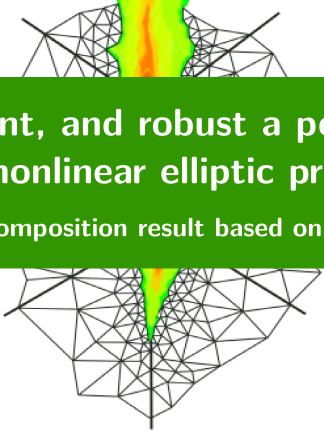

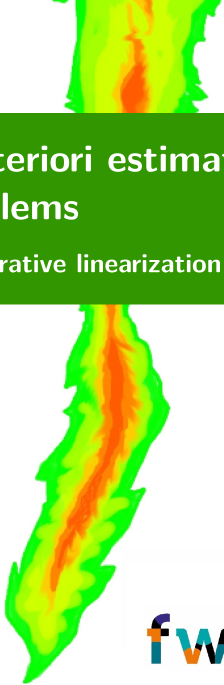

Reliable, efficient, and robust a posteriori estimates for nonlinear elliptic problems

An orthogonal decomposition result based on iterative linearization

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0 Outline

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1 Introduction

Nonlinear elliptic problems

For $d \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^d$ be an open and bounded polytope. Let $u \in H_0^1(\Omega)$ solve the **nonlinear** elliptic operator equation: for $\mathcal{R} : H_0^1(\Omega) \to H^{-1}(\Omega)$,

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Assumption 1 $\mathcal{R}$ is monotone & Lipschitz*

For a numerical approximation $u_\ell \in H_0^1(\Omega)$, and constants $\lambda_{\mathrm{M}} > \lambda_{\mathrm{m}} > 0$,

$$\lambda_{\mathrm{m}} \,\mathrm{dist}(u_\ell, u) \leq \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \mathcal{R}(u_\ell) - \mathcal{R}(u), \varphi \rangle}{\|\nabla \varphi\|} \leq \lambda_{\mathrm{M}} \,\mathrm{dist}(u_\ell, u).$$

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Then the estimate [Chaillou & Suri (2006), Kim (2007), Houston *et al* (2008), Garau *et al* (2011),...],

$$\lambda_{\mathrm{m}} \operatorname{dist}(u_\ell, u) \leq \eta(u_\ell) \leq C\lambda_{\mathrm{M}} \operatorname{dist}(u_\ell, u)$$

is not robust with respect to $\lambda_{\mathrm{M}}/\lambda_{\mathrm{m}}$

1 Dual norm of the residual estimate

Reliable, and **locally efficient** a posteriori error estimates **robust** with respect to the strength of the **nonlinearity** $\lambda_{\mathrm{M}}/\lambda_{\mathrm{m}}$

$$\|\mathcal{R}(u_\ell)\|_{H^{-1}(\Omega)} \leq \eta(u_\ell) \leq C\|\mathcal{R}(u_\ell)\|_{H^{-1}(\Omega)}$$

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- The dual norm of the residual might be too weak an error measure

1 A linear example

Consider the diffusion eq: $\langle \mathcal{R}(u), \varphi \rangle := (f, \varphi) - (\mathcal{D}\nabla u, \nabla\varphi) = 0$.

Let $\lambda_{\mathrm{m}} \, |\boldsymbol{y}|^2 \leq \boldsymbol{y}^{\mathrm{T}} \mathcal{D} \boldsymbol{y} \leq \lambda_{\mathrm{M}} \, |\boldsymbol{y}|^2$, for all $\boldsymbol{y} \in \mathbb{R}^d$.

