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Adaptive FEM with quasi-optimal cost for nonlinear PDEs

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Introduction

Model problem

$$-\operatorname{div} \mathcal{A}(\nabla u^\star) = 0 \quad \text{in } \Omega$$
$$u^\star = 0 \quad \text{on } \partial\Omega$$

Example for strongly monotone nonlinearities

- nonlinear material laws $\boldsymbol{M} = \chi(|\boldsymbol{H}|)\,\boldsymbol{H}$ in magnetostatics
- together with $\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M})$
- e.g., $\mathcal{A}(\nabla u) = \Big(1 + \dfrac{1}{1 + |\nabla u|}\Big)\,\nabla u$

Zeidler: Nonlinear functional analysis and its applications, Part II/B

Abstract setting

- $\mathcal{H}$ separable Hilbert space with norm $|||\cdot|||$, $\mathcal{X}_\ell$ closed subspace
- nonlinear operator $A \colon \mathcal{H} \to \mathcal{H}^*$ such that, for all $u, v \in \mathcal{H}$,

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- nonlinear operator $\boldsymbol{A}\colon \mathcal{H} \to \mathcal{H}^*$ such that, for all $u, v \in \mathcal{H}$,
  (O1) strongly monotone $\alpha\,\|u - v\|^2 \le \langle \boldsymbol{A}u - \boldsymbol{A}v\,,\,u - v\rangle$
  (O2) Lipschitz continuous $\|\boldsymbol{A}u - \boldsymbol{A}v\| \le L\,\|u - v\|$

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  - (O2) Lipschitz continuous $|\!|\!| \boldsymbol{A}u - \boldsymbol{A}v |\!|\!|_* \leq L \, |\!|\!| u - v |\!|\!|$

Main Theorem on Strongly Monotone Operators (Zarantonello '60)

- exists unique $u^\star \in \mathcal{H}$ such that $\langle \boldsymbol{A}u^\star \, , \, v \rangle = 0$ for all $v \in \mathcal{H}$

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Corollary

- exists unique $u_\ell^\star \in \mathcal{X}_\ell$ such that $\langle \boldsymbol{A}u_\ell^\star \, , \, v_\ell \rangle = 0$ for all $v_\ell \in \mathcal{X}_\ell$
- $\|u^\star - u_\ell^\star\| \le \dfrac{L}{\alpha} \min_{u_\ell \in \mathcal{X}_\ell} \|u^\star - u_\ell\|$

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Zarantonello iteration

- constructive proof by Banach fixpoint theorem

- $I_{\mathcal{H}} : \mathcal{H} \to \mathcal{H}^*$ Riesz mapping $\quad \langle I_{\mathcal{H}} u \, , \, v \rangle = (u \, , \, v)_{\mathcal{H}}$
- $\Phi(u) := u - \delta \, I_{\mathcal{H}}^{-1} \boldsymbol{A} u \quad$ with $\quad 0 < \delta < 2\alpha/L^2$

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$\Longrightarrow \quad \|\Phi(u) - \Phi(v)\| \le \kappa\, \|u - v\| \quad$ with $\quad \kappa := \big[\,1 - \delta(2\alpha - \delta L^2)\,\big]^{1/2}$

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1. Φ has unique fixpoint $u^\star \in \mathcal{H} \quad (\iff \quad \boldsymbol{A} u^\star = 0 \ )$

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2. $u^0 \in \mathcal{H}$ arbitrary, $\quad u^k := \Phi(u^{k-1})$

$$\Longrightarrow \quad \|u^\star - u^k\| \le \frac{\kappa}{1-\kappa}\, \|u^k - u^{k-1}\| \le \frac{\kappa^k}{1-\kappa}\, \|u^1 - u^0\|$$

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- $u_\ell^0 \in \mathcal{X}_\ell$ arbitrary, $u_\ell^k := \Phi_\ell(u_\ell^{k-1})$

Congreve, Wihler: J. Comput. Appl. Math., 311 (2017)

