Propagating Hyperbolic Solutions Through Unstructured Tents

Jay Gopalakrishnan

Portland State University

In collaboration with

Dow Drake, Joachim Schöberl, Zheng Sun & Christoph Wintersteiger

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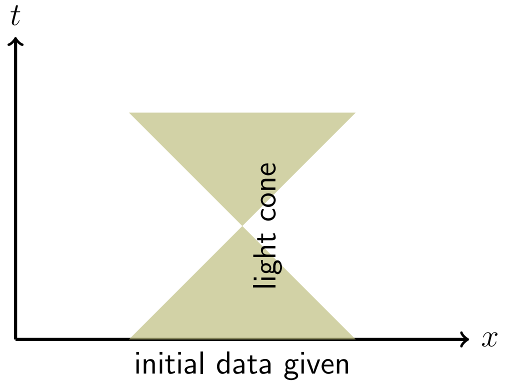
Thanks: NSF DMS

Tents

Hyperbolic solutions have finite propagation speed.



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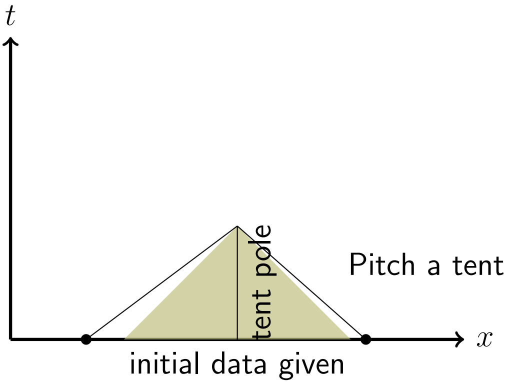

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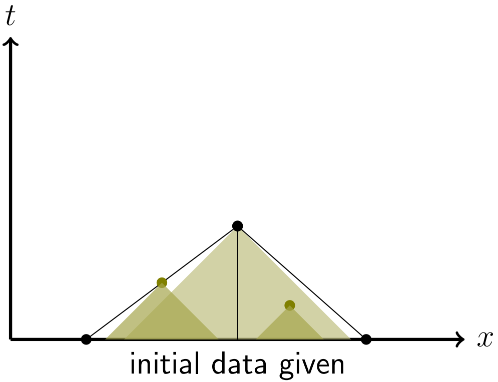

The hyperbolic problem is solvable in the tent provided the tent pole is not too high.

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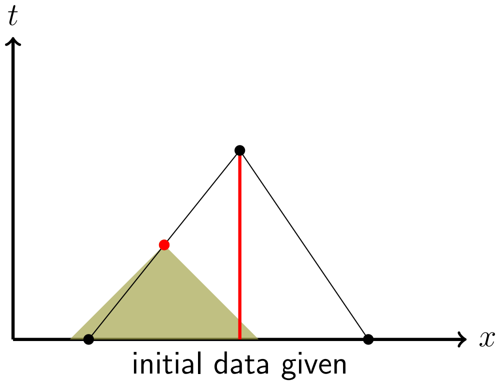

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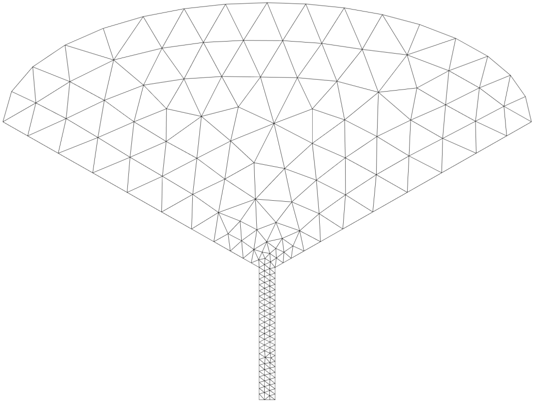
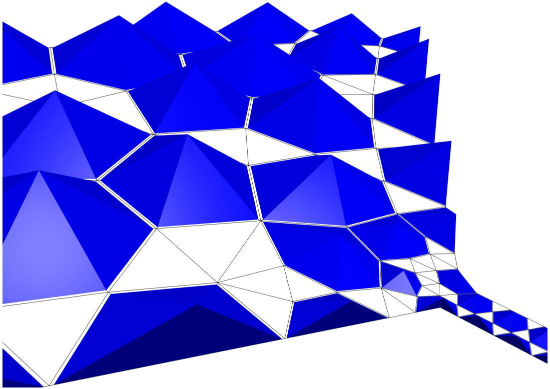
"Tent pitching" in spacetime

Time marching with unstructured spatial mesh and varying time step sizes using tents were pursued by many: [Richter 1994] [Lorie+Roe+van Leer 1995] [Falk+Richter 1999] [Yin+Acharia+Sobh+Haber+Tortorelli 2000] [Üngör+Sheffer 2000] [Monk+Richter 2005] [Abedi+Petracovici+Haber 2006] [Abedi+Haber 2018]