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$$\|\nabla(u - u_\ell)\| \leq \frac{\lambda_{\mathrm{M}}}{\lambda_{\mathrm{m}}} \|\nabla(u - \varphi_\ell)\| \quad \forall \, \varphi_\ell \in V_\ell.$$

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However, defining the energy norm $|||\varphi|||_{1,\mathcal{D}} = \|\mathcal{D}^{\frac{1}{2}} \nabla \varphi\|$ one has

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This motivates rather the error measure

$$|||\mathcal{R}(u_\ell)|||_{-1,\mathcal{D}} := \sup_{\varphi \in H_0^1(\Omega)} \frac{\langle \mathcal{R}(u_\ell), \varphi \rangle}{|||\varphi|||_{1,\mathcal{D}}} = |||u - u_\ell|||_{1,\mathcal{D}}$$

which also results in robust estimates

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Example (nonlinear diffusion): $\langle \mathcal{R}(u), \varphi \rangle := (f, \varphi) - (\mathcal{D}(u)\nabla u, \nabla \varphi) = 0.$

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Then $|||\mathcal{R}(\cdot)|||_{-1,\mathcal{D}(u)}$ cannot be defined since $u \in H_0^1(\Omega)$ is unknown.

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Linearization iterations

We generally solve nonlinear equations by linearization iterations, i.e., by finding a sequence $\{u_\ell^i\}_{i\in\mathbb{N}} \subset V_\ell \subset H_0^1(\Omega)$.

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Example (Fixed point iteration) For each $i \in \mathbb{N}$ and $u_\ell^i \in V_\ell$, let $u_\ell^{i+1} \in V_\ell$ solve $(\mathcal{D}(u_\ell^i)\nabla u_\ell^{i+1}, \nabla \varphi_\ell) = (f, \varphi_\ell)$ for all $\varphi_\ell \in V_\ell$.

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Then defining the **iteration-dependent energy norm**

$$\begin{cases} |||\varphi|||_{1,u_\ell^i} := \|\mathcal{D}(u_\ell^i)^{\frac{1}{2}}\nabla \varphi\| & \text{for } \varphi \in H_0^1(\Omega), \\ |||\varsigma|||_{-1,u_\ell^i} = \sup_{\varphi\in H_0^1(\Omega)} \langle \varsigma, \varphi\rangle / |||\varphi|||_{1,u_\ell^i} & \text{for } \varsigma \in H^{-1}(\Omega), \end{cases}$$

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we have (under conditions) robust estimates of
$\left|\left|\left| \mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1}) \right|\right|\right|_{-1,u_\ell^i} = \left|\left|\left| u_{\langle \ell \rangle}^{i+1} - u_\ell^{i+1} \right|\right|\right|_{1,u_\ell^i}$

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Noting that

$$\langle \mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1}), \varphi \rangle := -(\mathcal{D}(u_\ell^i)\nabla (u_\ell^{i+1} - u_\ell^i), \nabla \varphi) + \langle \mathcal{R}(u_\ell^i), \varphi \rangle$$

can we provide a robust estimate for $|||\mathcal{R}(u_\ell^i)|||_{-1,u_\ell^i}$?

2 An orthogonal decomposition result

Theorem 1 Decomposition of the total error

Under Assumption 1, provided that the linearization iterations $\{u_\ell^i\}_{i\in\mathbb{N}} \subset V_\ell$ are generated by FE approximations of $u_{\langle\ell\rangle}^i \in H_0^1(\Omega)$ solving

$$\langle \mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_{\langle\ell\rangle}^{i+1}), \varphi\rangle := -\mathfrak{L}(u_\ell^i; u_{\langle\ell\rangle}^{i+1} - u_\ell^i, \varphi) + \langle \mathcal{R}(u_\ell^i), \varphi\rangle = 0 \quad \forall\, \varphi \in H_0^1(\Omega)$$

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Remark We would consider $\mathfrak{L} : H_0^1(\Omega) \times H_0^1(\Omega) \mapsto \mathbb{R}$ corresponding to **linear reaction-diffusion** problems, i.e,

$$\mathfrak{L}(u_\ell^i; v, w) := (\underbrace{L(\boldsymbol{x}, u_\ell^i)}_{\text{known reaction coeff.}}\, v, w) + (\underbrace{\mathfrak{a}(\boldsymbol{x}, u_\ell^i)}_{\text{known diffusion coeff.}}\nabla v, \nabla w).$$

