Adaptive Virtual Element Method

Marco Verani

MOX, Department of Mathematics, Politecnico di Milano

Joint work with:
L. Beirao da Veiga (Università di Milano Bicocca)
C. Canuto (Politecnico di Torino)
R.H. Nochetto (University of Maryland)
G. Vacca (Università di Bari)

Interplay of discretization and algebraic solvers: a posteriori error estimates and adaptivity, Paris, 8-10 June 2022

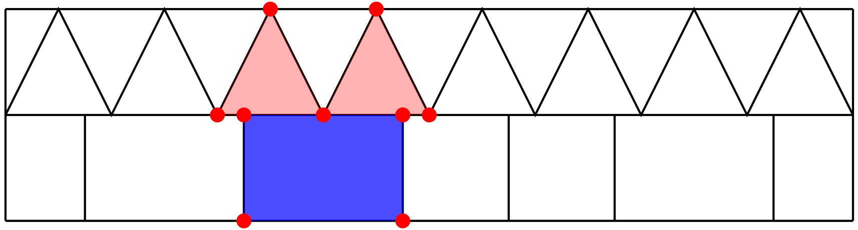
Outline

Introduction

- Many different methods to solve PDEs on polytopal (i.e. polygonal/polyedral) meshes: Polytopal Finite Elements, Mixed/Hybrid Finite Volumes, Mimetic Finite Differences, Virtual Elements, Hybrid High-Order, Hybrid Discontinuous Galerkin, Polytopal Discontinuous Galerkin, Weak Galerkin, BEM-based polytopal FEM ...

Why polygons/polyhedra?

Gluing meshes



Conforming mesh: no hanging nodes

Complex Geometries

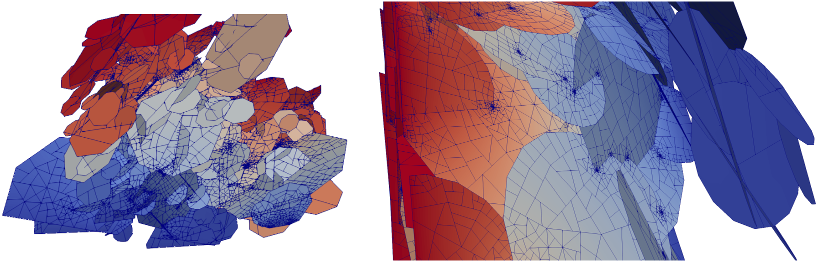


Figure: Polygonal mesh on a system of fractures (Courtesy of S. Berrone and A. D'Auria (Politecnico di Torino))

Moving Geometries

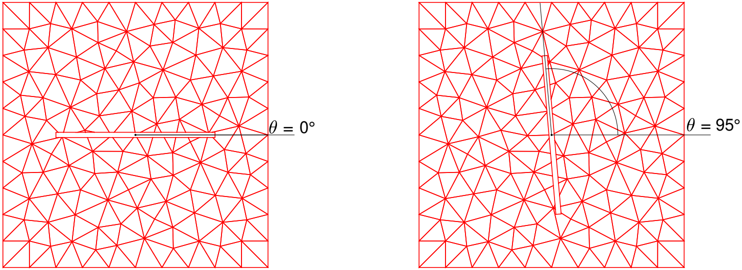
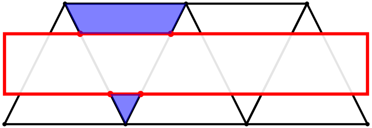


Figure: A rotating a bar on a triangular background mesh induces a polygonal mesh (from [Antonietti, Mascotto, V., Zonca, 2021])



Adaptive Mesh refinement

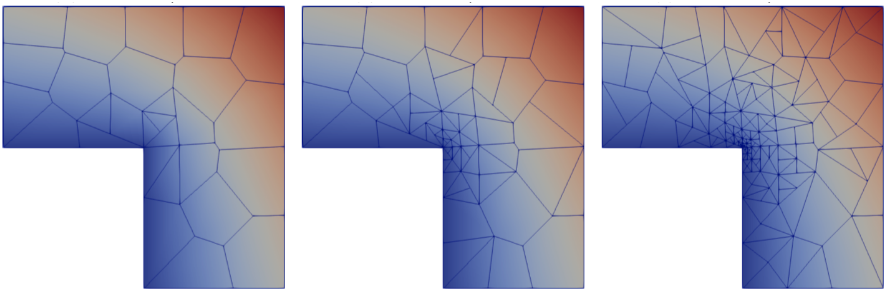


Figure: Polygonal refinement strategy based on preferential cutting direction (Courtesy of S. Berrone and A. D'Auria (Politecnico di Torino))

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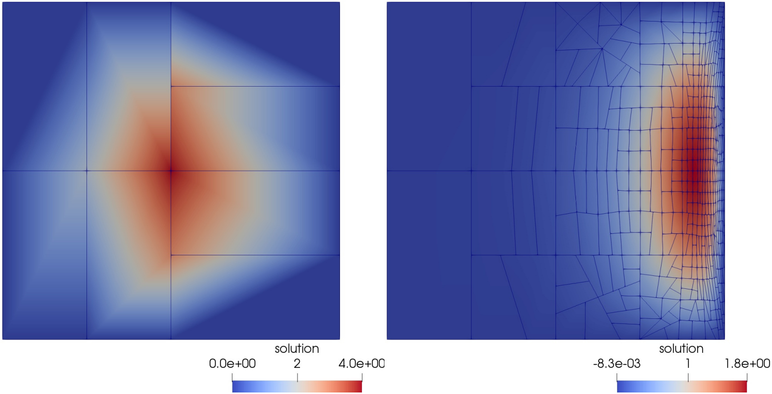
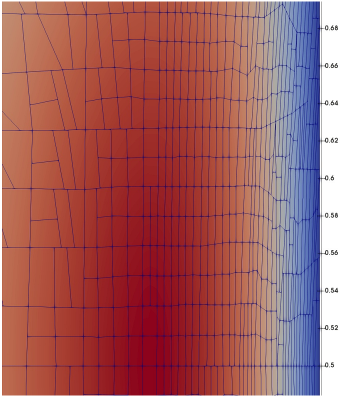