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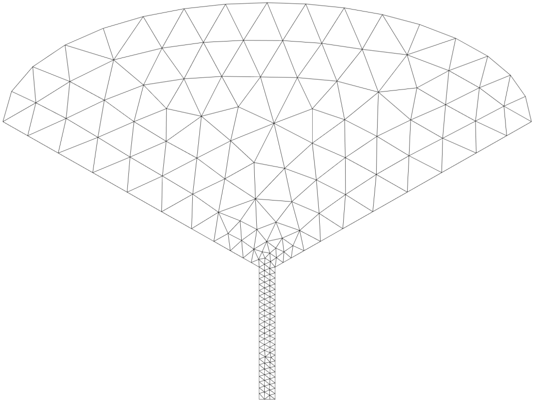

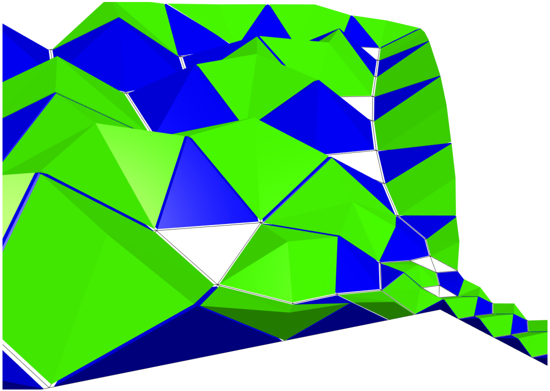

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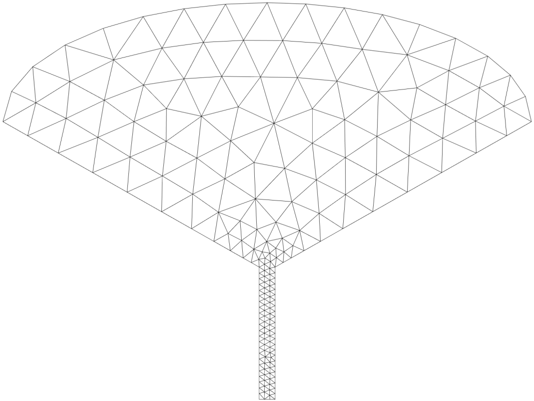

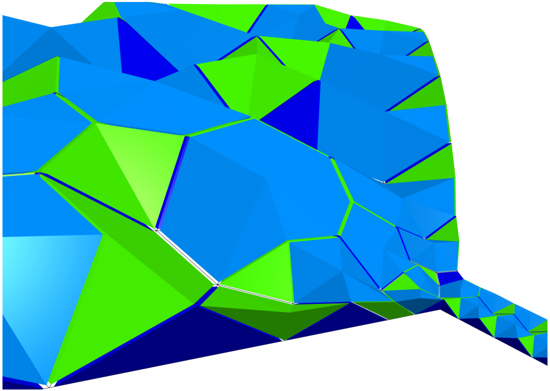

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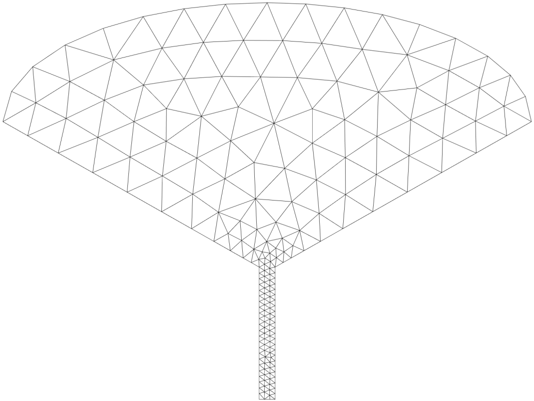

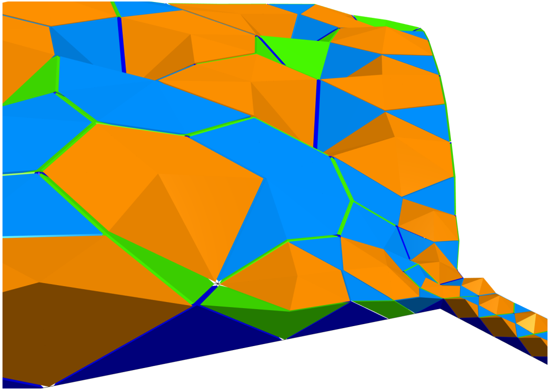

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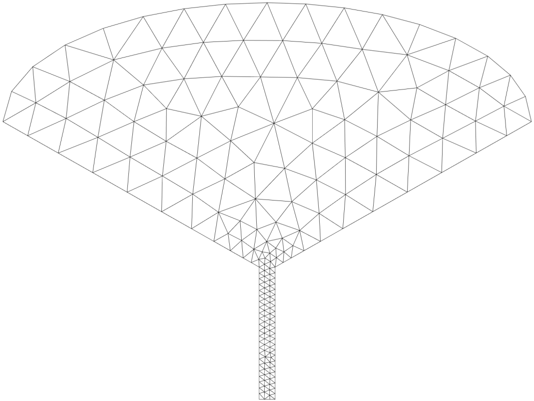