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we have

$$\underbrace{|\!|\!|\mathcal{R}(u_\ell^i)|\!|\!|_{-1,u_\ell^i}^2}_{\text{total error}} = \underbrace{\left|\!\left|\!\left|\mathcal{R}_{\text{lin}}^{u_\ell^i}(u_\ell^{i+1})\right|\!\right|\!\right|_{-1,u_\ell^i}^2}_{\text{discretization error of the linerization step}} + \underbrace{|\!|\!|u_\ell^{i+1} - u_\ell^i|\!|\!|_{1,u_\ell^i}^2}_{\text{linearization error}}.$$

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Total error: $\underbrace{}_{\left|\left|\left|u_\ell^i - u_{\langle\ell\rangle}^{i+1}\right|\right|\right|_{1,u_\ell^i}^2}$; discretization error of the linerization step: $\underbrace{}_{\left|\left|\left|u_{\langle\ell\rangle}^{i+1} - u_\ell^{i+1}\right|\right|\right|_{1,u_\ell^i}^2}$

Proof: Since $u_\ell^{i+1} - u_\ell^i \in V_\ell$,

$$\left|\left|\left|\mathcal{R}(u_\ell^i)\right|\right|\right|_{-1,u_\ell^i}^2 = \left|\left|\left|u_\ell^i - u_{\langle\ell\rangle}^{i+1}\right|\right|\right|_{1,u_\ell^i}^2 = \left|\left|\left|(u_\ell^i - u_\ell^{i+1}) + (u_\ell^{i+1} - u_{\langle\ell\rangle}^{i+1})\right|\right|\right|_{1,u_\ell^i}^2$$

$$= \left|\left|\left|u_{\langle\ell\rangle}^{i+1} - u_\ell^{i+1}\right|\right|\right|_{1,u_\ell^i}^2 + \left|\left|\left|u_\ell^{i+1} - u_\ell^i\right|\right|\right|_{1,u_\ell^i}^2 + 2\underbrace{\mathfrak{L}(u_\ell^i; u_{\langle\ell\rangle}^{i+1} - u_\ell^{i+1}, u_\ell^{i+1} - u_\ell^i)}_{=0,\text{ due to Galerkin orthogonality}}$$

$$= \left|\left|\left|\mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1})\right|\right|\right|_{-1,u_\ell^i}^2 + \left|\left|\left|u_\ell^{i+1} - u_\ell^i\right|\right|\right|_{1,u_\ell^i}^2.$$

Proof: Since $u_\ell^{i+1} - u_\ell^i \in V_\ell$,

$$
\begin{aligned}
{|||\mathcal{R}(u_\ell^i)|||}^2_{-1,u_\ell^i} &= {|||u_\ell^i - u_{\langle\ell\rangle}^{i+1}|||}^2_{1,u_\ell^i} = {|||(u_\ell^i - u_\ell^{i+1}) + (u_\ell^{i+1} - u_{\langle\ell\rangle}^{i+1})|||}^2_{1,u_\ell^i} \\
&= {|||u_{\langle\ell\rangle}^{i+1} - u_\ell^{i+1}|||}^2_{1,u_\ell^i} + {|||u_\ell^{i+1} - u_\ell^i|||}^2_{1,u_\ell^i} + 2\underbrace{\mathfrak{L}(u_\ell^i; u_{\langle\ell\rangle}^{i+1} - u_\ell^{i+1}, u_\ell^{i+1} - u_\ell^i)}_{=0,\text{ due to Galerkin orthogonality}} \\
&= {|||\mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1})|||}^2_{-1,u_\ell^i} + {|||u_\ell^{i+1} - u_\ell^i|||}^2_{1,u_\ell^i}.
\end{aligned}
$$

- The linerization error is computed directly, we define

$$\eta^i_{\mathrm{lin},\Omega} := |||u_\ell^{i+1} - u_\ell^i|||_{1,u_\ell^i}.$$

Proof: Since $u_\ell^{i+1} - u_\ell^i \in V_\ell$,

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- The linerization error is computed directly, we define

$$\eta_{\mathrm{lin},\Omega}^i := \left|\left|\left|u_\ell^{i+1} - u_\ell^i\right|\right|\right|_{1,u_\ell^i}.$$

- For estimating $\left|\left|\left|\mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1})\right|\right|\right|_{-1,u_\ell^i}$ we introduce $\eta_{\mathrm{disc},\Omega}^i$, following the analysis on robust estimates of **singularly perturbed reaction -diffusion problems** in [Verfürth (1998)], [Ainsworth & Vejchodský (2011, 2014)] [Smears & Vohralík (2020)]