Anisotropic polygonal refinement

[Antonietti,Berrone,Borio,D'Auria,V.,Weisser 2021]

Adaptive Mesh refinement



Zoom of the refined mesh

Agglomeration

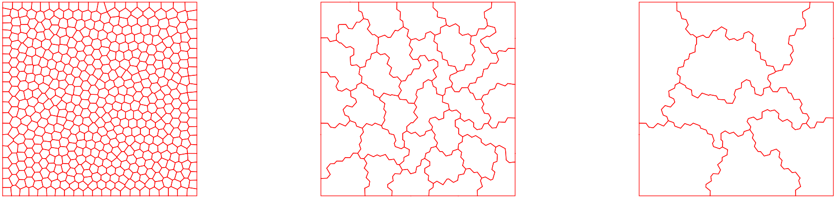


Figure: Agglomerated polygonal meshes with $N_{el} = 512, 32, 8$

Agglomeration useful, e.g., in (adaptive) de-refinement mesh strategies

Introduction

In this Talk we focus on the Virtual Element Method (VEM)

[Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, '13]

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[Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, '13]

Idea of VEM: Galerkin method where the explicit knowledge of the basis functions on polygons is not needed to assemble the algebraic problem (only DOFS are needed).

Introduction

In this Talk we focus on the Virtual Element Method (VEM)

[Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, '13]

Intense research activity on VEM (very incomplete list ...):

- Methods:
  Conforming and nonconforming approximation; mixed formulation; serendipity spaces; divergence-free elements; Trefftz methods; *hp*-approximation; a posteriori error estimates and adaptivity; curved faced/edges, divergence-free elements; preconditioners; ...
- Applications:
  fluidynamic problems; structural mechanics problems; contact mechanics and elasto-plastic deformation problems; phase-field models of isotropic brittle fractures; cracks in materials; elastic wave propagation phenomena; underground flows and discrete fracture networks; propagation and scattering of time-harmonic waves; eigenvalue problems; Maxwell equation; Schrodinger equation; Laplace-Beltrami equation; Cahn-Hilliard equation; obstacle and minimal surface problems; topology optimization problems; nonlocal reaction-diffusion systems describing the cardiac electric field; ...

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Flexibility of VEM allows:

- to deal with general polygonal/polyedral meshes;
- to easily incorporate additional features and regularity properties into the discrete space (divergence free, C^k, ...).

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In this Talk:

- Beirao da Veiga, Canuto, Nochetto, Vacca, V., Adaptive VEM: Stabilization-Free A Posteriori Error Analysis, arXiv:2111.07656, 2021
- Beirao da Veiga, Canuto, Nochetto, Vacca, V., Adaptive VEM: convergence analysis, in preparation.
- Beirao da Veiga, Canuto, Nochetto, Vacca, V., Adaptive VEM: optimality analysis, in preparation.

Literature on VEM and adaptivity

A posteriori error estimates and numerical tests of AVEM

- Residual based h-estimator: [Berrone,Borio, 2017], [Cangiani, Georgoulis,Pryer,Sutton, 2017]
- Residual based hp-estimator: [Beirao,Manzini,Mascotto, 2019]
- Residual based anisotropic estimator: [Antonietti,Berrone,Borio,D'Auria,V., 2021]
- Mixed-VEM: [Cangiani,Munar, 2019], [Munar, Sequeira, 2020]
- Gradient recovery: [Chi,Beirao,Paulino, 2019]
- Equilibrated flux: [Dassi,Gedicke,Mascotto, 2020], [Dassi,Gedicke,Mascotto, 2021]

Polytopal meshes: quality and refinement

- 2d: [Beirao,Manzini, 2015], [Hoshina,Menezes,Pereira, 2018], [Berrone,Borio,D'Auria, 2021], [Berrone,D'Auria, 2021], [Attene et al., 2021],[Antonietti,Manuzzi, 2022], [Sorgente,Biasotti,Manzini, Spagnuolo, 2022], ...
- 3d: [D'Auria, PhD thesis, 2020], [Antonietti,Dassi,Manuzzi, 2022], ...

Continuous problem and virtual discretization

Continuous problem

$$\boxed{-\nabla \cdot (A \nabla u) + cu = f \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on } \partial\Omega}$$

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where

$A \in (L^\infty(\Omega))^{2\times 2}$ is symmetric and uniformly positive-definite in Ω,

$c \in L^\infty(\Omega)$ is non-negative in Ω,

$f \in L^2(\Omega)$.

