, then the multigrid iteration has a contraction number uniformly bounded below 1.

arXiv 2104.00118

Assumptions LS1–LS6

	LDG-H	RT-H	BDM-H
LS1	1	3	like RT-H
LS2	1	4	use 6
LS3	2	2	2
$<$ LS4	1	easy	easy
LS5	use 1	4	use 4
LS6	3	5	use 6

1. Cockburn, Dubois, Gopalakrishnan, and S. Tan 2013
2. Chen, Lu, and Xu 2014
3. S. Tan 2009
4. Cockburn and Gopalakrishnan 2005
5. Gopalakrishnan 2003
6. Cockburn and Gopalakrishnan 2004

Convergence continuous injection

mesh level		2		3		4		5		6		7	
smoother		1	2	1	2	1	2	1	2	1	2	1	2
$p=1$	$\tau=\frac{1}{h}$	33	17	39	20	38	19	36	19	35	18	35	18
	$\tau=1$	33	17	39	19	36	18	35	18	34	17	33	17
$p=2$	$\tau=\frac{1}{h}$	13	08	12	07	11	07	10	06	10	06	09	05
	$\tau=1$	13	08	12	07	11	07	10	06	10	06	09	05
$p=3$	$\tau=\frac{1}{h}$	24	15	25	15	25	15	25	15	25	15	25	15
	$\tau=1$	24	15	25	15	25	15	25	15	25	15	25	15

Convergence linear interpolation

		mesh level 2		3		4		5		6		7	
	smoother	1	2	1	2	1	2	1	2	1	2	1	2
$p=1$	$\tau=\frac{1}{h}$	18	10	22	12	22	12	23	12	23	12	23	12
	$\tau=1$	18	10	21	12	22	12	22	12	22	12	23	12
$p=2$	$\tau=\frac{1}{h}$	13	08	13	07	12	07	12	07	12	07	12	07
	$\tau=1$	13	08	13	07	12	07	12	07	12	07	12	07
$p=3$	$\tau=\frac{1}{h}$	17	11	17	10	17	10	17	10	17	10	17	10
	$\tau=1$	17	11	17	10	17	10	17	10	17	10	17	10

Convergence local reconstruction

mesh level		2		3		4		5		6		7	
smoother		1	2	1	2	1	2	1	2	1	2	1	2
$p=1$	$\tau=\frac{1}{h}$	18	10	22	12	22	12	23	12	23	12	23	12
	$\tau=1$	18	10	21	12	22	12	22	12	22	12	23	12
$p=2$	$\tau=\frac{1}{h}$	11	08	11	07	11	07	11	07	11	07	11	07
	$\tau=1$	11	08	11	07	11	07	11	07	11	07	11	07
$p=3$	$\tau=\frac{1}{h}$	17	11	17	10	17	10	17	10	17	10	17	10
	$\tau=1$	17	11	17	10	17	10	17	10	17	10	17	10

Conclusions

- We presented conditions such that HDG multigrid methods converge uniformly
- Consistency of injection with conforming FEM is crucial
- Several injection operators fit into the analysis
- Several cellwise operators are available

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Convergence theorem

Convergence Theorem (Lu/Rupp/Kanschat 2021)

If the assumptions on the previous slides are fulfilled and $\tau_\ell h_\ell \lesssim 1$, then the multigrid iteration has a contraction number uniformly bounded below 1.

- The previous theorem needed $\tau_\ell h_\ell \sim 1$
- Methods covered: LDG-H, RT-H, BDM-H
  - LS1-LS6 were already proven by other people except BDM-H
  - BDM-H simple extension of work by Cockburn, Gopalakrishnan et al.

Experimental setup

$$-\Delta u = 1 \qquad \text{in } \Omega = (0,1)^2$$

Computations with OpenFFW (Byfut, Gedicke, Günther, Reininghaus, Wiedemann)