Theorem 2 Reliable, efficient, and robust a posteriori estimates

Global reliability

$$|||\mathcal{R}(u_\ell^i)|||^2_{-1,u_\ell^i} \leq [\eta_\Omega^i]^2 := \sum_{K\in\mathcal{T}_\ell} \left([\eta^i_{\mathrm{disc},K}]^2 + [\eta^i_{\mathrm{lin},K}]^2\right).$$

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Global efficiency

$$[\eta_\Omega^i]^2 \lesssim \left|\left|\left|\mathcal{R}(u_\ell^i)\right|\right|\right|_{-1,u_\ell^i}^2 + \text{ (data oscillation terms)}.$$

Local efficiency

For $\omega \subset \Omega$, there exists a neighbourhood $\mathfrak{T}_\omega \subseteq \Omega$ such that

$$[\eta_\omega^i]^2 \lesssim \left|\left|\left|\mathcal{R}(u_\ell^{i+1})\right|\right|\right|_{-1,u_\ell^i,\mathfrak{T}_\omega}^2 + [\eta_{\mathrm{lin},\mathfrak{T}_\omega}^i]^2 + \text{(data oscillation terms)}.$$

3 Outline

Class 1: gradient-dependent diffusivity problems

For all $\varphi \in H_0^1(\Omega)$, $\mathcal{R} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined as

$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(\boldsymbol{x}, u), \varphi \rangle - (\boldsymbol{\sigma}(\boldsymbol{x}, \nabla u), \nabla \varphi)$$

3 Class of problems

Class 1: gradient-dependent diffusivity problems

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$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(\boldsymbol{x}, u), \varphi \rangle - (\boldsymbol{\sigma}(\boldsymbol{x}, \nabla u), \nabla \varphi)$$

Assumption 1 is satisfied if $f(\boldsymbol{x}, \cdot)$, $\boldsymbol{\sigma}(\boldsymbol{x}, \cdot)$ are monotone and Lipschitz

$$(\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{z})) \cdot (\boldsymbol{y} - \boldsymbol{z}) \geq \lambda_{\mathrm{m}} |\boldsymbol{y} - \boldsymbol{z}|^2 \quad \text{for } \boldsymbol{x} \in \Omega \text{ and } \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d,$$

$$|\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) - \boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{z})| \leq \lambda_{\mathrm{M}} |\boldsymbol{y} - \boldsymbol{z}| \quad \text{for } \boldsymbol{x} \in \Omega \text{ and } \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^d.$$

with

$$\mathrm{dist}(u, v) = \|\nabla(u - v)\|$$

3 Class of problems

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with

$$\mathrm{dist}(u, v) = \|\nabla(u - v)\|$$

Example (Mean curvature flow) For $a(\cdot)$ satisfying ellipticity condition and $b(\cdot) > 0$: $\boldsymbol{\sigma}(\boldsymbol{x}, \boldsymbol{y}) = a(\boldsymbol{x}) + \frac{b(\boldsymbol{x})\boldsymbol{y}}{(1+|\boldsymbol{y}|^2)^{\frac{1}{2}}}$

Linearization operator

Considering the linearization operator

$$\mathfrak{L}(u_\ell^i; v, w) := (L(\boldsymbol{x}, u_\ell^i)\, v, w) + (\mathfrak{a}(\boldsymbol{x}, u_\ell^i)\nabla v, \nabla w),$$

the coefficient functions for commonly used linearization schemes are

Scheme	$L(\boldsymbol{x}, v)$	$\mathfrak{a}(\boldsymbol{x}, v)/\tau$
Kačanov (fixed point)	$\partial_\xi f(\boldsymbol{x}, v)$	$A(\boldsymbol{x}, \lvert\nabla v\rvert)$
Zarantonello	0	Λ (constant) > 0

Class 2: gradient-independent diffusivity problems

For all $\varphi \in H_0^1(\Omega)$, $\mathcal{R} : H_0^1(\Omega) \to H^{-1}(\Omega)$ is defined as