Continuous problem

$$-\nabla \cdot (A\nabla u) + cu = f \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad u = 0 \quad \text{on } \partial\Omega$$

The variational formulation is

$$u \in \mathbb{V} \ : \ \mathcal{B}(u,v) = (f,v)_\Omega \qquad \forall v \in \mathbb{V} = H^1_0(\Omega)$$

with $\mathcal{B}(u,v) := a(u,v) + m(u,v)$ where

$$a(u,v) := \int_\Omega (A\nabla u) \cdot \nabla v\,, \quad m(u,v) := \int_\Omega c\,u\,v$$

Virtual Discretization

Local Virtual Space (on polygon $E \in \mathcal{T}$)

$$V_{\mathcal{T}}(E) = \{v_{\mathcal{T}} \in H^1(E) : \Delta v_{\mathcal{T}} = 0 \text{ in } E,\ v_{\mathcal{T}} \in \mathbb{P}^1(e)\ \forall e \in \partial E\}$$

$v_{\mathcal{T}}$ virtual solution of Laplace problem with prescribed boundary datum

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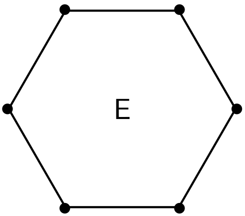

LOCAL DOFS: $v_{\mathcal{T}}$ at vertices

$v_{\mathcal{T}}$ virtual solution of Laplace problem with prescribed boundary datum

- $v_{\mathcal{T}} \in C^0(\partial E)$

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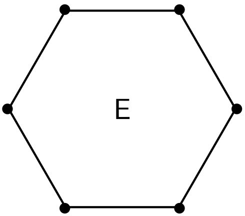

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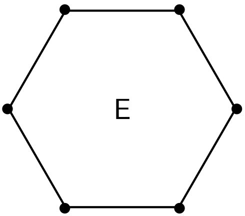

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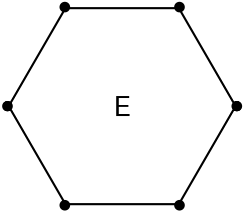

LOCAL DOFS: $v_{\mathcal{T}}$ at vertices

$v_{\mathcal{T}}$ virtual solution of Laplace problem with prescribed boundary datum

- $v_{\mathcal{T}} \in C^0(\partial E)$
- DOFS are unisolvent
- $\mathbb{P}^1(E) \subset V_{\mathcal{T}}(E)$
- On triangles: $V_{\mathcal{T}}(E) = \mathbb{P}^1(E)$

Virtual Discretization

Global Virtual Space

$$V_{\mathcal{T}} = \{v_{\mathcal{T}} \in H_0^1(\Omega) : v_{\mathcal{T}}|_E \in V_{\mathcal{T}}(E) \ \forall E \in \mathcal{T}\}$$

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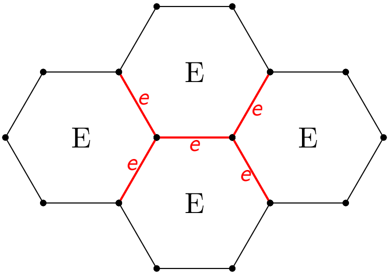

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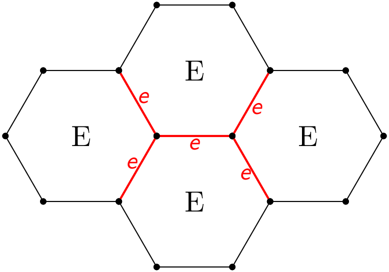

GLOBAL DOFS: $v_{\mathcal{T}}$ at vertices

- Local spaces $V_{\mathcal{T}}(E)$ are C^0-glued:

  C^0-continuity at vertices (same point values of $v_{\mathcal{T}}$);

  C^0-continuity across edges e (same polynomial functions $v_{\mathcal{T}}$).

Virtual Discretization

Weak formulation

$$u \in \mathbb{V} \ : \ \mathcal{B}(u,v) = (f,v)_\Omega \qquad \forall v \in \mathbb{V} = H^1_0(\Omega)$$

with $\mathcal{B}(u,v) := a(u,v) + m(u,v)$ where

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One would be tempted to write the virtual discrete problem as:

$$u_\mathcal{T} \in V_\mathcal{T} : \mathcal{B}(u_\mathcal{T}, v_\mathcal{T}) = (f, v_\mathcal{T})_\Omega \quad \forall v_\mathcal{T} \in V_\mathcal{T}$$

BUT

this would require the explicit expression of the virtual functions in each polygon E that we do not want to employ. To set up the linear system we want to use only the DOFS.

Virtual discretization

Step 0: $\Pi_E^\nabla : V_{\mathcal{T}}(E) \to \mathbb{P}_1(E)$ is the energy projector:

$$(\nabla(v - \Pi_E^\nabla v), \nabla w)_E = 0 \quad \forall w \in \mathbb{P}_1(E), \qquad \int_{\partial E} (v - \Pi_E^\nabla v) = 0\,.$$

computable using DOFS only.

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Step 1: Define the local discrete bilinear form as

$$\boxed{\mathcal{B}_{\mathcal{T}}^E(u_{\mathcal{T}}, v_{\mathcal{T}}) := a^E(\Pi_E^\nabla u_{\mathcal{T}}, \Pi_E^\nabla v_{\mathcal{T}}) + m^E(\Pi_E^\nabla u_{\mathcal{T}}, \Pi_E^\nabla v_{\mathcal{T}}) + \gamma S^E(u_{\mathcal{T}}, v_{\mathcal{T}})}$$

where

- $\gamma > 0$ stabilization parameter
- $S^E(v_{\mathcal{T}}, v_{\mathcal{T}}) \simeq |v_{\mathcal{T}} - \Pi_E^\nabla v_{\mathcal{T}}|_{1,E}$ stabilizing form.

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- $\gamma > 0$ stabilization parameter
- $S^E(v_\mathcal{T}, v_\mathcal{T}) \simeq |v_\mathcal{T} - \Pi_E^\nabla v_\mathcal{T}|_{1,E}$ stabilizing form.