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$\Longrightarrow$ $\|u_\ell^\star - u_\ell^k\| \le \kappa \, \|u_\ell^\star - u_\ell^{k-1}\|$ for all $k \in \mathbb{N}$

- triangle inequality $\Longrightarrow$ $\dfrac{1-\kappa}{\kappa} \|u_\ell^\star - u_\ell^k\| \le \|u_\ell^k - u_\ell^{k-1}\| \le (1+\kappa) \, \|u_\ell^\star - u_\ell^{k-1}\|$

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Discrete Zarantonello iteration

- solve $\quad (u_\ell^k \, , \, v_\ell)_{\mathcal{H}} = (u_\ell^{k-1} \, , \, v_\ell)_{\mathcal{H}} + \dfrac{\alpha}{L^2} \, \langle \boldsymbol{A} u_\ell^{k-1} \, , \, v_\ell \rangle \quad$ for all $v_\ell \in \mathcal{X}_\ell$

- i.e., each step of Zarantonello iteration solves one Laplace problem

Congreve, Wihler: J. Comput. Appl. Math., 311 (2017)

Quasi-optimal computational cost

- $k = 1, \ldots, K(\ell)$ steps of Zarantonello iteration $u_\ell^k \approx u_\ell^\star$ per mesh $\mathcal{T}_\ell$
- index set $(\ell, k) \in \mathcal{Q} \subset \mathbb{N}_0^2$ of discrete approximations u_ℓ^k
- suppose cost $\mathcal{O}(\#\mathcal{T}_\ell)$ for linear solve / estimate / mark / refine on mesh $\mathcal{T}_\ell$

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Aim for thorough proof of

$$\|u^\star\|_{\mathbb{A}_s} := \sup_{N \ge \#\mathcal{T}_0} \Big(N^s \min_{\#\mathcal{T}_{\rm opt} \le N} \big[\, |||u^\star - u^\star_{\rm opt}||| + \eta_{\rm opt}(u^\star_{\rm opt}) \,\big] \Big) \qquad \text{for all } s > 0$$

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$$\overset{\checkmark}{\simeq} \sup_{(\ell,0) \in \mathcal{Q}} (\#\mathcal{T}_\ell)^s \big[|||u^\star - u_\ell^\star||| + \eta_\ell(u_\ell^\star) \big]$$

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$$\overset{?}{\simeq} \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ (\ell',k')\leq(\ell,k)}} \mathcal{T}_{\ell'} \Big)^s \big[\, |\!|\!| u^\star - u_\ell^k |\!|\!| + \eta_\ell(u_\ell^k) \big]$$

AFEM with exact solver

Well-known AFEM algorithm

Input: $\mathcal{T}_0, \quad 0 < \theta \leq 1$

For each $\ell = 0, 1, 2, \ldots$ do

Dörfler: SIAM J. Numer. Anal., 33 (1996)

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NumPDEs ASC ▸ TUWIEN

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Output: Discrete solutions $u_\ell^\star$ and corresponding estimators $\eta_\ell(u_\ell^\star)$

- in practice: $\mathcal{M}_\ell$ is always chosen with quasi-minimal cardinality

Dörfler: SIAM J. Numer. Anal., 33 (1996)

Stevenson: Found. Comput. Math., 7 (2007)

Plain convergence

- recall: (O1) strongly monotone & (O2) Lipschitz continuous

Proposition

- (O1) & (O2) & residual error estimator
- arbitrary $0 < \theta \le 1$

$\Longrightarrow$ exists $0 < q < 1$ and $C > 0$ such that

$$\|u^\star - u_\ell^\star\| \lesssim \eta_\ell(u_\ell^\star) \le q\,\eta_{\ell-1}(u_{\ell-1}^\star) + C\,\|u_\ell^\star - u_{\ell-1}^\star\| \xrightarrow{\ell\to\infty} 0$$

Morin, Siebert, Veeser: Math. Models Methods Appl. Sci., 18 (2008)