$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(\boldsymbol{x}, u), \varphi \rangle - \tau(\bar{\mathbf{K}}(\boldsymbol{x})(\mathcal{D}(\boldsymbol{x}, u)\nabla u + \boldsymbol{q}(\boldsymbol{x}, u)), \nabla \varphi)$$

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Assumption 1 is satisfied if $\tau > 0$ is small and

- $\mathcal{D} : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is bounded and Lipschitz
- $\bar{\mathbf{K}} : \Omega \to \mathbb{R}^{d \times d}$ is symmetric positive definite
- $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is monotone and Lipschitz upto the boundary
- $\boldsymbol{q} : \Omega \times \mathbb{R} \to \mathbb{R}^d$ is bounded and satisfies a Lipschitz condition*

with

$$\mathrm{dist}(u, v) = \left\| \bar{\mathbf{K}}^{\frac{1}{2}} \nabla \int_u^v \mathcal{D} \right\|$$

Class 2: gradient-independent diffusivity problems

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$$\langle \mathcal{R}(u), \varphi \rangle := \langle f(\boldsymbol{x}, u), \varphi \rangle - \tau(\bar{\mathbf{K}}(\boldsymbol{x})(\mathcal{D}(\boldsymbol{x}, u)\nabla u + \boldsymbol{q}(\boldsymbol{x}, u)), \nabla \varphi)$$

Semilinear equations $-\Delta u = f(\boldsymbol{x}, u)$
Such equations pop up in quantum mechanics (special solutions to nonlinear Klein–Gordon equations), gravitation influences on stars, membrane buckling problems...

Time-discrete nonlinear advection-reaction-diffusion equations

with time-step $\tau > 0$, the following evolutions equations reduce to this case

poro-Fischer equations:	$\partial_t u = \Delta u^m + \lambda\, u\,(1 - u)$
the Richards equation:	$\partial_t S(u) = \nabla \cdot [\bar{\mathbf{K}}(\boldsymbol{x})\kappa(S(u))(\nabla u + \boldsymbol{g})] + f(\boldsymbol{x}, u)$
biofilm equations:	$\partial_t u_k = \mu_k \Delta \Phi_k(u_k) + f_k((u_k)_{k=1}^n)$

3 Linearization schemes: practical examples

Abstract linearization

Considering the linearization operator

$$\mathfrak{L}(u_\ell^i; v, w) := (L(\boldsymbol{x}, u_\ell^i)\, v, w) + (\mathfrak{a}(\boldsymbol{x}, u_\ell^i)\nabla v, \nabla w),$$

the coefficient functions for commonly used linearization schemes are

Scheme	$L(\boldsymbol{x}, v)$	$\mathfrak{a}(\boldsymbol{x}, v)/\tau$
Picard (fixed point)	$\partial_\xi f(\boldsymbol{x}, v)$	$\bar{\mathbf{K}}(\boldsymbol{x})\,\mathcal{D}(\boldsymbol{x}, v)$
Jäger–Kačur	$\max_{\xi\in\mathbb{R}}\left(\frac{f(\boldsymbol{x},\xi)-f(\boldsymbol{x},v)}{\xi-v}\right)$	$\bar{\mathbf{K}}(\boldsymbol{x})\,\mathcal{D}(\boldsymbol{x}, v)$
L-scheme	L (constant) $\geq \frac{1}{2}\sup\partial_\xi f$	$\bar{\mathbf{K}}(\boldsymbol{x})\,\mathcal{D}(\boldsymbol{x}, v)$
M-scheme	$\partial_\xi f(\boldsymbol{x}, v) + M\tau$ (constant)	$\bar{\mathbf{K}}(\boldsymbol{x})\,\mathcal{D}(\boldsymbol{x}, v)$

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Effectivity indices

Global effectivity index: $\text{Eff. Ind.} := \eta_\Omega^i / \left|\!\left|\!\left| \mathcal{R}(u_\ell^i) \right|\!\right|\!\right|_{-1,u_\ell^i}$