Properties:

- Consistency: $\mathcal{B}_\mathcal{T}^E(q, v_\mathcal{T}) = \mathcal{B}^E(q, v_\mathcal{T}) \quad \forall q \in \mathbb{P}^1(E), \ \forall v_\mathcal{T} \in V_\mathcal{T}(E)$
- Stability: $\mathcal{B}_\mathcal{T}^E(v_\mathcal{T}, v_\mathcal{T}) \simeq \mathcal{B}^E(v_\mathcal{T}, v_\mathcal{T}) \quad \forall v_\mathcal{T} \in V_\mathcal{T}(E)$

Virtual Element discretization

Step 2: Discrete problem. Find $u_{\mathcal{T}} \in V_{\mathcal{T}}$ such that

$$\mathcal{B}_{\mathcal{T}}(u_{\mathcal{T}}, v_{\mathcal{T}}) = (f, v_{\mathcal{T}})_{\mathcal{T}} \quad \forall v_{\mathcal{T}} \in V_{\mathcal{T}}$$

where

- $\mathcal{B}_{\mathcal{T}}(u_{\mathcal{T}}, v_{\mathcal{T}}) = \sum_{E \in \mathcal{T}} \mathcal{B}_{\mathcal{T}}^{E}(u_{\mathcal{T}}, v_{\mathcal{T}})$
- $(f, v_{\mathcal{T}})_{\mathcal{T}} = \sum_{E \in \mathcal{T}} \int_E f \Pi_E^{\nabla} v_{\mathcal{T}}$

$\rightarrow$ Optimal a priori error estimates in energy norm under suitable mesh assumptions

Typical mesh assumptions (for theory)

E polygonal element of a partition $\mathcal{T}$

(a) E is a star-shaped polygon with respect to a circle of radius ρ and center $z \in E$.
(b) The aspect ratio is uniformly bounded from above by σ, i.e. $h_E/\rho < \sigma$, being h_E the diameter of E.
(c) For every edge $e \subset \partial E$ it holds $h_E \leq ch_e$, being h_e the length of e.

Assumptions can be weakened (small edges):

[Beirão da Veiga, Lovadina, Russo, 2017], [Brenner, Sung, 2018]

Modified Local Space

Local **Enhanced** Virtual Space (on polygon E)

$$
\begin{aligned}
V_{\mathcal{T}}(E) \;=\; \{v_{\mathcal{T}} \in H^1(E) : &\overbrace{\Delta v_{\mathcal{T}} \in \mathbb{P}^1(E)}^{ADD\ DOFS},\; v_{\mathcal{T}} \in \mathbb{P}^1(e)\ \forall e \in \partial E \\
&\underbrace{(\Pi^{\nabla}_E v_{\mathcal{T}}, q)_{L^2(E)} = (v_{\mathcal{T}}, q)_{L^2(E)}\ \forall q \in \mathbb{P}^1(E)}_{ADD\ CONSTRAINTS}\}
\end{aligned}
$$

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DOFS $=$ vertex values

L^2-projection on $\mathbb{P}^1$ is computable using DOFS only .

Adaptive Virtual Element Method (AVEM)

Assumption (Coefficients and right-hand side of the equation)

The coefficients A and c and the right-hand side f are constant in each element of the polygonal mesh $\mathcal{T}$.

Study convergence and optimality properties of AVEM:

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

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We follow the framework developed for AFEM (in particular, adaptive DGFEM: [Karakashian, Pascal, 2003], [Bonito, Nochetto, 2010])

Study convergence and optimality properties of AVEM:

SOLVE $\longrightarrow$ ESTIMATE $\longrightarrow$ MARK $\longrightarrow$ REFINE

Crucial Questions:

Study convergence and optimality properties of AVEM:

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Crucial Questions:

Q1: Is it possible to systematically refine general polytopes and preserve shape regularity?

Study convergence and optimality properties of AVEM:

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Q1: Is it possible to systematically refine general polytopes and preserve shape regularity?
Shape regularity is critical to have robust interpolation estimates regardless of the resolution level.

Study convergence and optimality properties of AVEM:

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Crucial Questions:

Q1: Is it possible to systematically refine general polytopes and preserve shape regularity?

At the moment, there is no general positive answer.

$\leadsto$ see Paola Antonietti's talk

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Q2: Is it possible to prove that **Error (+ Estimator)** reduces between consecutive adaptive iterations?

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This is crucial to show that AVEM converges.

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Crucial Questions:

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Comparing the stabilization terms under refinement is crucial (and problematic).

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Crucial Questions:

Q3: Is the number of elements generated by REFINE proportional to the number of elements collectively selected by MARK?

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Crucial Questions:

Q3: Is the number of elements generated by REFINE proportional to the number of elements collectively selected by MARK?

This is crucial to show that AVEM leads to an error decay comparable with the best approximation in terms of degrees of freedom.

Study convergence and optimality properties of AVEM:

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Crucial Questions:

Q3: Is the number of elements generated by REFINE proportional to the number of elements collectively selected by MARK?

YES, if the refinement is local

Drawback: unlimited growth of nodes per element

Study convergence and optimality properties of AVEM:

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$\leadsto$ **Restrict the number of hanging nodes per edge**

Assumptions

In view of Q1, Q2 and Q3 we adopt the following framework:

- Polygonal mesh made of triangles with hanging nodes;
- Refinement based on "newest-vertex element bisection";
- Condition preventing unbounded number of hanging nodes per edge.

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On triangles with hanging nodes:

$$\textbf{VEM} \neq \textbf{FEM}$$

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Alternatively: Polygonal mesh made of squares with hanging nodes (standard quad-tree refinement), or heterogeneous mesh made of triangles and squares with (bounded number of) hanging nodes .

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Quite restrictive framework! **However**, it allows:

- to prove novel convergence and optimality results for AVEM;
- to sketch roadmap (and identify obstructions) to tackle more general situations.

Recall the structure of AVEM:

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Recall the structure of AVEM:

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

REFINE: This module refines all marked elements and keeps the mesh admissible (bounded number of hanging nodes) via a routine named `MAKE-ADMISSIBLE` with optimal complexity.