Gantner, Haberl, Praetorius, Stiftner: IMA J. Numer. Anal., 38 (2018)

Energy minimization

- suppose that $\boldsymbol{A}$ has a potential, i.e.,

(O3) $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ such that $\langle \boldsymbol{A}u\,,\,v\rangle = \lim_{r\to 0} \dfrac{\mathcal{J}(u+rv)-\mathcal{J}(u)}{r}$ for all $u, v \in \mathcal{H}$

Lemma

- (O1) & (O2) & (O3)
- $v_\ell \in \mathcal{X}_\ell$

$$\Longrightarrow \quad \frac{\alpha}{2}\,\|u_\ell^\star - v_\ell\|^2 \le \mathcal{J}(v_\ell) - \mathcal{J}(u_\ell^\star) \le \frac{L}{2}\,\|u_\ell^\star - v_\ell\|^2$$

- i.e., Galerkin formulation is equivalent to minimization of $\mathcal{J}$ over $\mathcal{X}_\ell$

$$\Longrightarrow \quad \|u_{\ell+1}^\star - u_\ell^\star\|^2 \simeq \mathcal{J}(u_\ell^\star) - \mathcal{J}(u_{\ell+1}^\star) = [\mathcal{J}(u_\ell^\star) - \mathcal{J}(u^\star)] - [\mathcal{J}(u_{\ell+1}^\star) - \mathcal{J}(u^\star)]$$

Zeidler: Nonlinear functional analysis and its applications, Part II/B

Linear convergence

Theorem (Diening, Kreuzer '08; ... ; Carstensen, Feischl, Page, P. '14)

- (O1) & (O2) & (O3) & residual error estimator
- arbitrary $0 < \theta \le 1$

$\Longrightarrow$ exists $0 < q < 1$ such that

$$\|u^\star - u_\ell^\star\| \lesssim \eta_\ell(u_\ell^\star) \lesssim q^{\ell-\ell'}\eta_{\ell'}(u_{\ell'}^\star) \quad \text{for all } \ell' \le \ell$$

- required: certain Pythagoras-type quasi-orthogonality
- here: energy

Diening, Kreuzer: SIAM J. Numer. Anal., 46 (2008)
Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)
Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

Optimal convergence rates

Theorem (Stevenson '07; ... ; Carstensen, Feischl, Page, P. '14)

- (O1) & (O2) & (O3) & residual error estimator
- arbitrary $s > 0$
- $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} N^s \, \eta_{\rm opt}(u^\star_{\rm opt}) < \infty$
- sufficiently small $0 < \theta \leq 1$

$\Longrightarrow$ $\sup_{\ell \in \mathbb{N}} (\#\mathcal{T}_\ell)^s \, \eta_\ell(u^\star_\ell) \simeq \|u^\star\|_{\mathbb{A}_s}$

- note: $\eta_{\rm opt}(u^\star_{\rm opt}) \simeq \big[\, |\!|\!| u^\star - u^\star_{\rm opt} |\!|\!| + \eta_{\rm opt}(u^\star_{\rm opt}) \,\big]$
- required: linear convergence of $\eta_\ell(u^\star_\ell)$

Stevenson: Found. Comput. Math., 7 (2007)
Garau, Morin, Zuppa: Numer. Math. Theory Methods Appl., 5 (2012)
Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

AFEM with iterative solver

Stopping criterion for linearization

- reliability & stability of residual error estimator

$$
\begin{aligned}
|||u^\star - u_\ell^k||| \le |||u^\star - u_\ell^\star||| + |||u_\ell^\star - u_\ell^k||| &\overset{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^\star) + |||u_\ell^\star - u_\ell^k||| \\
&\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + |||u_\ell^\star - u_\ell^k||| \\
&\overset{\text{solver}}{\lesssim} \eta_\ell(u_\ell^k) + |||u_\ell^k - u_\ell^{k-1}|||
\end{aligned}
$$