Local effectivity index: $(\text{Eff. Ind.})_K := \eta_K^i / \left|\!\left|\!\left| \mathcal{R}(u_\ell^i) \right|\!\right|\!\right|_{-1,u_\ell^i,K}, \qquad K \in \mathcal{T}_\ell,$

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4 Gradient-independent diffusivity case: the Richards equation

For $\Omega = (0,1) \times (0,1)$ we study

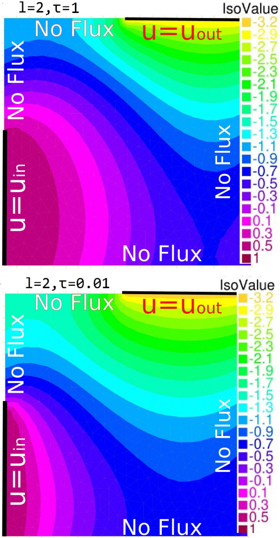
$$\langle \mathcal{R}(u_\ell), \varphi \rangle = (S(\bar{u}) - S(u_\ell), \varphi) \\ -\tau(\bar{\mathbf{K}}\kappa(S(u_\ell))[\nabla u_\ell - \boldsymbol{g}], \nabla\varphi)$$

where the van Genuchten parametrization for S, κ is used:

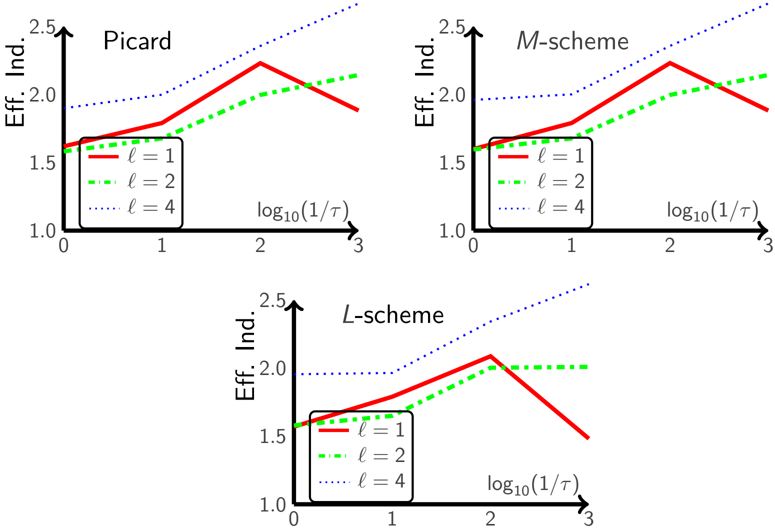
$$\begin{cases} S(\xi) := \left(1 + (2-\xi)^{\frac{1}{1-\lambda}}\right)^{-\lambda}, \\ \kappa(s) := \sqrt{s}\left(1 - (1 - s^{\frac{1}{\lambda}})^{\lambda}\right)^2, \end{cases}$$

with $\lambda = 0.5$, $u_\ell^0 = 0$,

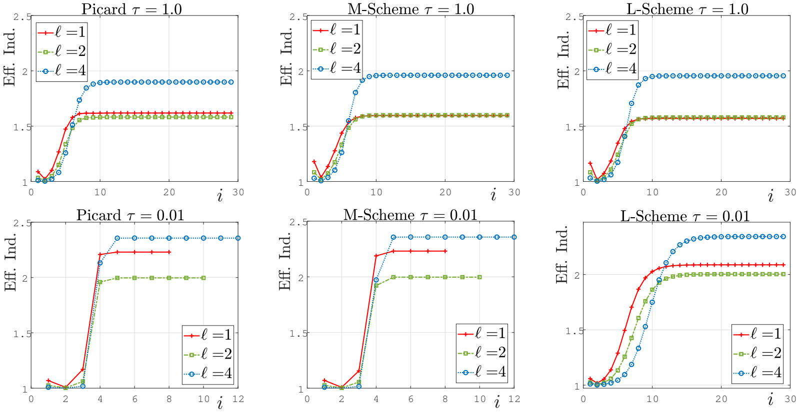
$$\bar{\mathbf{K}} = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}, \text{ and } \boldsymbol{g} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$