Recall the structure of AVEM:

$$\text{SOLVE} \quad \longrightarrow \quad \text{ESTIMATE} \quad \longrightarrow \quad \text{MARK} \quad \longrightarrow \quad \text{REFINE}$$

a posteriori error estimate

Proposition ([Beirao,Canuto,Nochetto,Vacca,V. 2021])

$$|u - u_{\mathcal{T}}|^2_{1,\Omega} \leq C_{apost} \left(\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) + S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \right)$$

$$c_{apost}\, \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} + S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \, .$$

Cf. [Cangiani, Georgoulis, Pryer, Sutton, 2017]

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$$c_{apost}\, \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} + S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}})\,.$$

where

$$r_{\mathcal{T}}(E; v, \mathcal{D}) := f_E \; - \; c_E\, \Pi^{\nabla}_E v\,, \qquad j_{\mathcal{T}}(e; v, \mathcal{D}) := \left[\!\left[A_E \nabla\, \Pi^{\nabla}_{\mathcal{T}} v \right]\!\right]_e$$

$$\eta^2_{\mathcal{T}}(E; v, \mathcal{D}) := h_E^2 \| r_{\mathcal{T}}(E; v, \mathcal{D})\|^2_{0,E} \; + \; \tfrac{1}{2} \textstyle\sum_{e \in \mathcal{E}_E} h_E \| j_{\mathcal{T}}(e; v, \mathcal{D})\|^2_{0,e}$$

$$\eta^2_{\mathcal{T}}(v, \mathcal{D}) := \textstyle\sum_{E \in \mathcal{T}} \eta^2_{\mathcal{T}}(E; v, \mathcal{D})\,.$$

Cf. [Cangiani, Georgoulis, Pryer, Sutton, 2017]

Stabilization free a posteriori error estimate

Proposition ([Beirao,Canuto,Nochetto,Vacca,V. 2021])

$$|u - u_{\mathcal{T}}|^2_{1,\Omega} \leq C_{apost} \left(\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) + S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \right)$$

$$c_{apost}\, \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} + S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}})\,.$$

Proposition ([Beirao,Canuto,Nochetto,Vacca,V. 2021])

$$\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B\, \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$$

Stabilization free a posteriori error estimate

Proposition ([Beirao,Canuto,Nochetto,Vacca,V. 2021])

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Proposition ([Beirao,Canuto,Nochetto,Vacca,V. 2021])

$$\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B\, \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$$

Stabilization-free a posteriori error estimate:

$$\left(c_{\text{apost}} - \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} \leq C_{\text{apost}} \left(1 + \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$$

Stabilization-free a posteriori error estimate:

$$\left(c_{\text{apost}} - \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} \leq C_{\text{apost}} \left(1 + \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$$

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Stabilization term is important in a priori analysis, but it is not vital in a posteriori analysis.

Stabilization-free a posteriori error estimate:

$$\left(c_{\mathrm{apost}} - \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} \leq C_{\mathrm{apost}} \left(1 + \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$$

Stabilization term is important in a priori analysis, but it is not vital in a posteriori analysis.

Stabilization-free a posteriori estimates opens the door to prove convergence of AVEM.

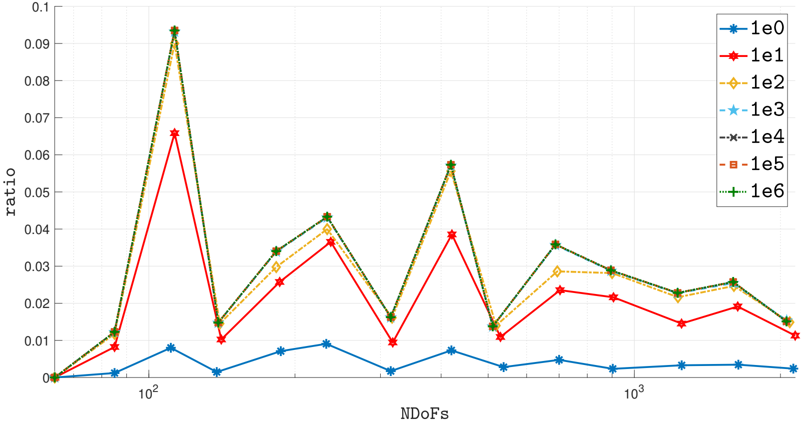
Stabilization-free a posteriori error estimate:

$$\left(c_{\text{apost}} - \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D}) \leq |u - u_{\mathcal{T}}|^2_{1,\Omega} \leq C_{\text{apost}} \left(1 + \frac{C_B}{\gamma^2}\right) \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$$

Technically speaking: to obtain the result is essential to have access to a **subspace $V^0_{\mathcal{T}}$ of $V_{\mathcal{T}}$ made of continuous piecewise affine function on $\mathcal{T}$**. This dictates our mesh assumptions!

details

- Recall the bound: $\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}}) \leq C_B\, \eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})$
- Employ AVEM with Dörfler parameter $\theta = 0.5$ for L-shaped domain problem with $A = I$, $c = 0$, $f = 1$ and vanishing boundary conditions.



$$\texttt{ratio} := \frac{\gamma^2 S_{\mathcal{T}}(u_{\mathcal{T}}, u_{\mathcal{T}})}{\eta^2_{\mathcal{T}}(u_{\mathcal{T}}, \mathcal{D})}$$ for different values of γ.

AVEM in action

Poisson problem with piecewise constant a, $A = aI$, $c = 0$ and $f = 0$ ($\to$ Kellogg's exact solution $u \in H^{1+\varepsilon}$, $\varepsilon < 0.1$).