$\Longrightarrow$ contraction allows for a-posteriori error control

Stopping criterion for linearization

- reliability & stability of residual error estimator

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$$\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{solver}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$$

$\Longrightarrow$ contraction allows for a-posteriori error control

- idea: equilibrate $\eta_\ell(u_\ell^k)$ and $\|u_\ell^k - u_\ell^{k-1}\|$

$\Longrightarrow$ stop linearization for $K = k$ with $\|u_\ell^K - u_\ell^{K-1}\| \le \lambda\, \eta_\ell(u_\ell^K)$

Stopping criterion for linearization

- reliability & stability of residual error estimator

$$\|u^\star - u_\ell^k\| \le \|u^\star - u_\ell^\star\| + \|u_\ell^\star - u_\ell^k\| \overset{\text{reliability}}{\lesssim} \eta_\ell(u_\ell^\star) + \|u_\ell^\star - u_\ell^k\|$$
$$\overset{\text{stability}}{\lesssim} \eta_\ell(u_\ell^k) + \|u_\ell^\star - u_\ell^k\|$$
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- nested iteration: $u_\ell^0 := u_{\ell-1}^K$

$\Longrightarrow$ a-posteriori error control for all u_ℓ^k but u_0^0

AFEM with contractive linearization

Input: $\mathcal{T}_0$, u_0^0, $0 < \theta \leq 1$, $\lambda > 0$

For each $\ell = 0, 1, 2, \ldots$ do

- **SOLVE & ESTIMATE:** For $k = 1, 2, 3, \ldots, K$, **repeat**
  - compute u_ℓ^k
  - compute $\eta_\ell(T, u_\ell^k)$ for all $T \in \mathcal{T}_\ell$

  until $\quad |||u_\ell^K - u_\ell^{K-1}||| \leq \lambda \, \eta_\ell(u_\ell^K)$

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- **MARK:** choose $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that $\theta \sum_{T \in \mathcal{T}_\ell} \eta_\ell(T, u_\ell^K)^2 \le \sum_{T \in \mathcal{M}_\ell} \eta_\ell(T, u_\ell^K)^2$
- **REFINE:** $\mathcal{T}_{\ell+1} := \mathrm{refine}(\mathcal{T}_\ell, \mathcal{M}_\ell)$, $u_{\ell+1}^0 := u_\ell^K$

Output: Discrete solutions u_ℓ^k and corresponding estimator $\eta_\ell(u_\ell^k) + \|u_\ell^k - u_\ell^{k-1}\|$

- note: number of solver steps $K = K(\ell)$ might vary with ℓ

Main result 1: Full linear convergence

Theorem (Gantner, Haberl, P., Schimanko '21)

- (O1) & (O2) & (O3) & residual error estimator
- $|\ell, k| \;:=\; \#\big\{(\ell', k') \in \mathcal{Q} \,:\, u_{\ell'}^{k'}$ computed earlier than $u_\ell^k\big\}$
- quasi-error $\quad \Delta_\ell^k := \big[|\!|\!| u^\star - u_\ell^k |\!|\!| + \eta_\ell(u_\ell^k) \big]$
- arbitrary $\quad u_0^0, \quad$ arbitrary $0 < \theta \le 1, \quad$ sufficiently small $\lambda > 0$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)
Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

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- quasi-error $\Delta_\ell^k := [\![u^\star - u_\ell^k]\!] + \eta_\ell(u_\ell^k)$
- arbitrary u_0^0, arbitrary $0 < \theta \leq 1$, sufficiently small $\lambda > 0$

$\Longrightarrow$ exists $0 < q < 1$ such that $|\!|\!| u^\star - u_\ell^k |\!|\!| \lesssim \Delta_\ell^k \lesssim q^{|\ell,k| - |\ell',k'|} \Delta_{\ell'}^{k'}$