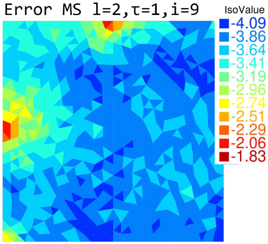
4 Robustness with respect to $\lambda_{\mathrm{M}}/\lambda_{\mathrm{m}}$ represented by $1/\tau$

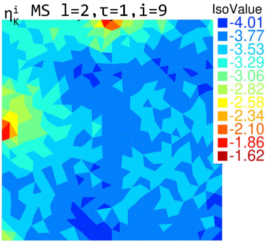


4 Global effectivity



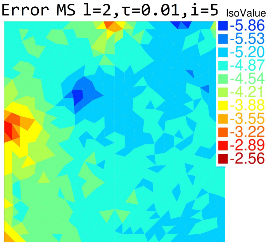
4 Distribution of error vs. estimates

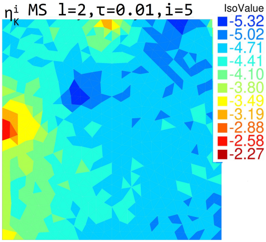




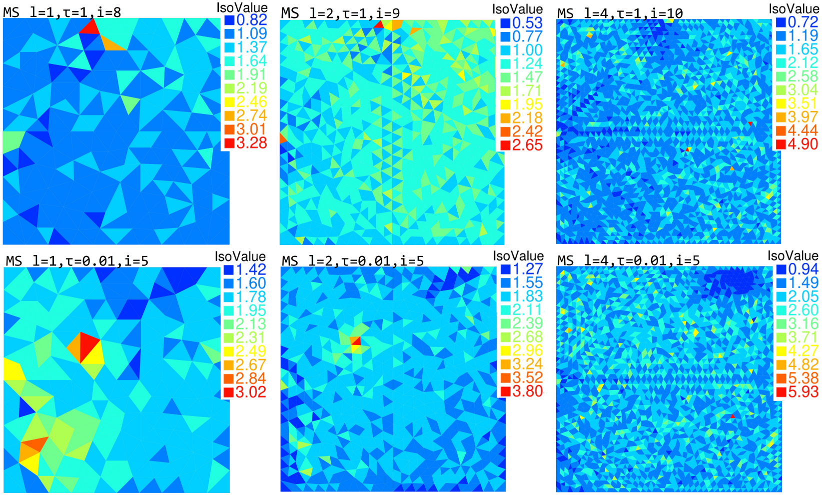
Error

Estimate

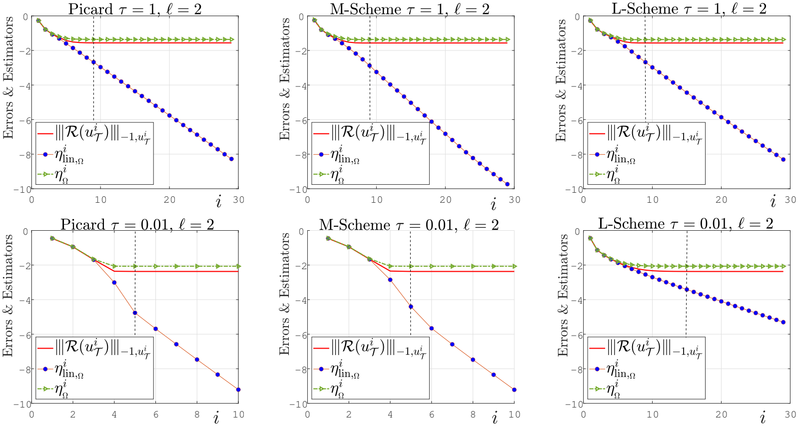




4 Local effectivity



4 Error with linearization iterations



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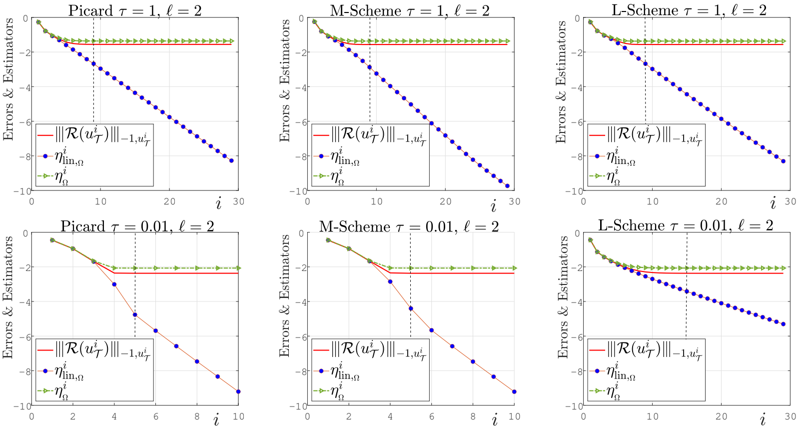

Adaptive iteration stopping criteria:

$$\eta^i_{\mathrm{lin},\Omega} \leq 0.05\,[\eta^i_{\Omega}].$$

4 The Newton scheme

For the Newton scheme, the linearization operator

$$\mathfrak{L}(u_\ell^i; v, w) := (L(\boldsymbol{x}, u_\ell^i)\, v, w) + (\mathfrak{a}(\boldsymbol{x}, u_\ell^i)\nabla v, \nabla w) + (\boldsymbol{w}(\boldsymbol{x}, u_\ell^i) v, \nabla w),$$

is non-symmetric.

For the Newton scheme, the linearization operator

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is non-symmetric. However, if for some $C_N \in [0, 2)$ we have