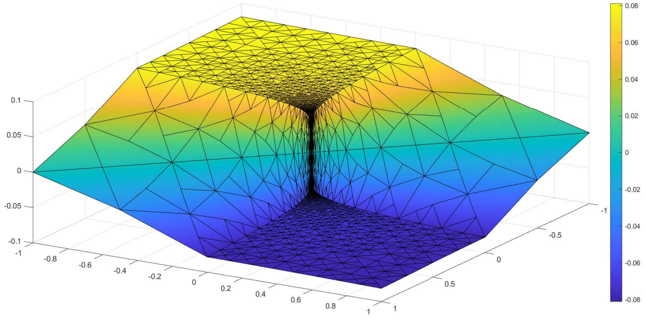
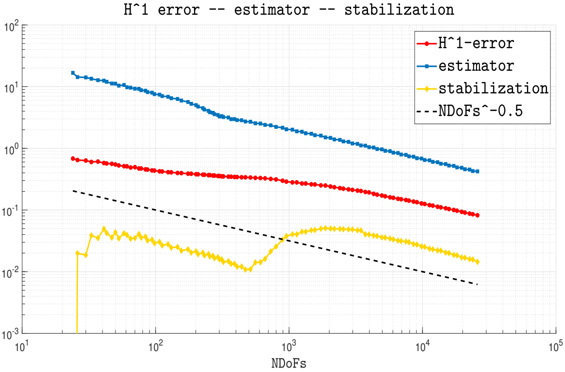
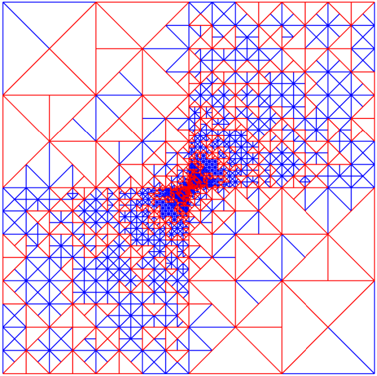


Figure: discrete solution

AVEM in action

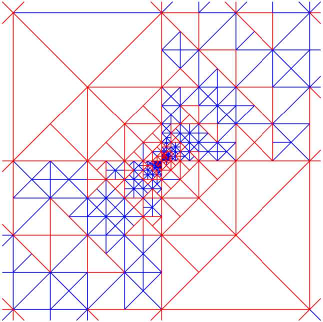


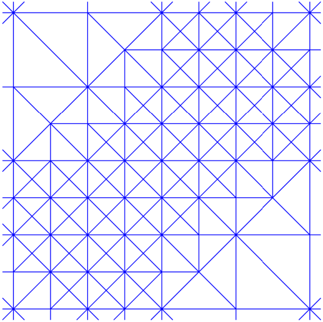
AVEM in action



Final grid. Mesh elements having more than three vertices are drawn in red

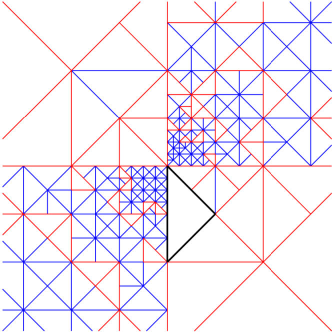
AVEM in action





Left: final grid $\mathcal{T}_{\texttt{VEM}}$. **Right**: final grid $\mathcal{T}_{\texttt{FEM}}$. Zoom to $(-10^{-9}, 10^{-9})^2$. VEM exhibits stronger grading at the singularity.

AVEM in action



Final grid $\mathcal{T}_{\texttt{VEM}}$, zoom to $(-10^{-10}, 10^{-10})^2$. The black element is a nonagon

AVEM with general data: the idea

AVEM with general data: the idea

Outer Loop

Approximate data (A, c, f) with piecewise constants up to tolerance ε_k

Update tolerance: $\varepsilon_k \to \varepsilon_{k+1} < \varepsilon_k$

Update Outer Loop counter : $k \to k+1$

cf. [Stevenson, 2007], [Bonito, DeVore, Nochetto, 2013]

AVEM with general data: the idea

Outer Loop

Approximate data (A, c, f) with piecewise constants up to tolerance ε_k

Inner Loop

Approximate the problem with piecewise constant data
by iterating:

$$\text{SOLVE} \longrightarrow \text{ESTIMATE} \longrightarrow \text{MARK} \longrightarrow \text{REFINE}$$

up to tolerance ε_k

Update tolerance: $\varepsilon_k \to \varepsilon_{k+1} < \varepsilon_k$
Update Outer Loop counter : $k \to k+1$

cf. [Stevenson, 2007], [Bonito, DeVore, Nochetto, 2013]

Convergence of Inner Loop

At each subiteration i we have :

$$\mathtt{InnerError}(i)^2 + \beta\, \mathtt{InnerEstimator}(i)^2 \lesssim \xi^i, \qquad \xi < 1$$

$\mathtt{InnerError}(i)$= difference between solution of the perturbed problem and its VEM approximation

Inner Loop: ε-approximation to ε-perturbed problem.

$+$

Outer Loop: reduces ε

Inner Loop: ε-approximation to ε-perturbed problem.

$+$

Outer Loop: reduces ε

$\rightarrow$

Convergence of AVEM towards original unperturbed problem

Inner Loop: ε-approximation to ε-perturbed problem.