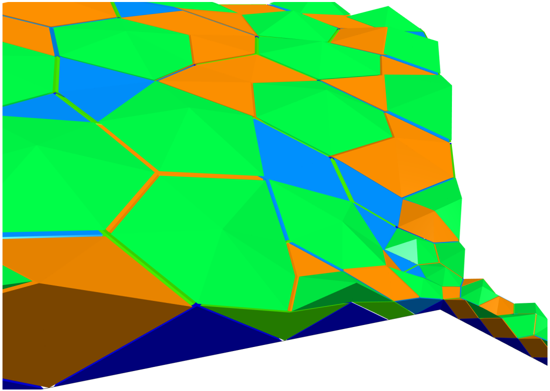

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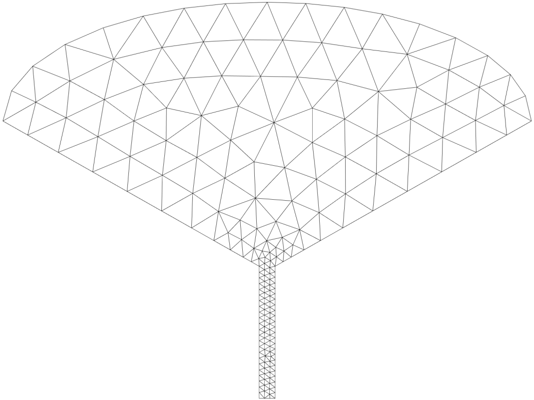

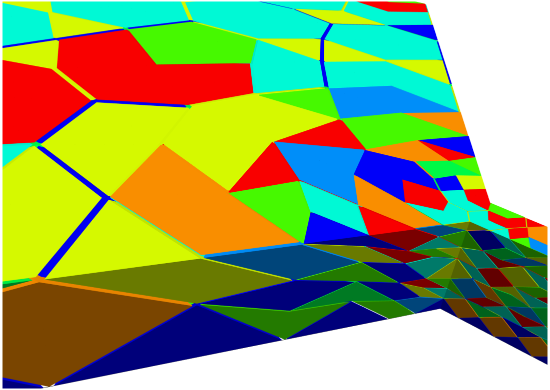

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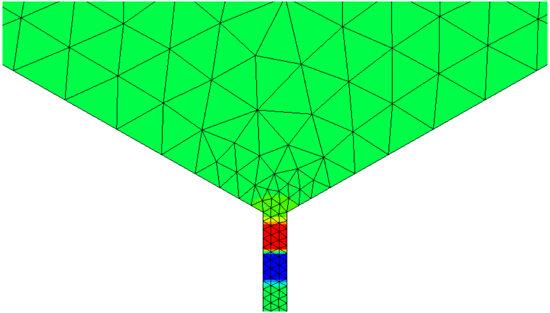

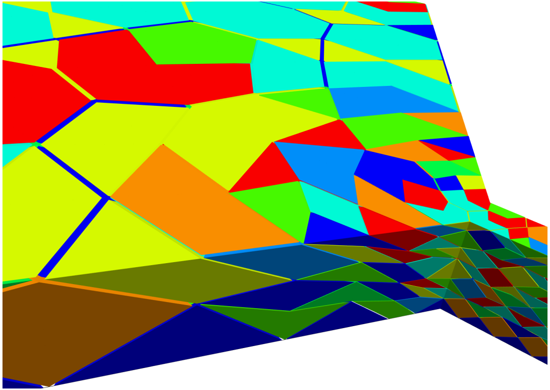

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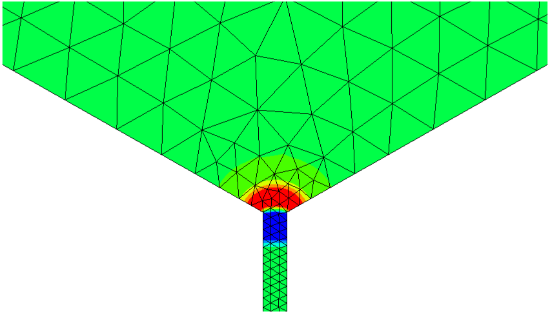

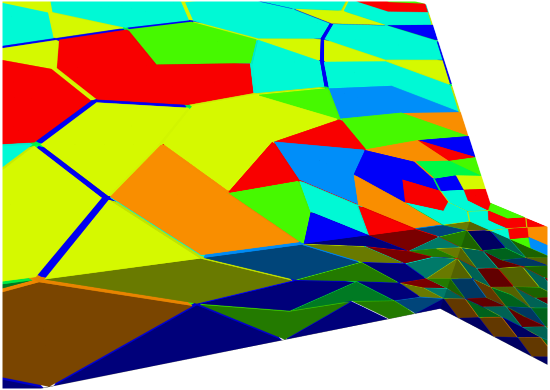

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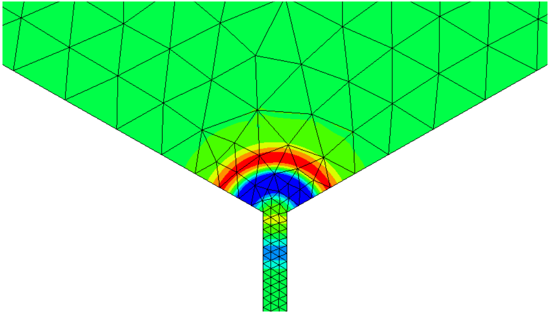