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- quasi-error $\Delta_\ell^k := [\![\![u^\star - u_\ell^k]\!]\!] + \eta_\ell(u_\ell^k)]$
- arbitrary u_0^0, arbitrary $0 < \theta \leq 1$, sufficiently small $\lambda > 0$

$\Longrightarrow$ exists $0 < q < 1$ such that $|\!|\!| u^\star - u_\ell^k |\!|\!| \lesssim \Delta_\ell^k \lesssim q^{|\ell,k|-|\ell',k'|} \Delta_{\ell'}^{k'}$

- note [CFPP14]: linear convergence $\iff$ $\displaystyle\sum_{\substack{(\ell,k)\in\mathcal{Q} \\ |\ell,k|>|\ell',k'|}} \Delta_\ell^k \lesssim \Delta_{\ell'}^{k'}$
- for exact solvers: $\Delta_\ell^\star \simeq \eta_\ell(u_\ell^\star)$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)
Carstensen, Feischl, Page, Praetorius: Comp. Math. Appl., 67 (2014)

Perturbation argument = Why sufficiently small λ?

- $\eta_\ell(u_\ell^K) \overset{\text{stability}}{\le} \eta_\ell(u_\ell^\star) + C_{\rm stab}\,|||u_\ell^\star - u_\ell^K||| \overset{\text{solver}}{\le} \eta_\ell(u_\ell^\star) + C_{\rm stab}\,\frac{\kappa}{1-\kappa}\,|||u_\ell^K - u_\ell^{K-1}|||$

  $\overset{\text{terminate}}{\le} \eta_\ell(u_\ell^\star) + C_{\rm stab}\,\frac{\kappa}{1-\kappa}\lambda\,\eta_\ell(u_\ell^K)$

- similarly: $\eta_\ell(u_\ell^\star) \le \eta_\ell(u_\ell^K) + C_{\rm stab}\,\frac{\kappa}{1-\kappa}\lambda\,\eta_\ell(u_\ell^K)$

$$\Longrightarrow \quad (1 - C\lambda)\,\eta_\ell(u_\ell^K) \le \eta_\ell(u_\ell^\star) \le (1 + C\lambda)\,\eta_\ell(u_\ell^K)$$

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$\Longrightarrow$ linear convergence of $\eta_\ell(u_\ell^K) \simeq \eta_\ell(u_\ell^\star)$

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$\Longrightarrow$ linear convergence of $\eta_\ell(u_\ell^K) \simeq \eta_\ell(u_\ell^\star)$

$\Longrightarrow$ summability of $\Delta_\ell^k = \|u^\star - u_\ell^k\| + \eta_\ell(u_\ell^k) \simeq \|u_\ell^\star - u_\ell^k\| + \eta_\ell(u_\ell^k)$ by geometric series

Main result 2: Optimal complexity

Theorem (Gantner, Haberl, P., Schimanko '21)

- arbitrary $s > 0$
- $\|u^\star\|_{\mathbb{A}_s} := \sup_{N \geq \#\mathcal{T}_0} \min_{\#\mathcal{T}_{\rm opt} \leq N} N^s \, \Delta^\star_{\rm opt} < \infty$
- sufficiently small $\quad 0 < \theta \leq 1 \quad$ and $\quad \lambda > 0$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

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- sufficiently small $0 < \theta \le 1$ and $\lambda > 0$

$$\Longrightarrow \quad \|u^\star\|_{\mathbb{A}_s} \lesssim \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \lesssim \max\big\{ \|u^\star\|_{\mathbb{A}_s}, \Delta_0^0 \big\}$$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

The wonder of full linear convergence!