+

Outer Loop: reduces ε

$\rightarrow$

Convergence of AVEM towards original unperturbed problem

Moreover, AVEM is quasi-optimal:

AVEM produces an approximation of u with N dofs that is comparable with the best N-term VEM approximation of u.

details

AVEM in action

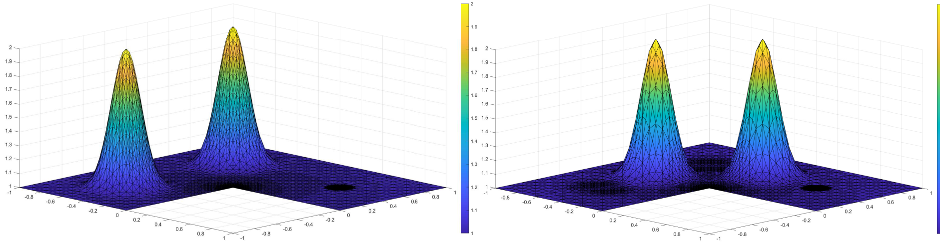
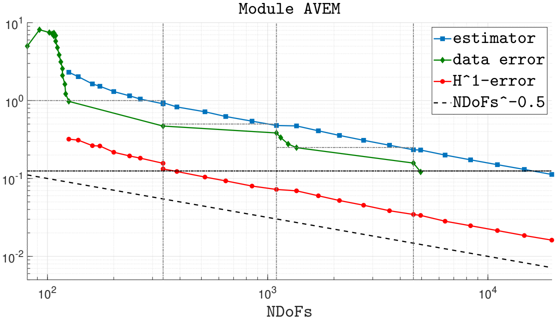


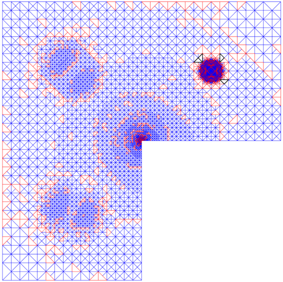
Figure: Left: graph of a ($A = aI$). Right: graph of c

Choose f so that

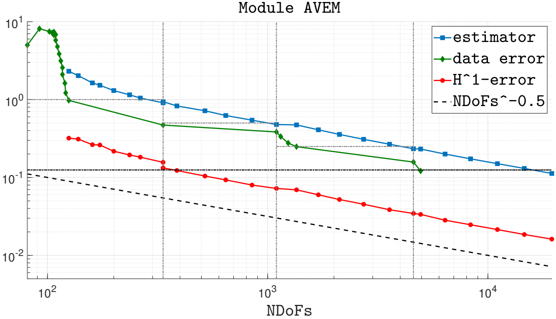
$$u_{\mathrm{ex}}(x,y) = r^{\frac{2}{3}} \sin(2\alpha/3) + \exp(-1000((x-0.5)^2 + (y-0.5)^2))$$

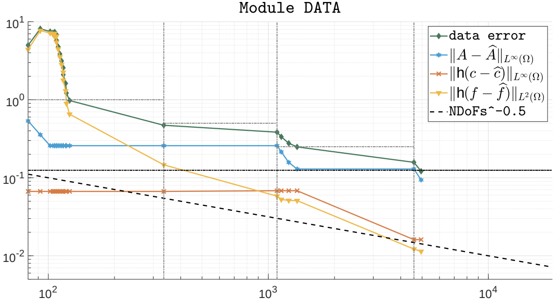
AVEM in action



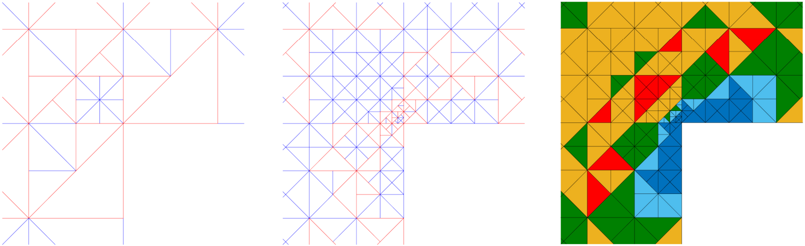


AVEM in action

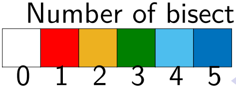




Numerical Experiment: Mesh Zoom to $(-10^{-2}, 10^{-2})^2$



- **Left**: final grid $\widehat{\mathcal{T}}_k$ obtained with `Outer Loop` (Data approximation).
- **Middle**: final grid $\mathcal{T}_{k+1}$ obtained with `Inner Loop`. Mesh elements having more than three vertices are drawn in red.
- **Right**: heat map representing the number of newest-vertex bisections needed to generate each $E \in \mathcal{T}_{k+1}$ starting from the mesh $\widehat{\mathcal{T}}_k$.
- Colorbar for the heat map:



Conclusions:

- We discussed convergence and optimality properties of AVEM;
- We obtained theoretical results under quite restrictive assumptions on the polygonal mesh (triangles/squares with hanging nodes);
- The analysis sheds light on obstructions to considering more general polygonal meshes.

Perspectives:

- Extending convergence and optimality analysis of AVEM to more general polygonal meshes. This seems to require (at least):
  1. Refinement strategy preserving shape regularity;
  2. Stabilization-free a posteriori error estimates.
- Extending convergence and optimality analysis of AVEM to higher order virtual elements.

Thanks for your attention!

The subspace $V_{\mathcal{T}}^0$

continue

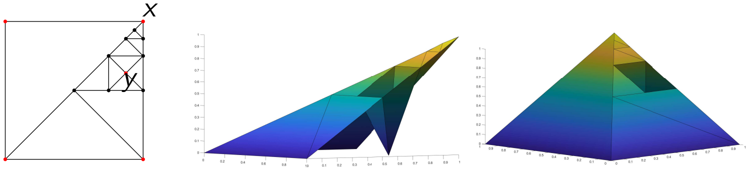
VEM space $V_{\mathcal{T}}$**:**

$$V_{\mathcal{T}} := \left\{ v \in \mathbb{V} : \ v_{|E} \in V_{\mathcal{T}}(E) \ \ \forall E \in \mathcal{T} \right\}.$$

Subspace $V_{\mathcal{T}}^0$ of continuous, piecewise linear functions on $\mathcal{T}$

$$V_{\mathcal{T}}^0 := \left\{ v \in V : \ v_{|E} \in \mathbb{P}_1(E) \ \ \forall E \in \mathcal{T} \right\}.$$

Basis functions in $V_{\mathcal{T}}^0$**:** Functions in $V_{\mathcal{T}}^0$ are uniquely determined by their value at the *proper nodes* of $\mathcal{T}$.