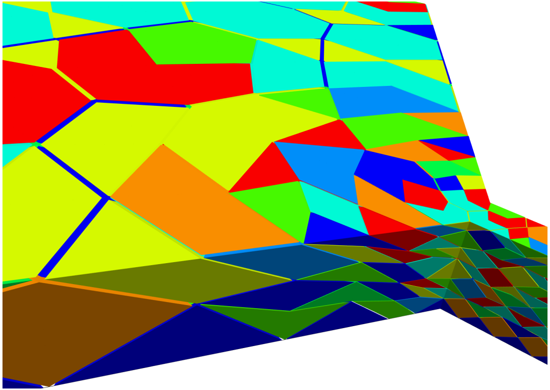

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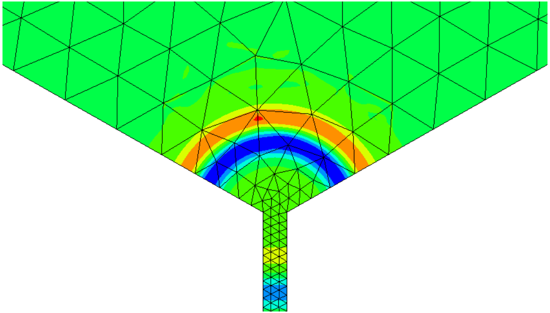
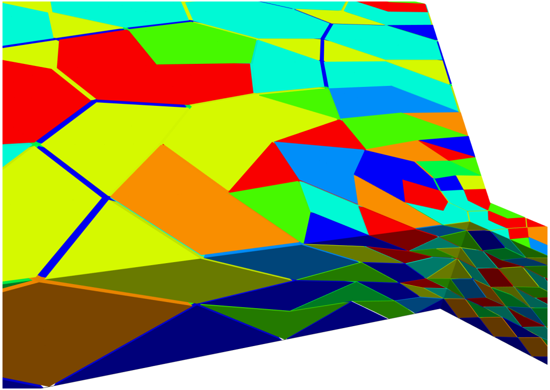




Opportunities

- ☺ A rational way to incorporate high order approximations, spatial adaptivity, and locally varying time steps, even on complex structures.
- ☺ Tent pole height restriction is a *local* causality constraint.
  - In contrast, in standard timestepping, time step is constrained by the *global* CFL constraint

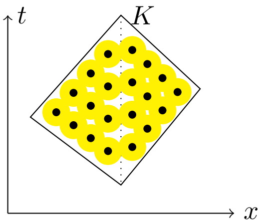
$$\frac{\text{minimal mesh size}}{(\text{maximal degree})^2} \times \frac{1}{\text{wave speed}}$$

- ☺ Very good candidate for task parallelism and hybrid parallel implementations.

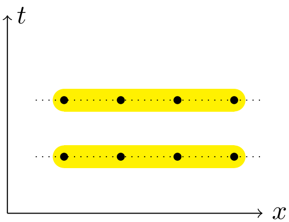
Limitation

☹ A tent domain is not a tensor product with a time interval.

- Cannot directly apply popular spatial discretizations.
- More coupling of tent degrees of freedom (than in explicit timestepping).



All spacetime unknowns within a tent are coupled.

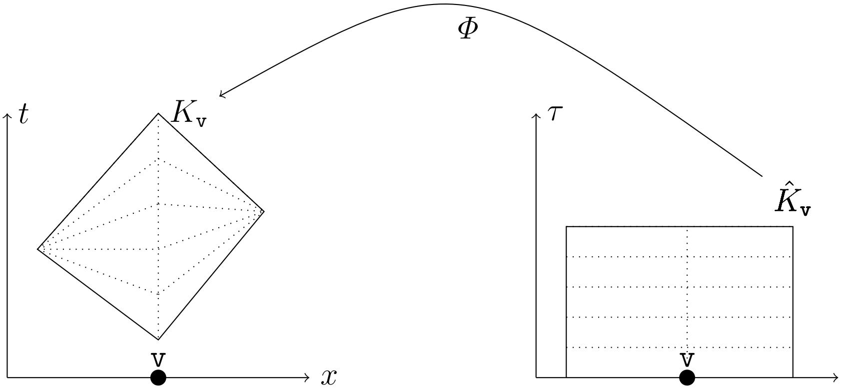


In traditional timestepping, standard spatial discretizations could be used.