- full linear convergence

$$\Longrightarrow \quad \sup_{(\ell,k)\in\mathcal{Q}} \Big(\sum_{\substack{(\ell',k')\in\mathcal{Q} \\ |\ell',k'|\le|\ell,k|}} \#\mathcal{T}_{\ell'} \Big)^s \Delta_\ell^k \simeq \sup_{(\ell,k)\in\mathcal{Q}} (\#\mathcal{T}_\ell)^s \Delta_\ell^k$$

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Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

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- sufficiently small $0 < \theta \le 1$

$$\Longrightarrow \quad \sup_{\ell\in\mathbb{N}_0} (\#\mathcal{T}_\ell)^s \eta_\ell(u_\ell^\star) \simeq \|u^\star\|_{\mathbb{A}_s}$$

Gantner, Haberl, Praetorius, Schimanko: Math. Comp., 90 (2021)

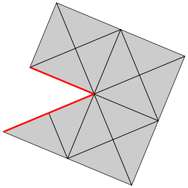
Numerical experiment

mixed BVP

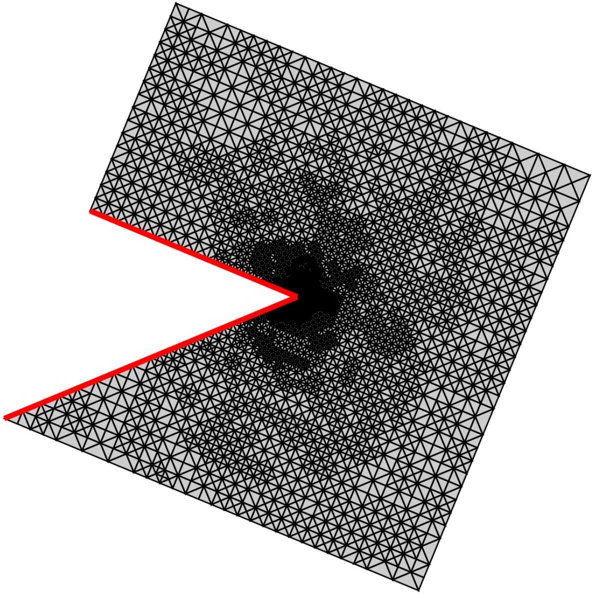
$$\begin{aligned} -\mathrm{div}\big[\,\mu(|\nabla u^\star|)\,\nabla u^\star\,\big] &= f \quad \text{in } \Omega \\ \mu(|\nabla u^\star|)\,\nabla u^\star \cdot \boldsymbol{n} &= g \quad \text{on } \Gamma_N \\ u^\star &= 0 \quad \text{on } \Gamma_D \end{aligned}$$

- $\mu(t) := 2 + \dfrac{1}{1+t} \implies \alpha = 2, \quad L = 3$ w.r.t. $|||\cdot||| = \|\nabla(\cdot)\|_{L^2(\Omega)}$

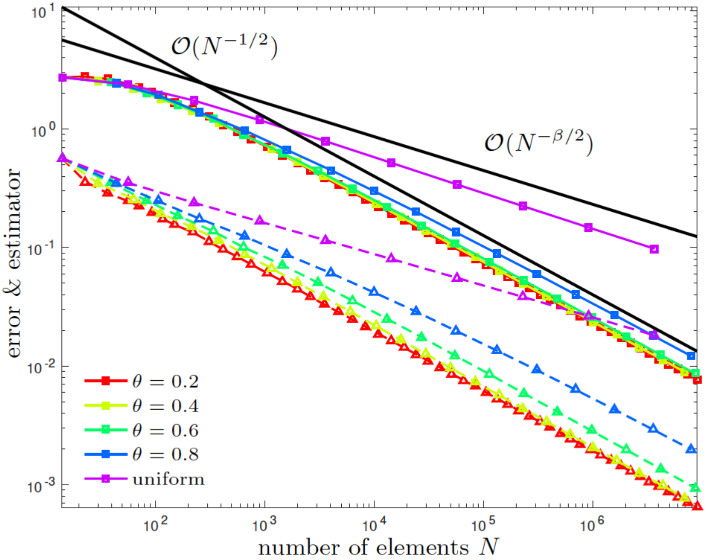
- prescribed singular solution $u(x) = r^\beta\,\cos(\beta\varphi)$
- $$\begin{aligned} \eta_\ell(T,u_\ell) = {} & h_T^2\,\|f + \mathrm{div}\big[\,\mu(|\nabla u_\ell|)\,\nabla u_\ell\,\big]\|_{L^2(T)}^2 \\ & + h_T\,\|g - \mu(|\nabla u_\ell|)\,\nabla u_\ell \cdot \boldsymbol{n}\|_{L^2(\partial T\cap\Gamma_N)}^2 \\ & + h_T\,\|\,[\![\,\mu(|\nabla u_\ell|)\,\nabla u_\ell \cdot \boldsymbol{n}\,]\!]\,\|_{L^2(\partial T\cap\Omega)}^2 \end{aligned}$$