Complexity of DATA (`Outer Loop`)

Implementation of DATA: Given a tolerance $\tau_k = \omega\varepsilon_k/3$, DATA refines

for A provided $\max_{E\in\mathcal{T}_k} \|A - A_E\|_{L^\infty(E)} > \tau_k$ (GREEDY algorithm);

for c provided $\max_{E\in\mathcal{T}_k} \|h_E(c - c_E)\|_{L^\infty(E)} > \tau_k$ (GREEDY algorithm);

for f provided $\left(\sum_{E\in\mathcal{T}_k} \|h_E(f - f_E)\|^2_{L^2(E)}\right)^{1/2} > \tau_k$ (Dörfler algorithm).

Convergence rate for A: If $A \in W^1_p(\Omega)$ for $p > 2$ piecewise in $\mathcal{T}_0$, then

$$\|A - A_k\|_{L^\infty(\Omega)} \leq \tau_k, \qquad \#\widehat{\mathcal{T}}_k - \#\mathcal{T}_k \lesssim |A|^2_{W^1_p(\Omega)}\tau_k^{-2}.$$

Convergence rate for f: If $f \in H^s(\Omega)$ for $s \in [0,1]$ piecewise in $\mathcal{T}_0$, then

$$\|h(f - f_k)\|_{L^2(\Omega)} \leq \tau_k, \qquad \#\widehat{\mathcal{T}}_k - \#\mathcal{T}_k \lesssim |f|^{\frac{2}{1+s}}_{H^s(\Omega)}\tau_k^{-\frac{2}{1+s}}.$$

Approximation Classes

Best approximation: If $\mathcal{E}_{\mathcal{T}}^2(v, v_{\mathcal{T}}) := \|v - v_{\mathcal{T}}\|^2 + |v_{\mathcal{T}} - \mathcal{I}_{\mathcal{T}} v_{\mathcal{T}}|_{1,\mathcal{T}}^2$, then

$$\mathcal{E}_{\mathcal{T}}(u, u_{\mathcal{T}}) \le C^{\dagger} \mathcal{E}_{\mathcal{T}}(u, v_{\mathcal{T}}) \qquad \forall v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}} \,.$$

Approximation classes: If $\mathbb{T}_N$ is the set of all Λ-admissible bisection refinements of $\mathcal{T}_0$ such that $\#\mathcal{T} - \#\mathcal{T}_0 \le N$, then let

$$\mathbb{A}_s := \Big\{ v \in H_0^1(\Omega) \;:\; \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v_{\mathcal{T}} \in \mathbb{V}_{\mathcal{T}}} \mathcal{E}_{\mathcal{T}}(v, v_{\mathcal{T}}) \lesssim N^{-s} \; \forall N \in \mathbb{N} \Big\},$$
$$\mathbb{A}_s^0 := \Big\{ v \in H_0^1(\Omega) \;:\; \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v_{\mathcal{T}}^0 \in \mathbb{V}_{\mathcal{T}}^0} \mathcal{E}_{\mathcal{T}}(v, v_{\mathcal{T}}^0) \lesssim N^{-s} \; \forall N \in \mathbb{N} \Big\}.$$

Class equivalence: For all $s > 0$ there holds

$$\mathbb{A}_s = \mathbb{A}_s^0.$$

Complexity of GALERKIN (Inner Loop)

continue

Subiterations of GALERKIN: The number of subiterations $J_k \leq J$ is bounded uniformly with respect to the outer iteration counter k.

Output of data: If $u_k^{\text{ex}} \in \mathbb{V}$ is the exact solution corresponding to data $\mathcal{D}_k$, then there exists a constant $D > 0$ such that

$$\| u - u_k^{\text{ex}} \| \leq D \, \omega \, \varepsilon_k.$$

Cardinality of $\mathcal{M}_k$: If $u \in \mathbb{A}_s$ and ω is sufficiently small relative to D, then

$$\# \mathcal{M}_k \lesssim J |u|_{\mathbb{A}_s}^{\frac{1}{s}} \varepsilon_k^{-\frac{1}{s}}.$$

Quasi-optimality of AVEM: If $u \in \mathbb{A}_s$ with $s \leq 1/2$ and data $\mathcal{D} \in \mathbb{A}_{s_\mathcal{D}}$ with $s_\mathcal{D} = 1/2$, then the Galerkin solution $u_{k+1} \in \mathbb{V}_{\mathcal{T}_{k+1}}$ and $\mathcal{T}_{k+1}$ satisfy

$$\| u - u_{k+1} \| \lesssim \varepsilon_k, \quad \# \mathcal{T}_{k+1} - \# \mathcal{T}_0 \lesssim \Big(|u|_{\mathbb{A}_s} + |\mathcal{D}|_{\mathbb{A}_{s_\mathcal{D}}} \Big) \varepsilon_k^{-\frac{1}{s}}.$$

($\# \mathcal{T}_{k+1} - \# \mathcal{T}_0 \lesssim \sum_{j=0}^{k} \# \mathcal{M}_j$ [Binev, Dahmen, DeVore, 04; Stevenson, 07])