MTP (Mapped Tent Pitching) Schemes

Spacetime Tent

Spacetime Cylinder



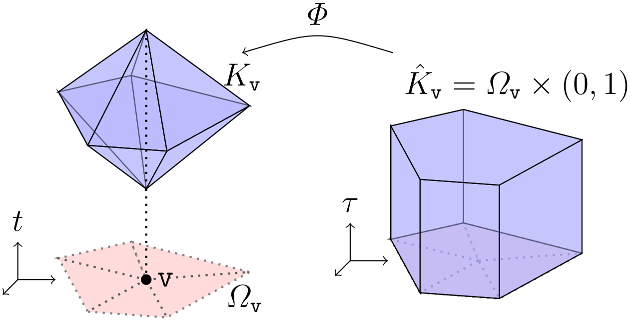
Instead of solving on spacetime tent $K_{\mathtt{v}}$, solve after pulling back the equation to a tensor-product spacetime cylinder $\hat{K}_{\mathtt{v}}$ where (pseudo)time τ and space x are separated.

Form of the (Duffy-like) map

The map is

$$\Phi \begin{pmatrix} x \\ \tau \end{pmatrix} = \begin{pmatrix} x \\ \varphi(x,\tau) \end{pmatrix}$$

where φ is defined as follows:



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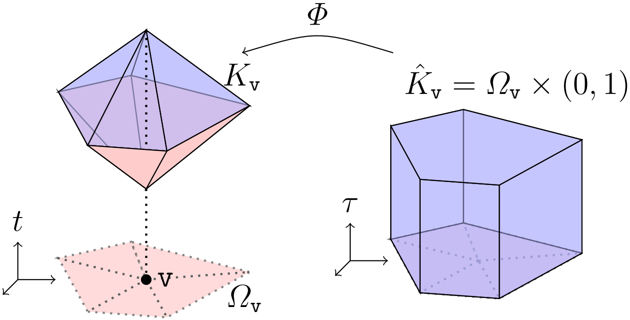
- tent bottom is the graph of φ_{bot},



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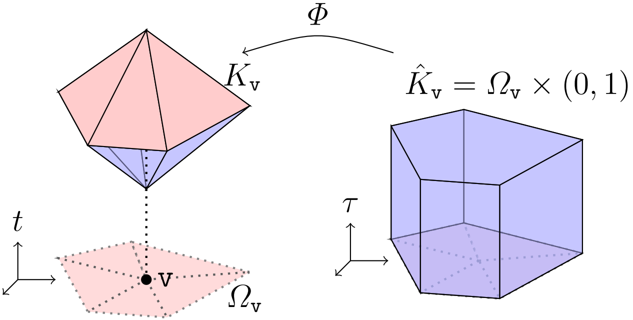
where φ is defined as follows: If

- tent bottom is the graph of φ_{bot},
- tent top is the graph of φ_{top},

then

$$\varphi = (1-\tau)\varphi_{\text{bot}} + \tau\varphi_{\text{top}}.$$



The height difference $\delta(x) = \varphi_{\text{top}}(x) - \varphi_{\text{bot}}(x)$ will appear as a weight function next.

Pullback of the conservation law

[G+Schöberl+Wintersteiger 2017]

$u : K_{\mathrm{v}} \to \mathbb{R}^L$ satisfies $\dfrac{\partial u}{\partial t} + \mathrm{div}_x f(u) = 0$	$\Longleftrightarrow$	$\hat{u} = u \circ \Phi : \hat{K}_{\mathrm{v}} \to \mathbb{R}^L$ satisfies $\dfrac{\partial}{\partial \tau}\left[\hat{u} - f(\hat{u})\,\mathrm{grad}_x \varphi\right] + \mathrm{div}_x\big(\delta\, f(\hat{u})\big) = 0.$

- In the tensor-product $\hat{K}_{\mathrm{v}}$, consider DG semidiscretization for $\hat{u}_h \approx \hat{u}$ of the form

$$\hat{u}_h(x,\tau) = \sum_j \underbrace{U_j(\tau)}_{\substack{\text{unknown}\\ \text{function of } \tau}} \underbrace{\psi_j(x)}_{\substack{\text{basis for spatial}\\ \text{DG space}}} .$$

- For all DG test functions v, the DG solution $\hat{u}_h$ solves

$$\int_{\Omega_{\mathrm{v}}} \frac{\partial}{\partial \tau}\left[\hat{u}_h - f(\hat{u}_h)\mathrm{grad}_x \varphi\right] \cdot v = \sum_{K \subset \Omega_{\mathrm{v}}} \left[\int_K \delta f(\hat{u}_h) : \mathrm{grad}_x v - \int_{\partial K} \delta \overbrace{\hat{F}^n_{\hat{u}_h}}^{\substack{\text{numerical}\\ \text{flux}}} \cdot v \right].$$

Tent ODE

$$\text{DG:} \quad \underbrace{\int_{\Omega_{\mathrm{v}}} \frac{\partial}{\partial \tau}\left[\hat{u}_h - f(\hat{u}_h)\,\mathrm{grad}_x\varphi\right]\cdot v}_{\frac{d}{d\tau}(M\hat{u}_h, v)} = \underbrace{\sum_{K\subset\Omega_{\mathrm{v}}}\left[\int_K \delta f(\hat{u}_h) : \mathrm{grad}_x v - \int_{\partial K} \delta \hat{F}^n_{\hat{u}_h}\cdot v\right]}_{(A\hat{u}_h, v)}$$

Since $\mathrm{grad}_x \varphi$ is linear in τ, using τ-independent operators M_0 and M_1 to write

$$M \equiv M(\tau) = M_0 - \tau\, M_1,$$

we obtain the local *tent ODE*

$$\boxed{(M\hat{u}_h)' = A\hat{u}_h}$$

with the (pseudo)time-varying mass term $M(\tau)$.