$$\boldsymbol{w}(\boldsymbol{x}, u_\ell^i)\, \mathfrak{a}^{-1}(\boldsymbol{x}, u_\ell^i)\, \boldsymbol{w}(\boldsymbol{x}, u_\ell^i) \leq C_N^2\, L(\boldsymbol{x}, u_\ell^i), \quad \forall\, \boldsymbol{x} \in \Omega, \text{ and } i \in \mathbb{N},$$

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then,

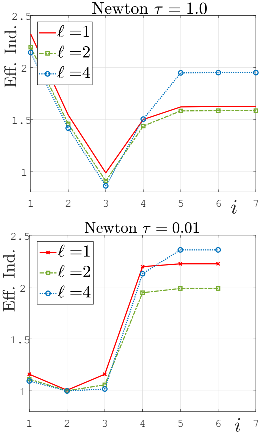
$$C_{\mathrm{m}}(C_N) \left[\left|\!\left|\!\left| \mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1}) \right|\!\right|\!\right|_{-1,u_\ell^i}^2 + |||u_\ell^{i+1} - u_\ell^i|||_{1,u_\ell^i}^2 \right] \leq |||\mathcal{R}(u_\ell^i)|||_{-1,u_\ell^i}^2$$

$$\leq C_{\mathrm{M}}(C_N) \left[\left|\!\left|\!\left| \mathcal{R}_{\mathrm{lin}}^{u_\ell^i}(u_\ell^{i+1}) \right|\!\right|\!\right|_{-1,u_\ell^i}^2 + |||u_\ell^{i+1} - u_\ell^i|||_{1,u_\ell^i}^2 \right]$$

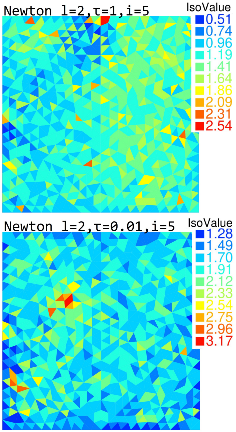
with $C_{\mathrm{m}}(C_N), C_{\mathrm{M}}(C_N) \to 1$ if $C_N \searrow 0$.

4 The Newton scheme: numerical results

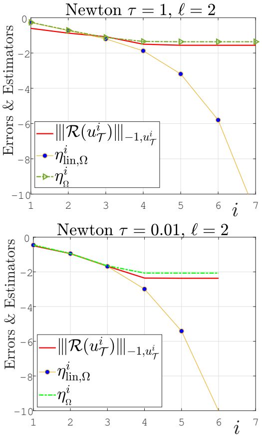
For gradient independent diffusivity case, we have



Global Effectivity



Local Effectivity



Error with iterations

4 Thank you for your time