Semidiscrete analysis

Restrict to symmetric linear hyperbolic systems. [Drake+G+Schöberl+Wintersteiger 2022]

Lemma A consequence of the causality condition

$M(\tau) \equiv M_0 - \tau M_1$ is (selfadjoint and) positive definite for all $0 \leq \tau \leq 1$.

Lemma A property of DG for MTP

For a large class of DG num. fluxes & b.c., $-D = A + A^t + M_1$ is negative semidefinite.

- A large class of DG numerical fluxes and boundary conditions can be treated at once using the Friedrichs' systems framework of [Ern+Guermond 2006–2008].
- $|v|_D^2 = (Dv, v) \;\sim\; \| \llbracket v \rrbracket \|^2_{L^2(\text{facets})} +$ dissipation through boundary conditions.

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Lemma — Stability of **semidiscretization in one tent**

DG solution $\hat{u}_h(\cdot, \tau) \equiv \hat{u}_h(\tau)$ satisfies $\|\hat{u}_h(\tau)\|_{M(\tau)} \leq \|\hat{u}_h(0)\|_{M(0)}$ for any $0 < \tau \leq 1$.

The lemma identifies a norm in which stability on spacetime fronts is attainable:

$$\|v\|_{M(\tau)} \equiv (M(\tau)v, v)^{1/2}.$$

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Theorem — **Global** semidiscrete error estimate

Suppose $\Omega \times (0, T)$ is meshed by m layers of tents whose layer heights sum up to $O(T)$. Then the exact solution at the final time, $u(T)$, and the semidiscrete solution $u_h(T)$ computed using DG discretization using degree p polynomials, satisfy

$$\|u(T) - u_h(T)\|_{L^2(\Omega)} = O(h^{p+1/2}).$$

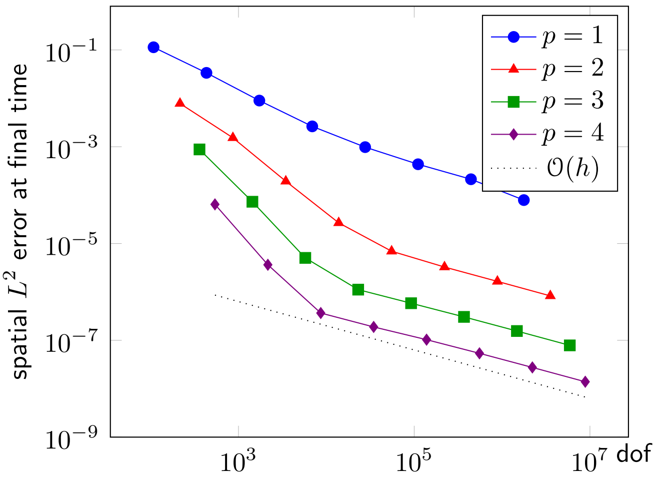
Full discretization: unexpected reduced rates

Solve $\boxed{(Mu)' = Au}$ by introducing $y = Mu$ and solving

$$y' = AM(\tau)^{-1}y.$$

But ...

When using the standard upwind spatial DG discretization for the wave equation and the *classical explicit RK4* scheme for timestepping, we observe that *the rate drops to first order.*



Fix: Structure-Aware Taylor (SAT) timestepping

Idea: Compute Taylor coefficients of the solution of of $y' = AM(\tau)^{-1}y$, or

$$y' = Au, \qquad y = M(\tau)u, \qquad u(0) = u_0,$$

- $y' = Au \implies y^{(k)}(0) = Au^{(k-1)}(0)$.
- $y = M(\tau)u \implies y^{(k)}(0) = M_0 u^{(k)}(0) - kM_1 u^{(k-1)}(0)$.
- $\implies$ the recursive formula $u^{(k)}(0) = M_0^{-1}(A + kM_1)u^{(k-1)}(0)$.
- Let $X_0 = I$, $X_k = M_0^{-1}(A + kM_1)X_{k-1}$. Then $u^{(k)}(0) = X_k u_0$.

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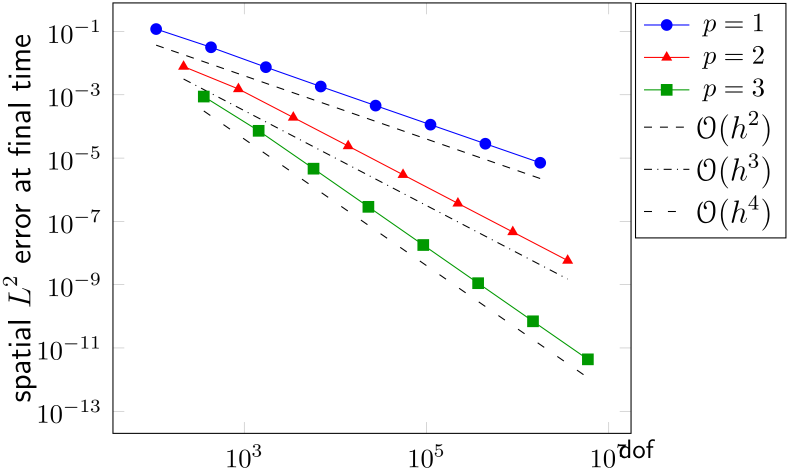
SAT timestepping with s stages: Compute $y^{(k)}(0) = AX_{k-1}u_0$,

$$\text{approximate } y(\tau) \text{ by} \quad y_s := \sum_{k=0}^{s} \frac{\tau^k}{k!} y^{(k)}(0), \text{ and}$$

$$\text{approximate } u(\tau) \text{ by} \quad R_s u_0 := M(\tau)^{-1} y_s.$$

Higher rates restored with $s = p + 1$ stage SAT

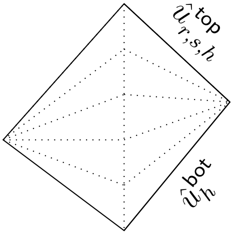


[G+Schöberl+Wintersteiger 2020]

Fully discrete analysis

1. Divide each tent into r subtents and apply the s-stage SAT scheme in each subtent:

$$\hat{u}_h^{\text{bot}} \equiv \hat{u}_h(0) \to \cdots r \text{ intermediate subtents } \cdots \to \hat{u}_{r,s,h}^{\text{top}}$$



2. Suppose tent-wise discrete weak stability holds in the sense that

$$\|\hat{u}_{r,s,h}^{\text{top}}\|_{M(1)} \le (1 + C_{\text{stab}} h) \|\hat{u}_h^{\text{bot}}\|_{M(0)}.$$

Theorem — Global error estimate for MTP with SAT

If the above stability condition holds, then (under the same assumptions as the previous theorem) the discrete solution $u_{r,s,h}(T)$ at the final time T satisfies

$$\|u(T) - u_{r,s,h}(T)\|_{L^2(\Omega)} = O(h^{p+1/2}) + O(h^{s-1/2}).$$

[Drake+G+Schöberl+Wintersteiger 2022]

Stability is a complex issue

For standard RK methods on method of lines discretizations (without tents):

- Textbook stability diagrams are misleading for $u' = Lu$ with non-normal L.
- RK2 and RK3 stable when $\Delta t \lesssim h$ if $(Lv, v) \le -\|Lv\|^2$. [Levy+Tadmor 1998]
- RK2 for advection with DG unstable when $\Delta t \lesssim h$. [Cockburn+Shu 2001]
- RK2 stable when $\Delta t \lesssim h^{4/3}$. [Zhang+Shu 2004][Burman+Ern+Fernández 2010]
- RK(s-stages, s^{th}order) stable when $s\%4 = 3$ under $\Delta t \lesssim h$. [Sun+Shu 2019]
- Stability for nonautonomous systems is still the wild west. [Ranocha+Ketcheson 2020]

For s-stage SAT scheme, dividing a tent into $r \sim h/\Delta t$ subtents:

- stable for $s = 2$ when $\Delta t \lesssim h^{3/2}$. [Drake+G+Schöberl+Wintersteiger 2022]

Stability of SAT schemes

Theorem [G+Sun (preprint)]

The s-stage SAT method is weakly stable if $\Delta t \lesssim h^{1+1/s}$ for any polynomial degree p.

Some ideas in the very technical stability proof:

1. Using the propagation operator $u_0 \mapsto R_s u_0$, it suffices to prove that $\boxed{\|R_s v\|_M \le (1 + C\tau^{1+s})\|v\|_{M_0}}$, since

$$
\begin{aligned}
\|\hat{u}^{\text{top}}_{r,s,h}\|_{M(1)} &\le (1 + C\tau^{1+s})^r \|\hat{u}^{\text{bot}}_h\|_{M(0)} && \text{applying it on } r \text{ subtents} \\
&\le (1 + C\tau^{1+s} r)\|\hat{u}^{\text{bot}}_h\|_{M(0)} \\
&\le (1 + C\tau^{s})\|\hat{u}^{\text{bot}}_h\|_{M(0)} && \text{since } \tau = r^{-1} \text{ on subtent top} \\
&\le (1 + Ch)\|\hat{u}^{\text{bot}}_h\|_{M(0)} && \text{when } \tau \lesssim h^{1/s}.
\end{aligned}
$$

2. Simplify the SAT expression to get $R_s v = \sum_{k=0}^{s} \frac{\tau^k}{k!} X_k v \; + \; \cdots$

   where $X_k = M_0^{-1}(A + kM_1)X_{k-1}$ and $X_0 = I$.

Proving $\|R_s v\|_M \le (1 + C\tau^{1+s})\|v\|_{M_0}$

3. $$\|R_s v\|_M^2 = \Big\| \sum_{k=0}^{s-1} \frac{\tau^k}{k!} X_k v \; + \; \cdots \Big\|^2_{M_0 - \tau M_1}$$
$$= \sum_{i,j=0}^{s-1} \frac{G_{ij}}{i!j!} - \sum_{i,j=0}^{s-1} \frac{F_{ij}}{i!j!} + \underbrace{\text{high-order term}}_{\rho}$$

Notation:
$F_{ij} = \tau^{i+j}(X_i v, X_j v)_{\tau M_1}$,
$G_{ij} = \tau^{i+j}(X_i v, X_j v)_{M_0}$,
$H_{ij} = \tau^{i+j}(X_i v, X_j v)_{\tau D}$.

Proving $\|R_s v\|_M \le (1 + C\tau^{1+s})\|v\|_{M_0}$

3. $$\|R_s v\|_M^2 = \Big\| \sum_{k=0}^{s-1} \frac{\tau^k}{k!} X_k v \; + \; \cdots \Big\|_{M_0 - \tau M_1}^2$$
$$= \sum_{i,j=0}^{s-1} \frac{G_{ij}}{i!j!} - \sum_{i,j=0}^{s-1} \frac{F_{ij}}{i!j!} + \underbrace{\text{high-order term}}_{\rho}$$

4. $$\|R_s v\|_M^2 = \sum_{i=0}^{s-1} \beta_i G_{ii} + \sum_{i,j=0}^{s-1} \eta_{ij} F_{ij} + \sum_{i,j=0}^{s-1} \gamma_{ij} H_{ij} + \rho$$

Notation:
$$F_{ij} = \tau^{i+j}(X_i v, X_j v)_{\tau M_1},$$
$$G_{ij} = \tau^{i+j}(X_i v, X_j v)_{M_0},$$
$$H_{ij} = \tau^{i+j}(X_i v, X_j v)_{\tau D}.$$

Key idea here is to use integration-by-parts-like identities:

$$G_{ij} = -\frac{1}{2} H_{ii} + \left(i + \frac{1}{2}\right) F_{ii}, \qquad \text{if } j = i+1,$$
$$G_{ij} = -G_{i+1,j-1} - H_{i,j-1} + (i+j) F_{i,j-1}, \qquad \text{if } j > i+1.$$

Proving $\|R_s v\|_M \le (1 + C\tau^{1+s})\|v\|_{M_0}$

3. $$\|R_s v\|_M^2 = \Big\| \sum_{k=0}^{s-1} \frac{\tau^k}{k!} X_k v \; + \; \cdots \Big\|^2_{M_0 - \tau M_1}$$
$$= \sum_{i,j=0}^{s-1} \frac{G_{ij}}{i!j!} - \sum_{i,j=0}^{s-1} \frac{F_{ij}}{i!j!} + \underbrace{\text{high-order term}}_{\rho}$$

Notation:
$F_{ij} = \tau^{i+j}(X_i v, X_j v)_{\tau M_1}$,
$G_{ij} = \tau^{i+j}(X_i v, X_j v)_{M_0}$,
$H_{ij} = \tau^{i+j}(X_i v, X_j v)_{\tau D}$.

4. $$\|R_s v\|_M^2 = \sum_{i=0}^{s-1} \beta_i G_{ii} + \sum_{i,j=0}^{s-1} \eta_{ij} F_{ij} + \sum_{i,j=0}^{s-1} \gamma_{ij} H_{ij} + \rho$$
5. $\beta_0 = 1,\ \beta_1 = \beta_2 = \cdots = \beta_{\lfloor s/2 \rfloor} = 0$, and $\beta_{\lfloor s/2 \rfloor + 1} \neq 0$,
6. $\eta_{ij} = 0$ for all $i + j \le s - 1$, and
7. γ_{ij} for $i, j < \lfloor (s+1)/2 \rfloor$ form a negative definite matrix.

   A few more technicalities finish the proof.

Conclusion

- Maps & Tents: [G+Schöberl+Wintersteiger 2017] *Mapped tent pitching schemes for hyperbolic systems*, SIAM J Sci. Comp., 39(6):B1043–B1063. MTP schemes, for the first time, allows *fully explicit* high order schemes (using standard DG) on unstructured spacetime meshes of causal tents.
- SAT timestepping: [G+Hochsteger+Schöberl+Wintersteiger 2020] *An explicit mapped tent pitching scheme for Maxwell equations*, Proc. ICOSAHOM, Lecture Notes in Computational Science and Engineering: 134: 359–369.
- Error analysis: [Drake+G+Schöberl+Wintersteiger 2022] *Convergence analysis of some tent-based schemes for linear hyperbolic systems*, Math. Comp. 91:699–733. Convergence analysis can be done at once for a large class of linear hyperbolic systems.
- Stability of timestepping: [G+Sun (preprint)] A proof of weak stability of Structure-Aware Taylor schemes of any order is now available.
- An NGSolve extension for tents under development at GitHub: `ngstents`