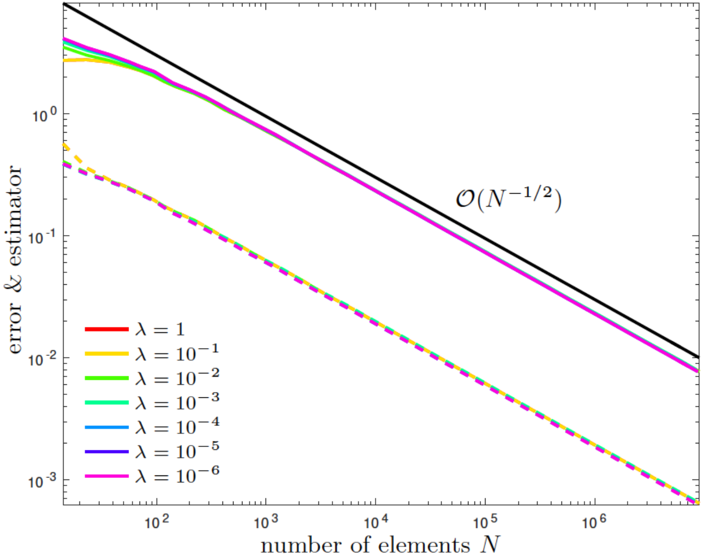
Adaptive mesh $\#\mathcal{T}_{20} = 7.657$



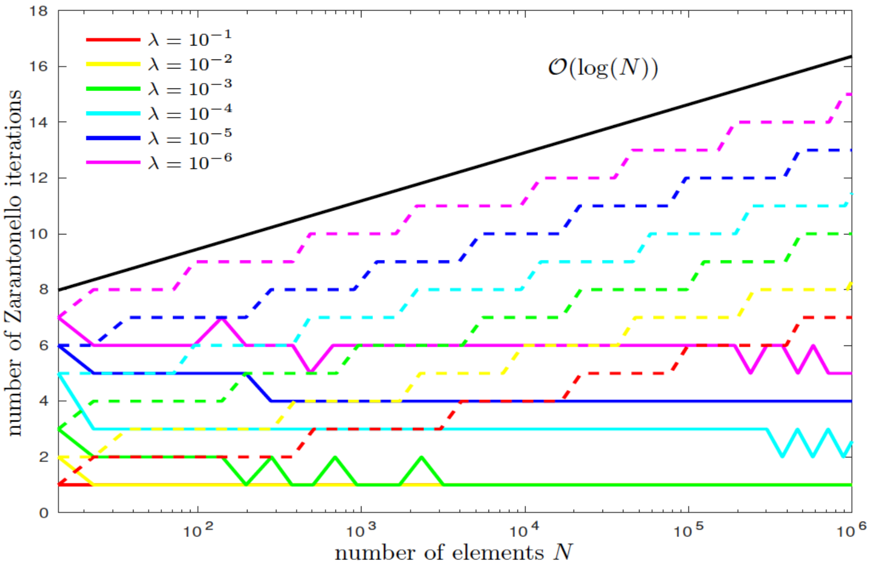
Dependence on θ for $\lambda = 10^{-2}$



Dependence on λ for $\theta = 0.2$



Nested iteration vs. $u_\ell^0 := 0$



Conclusion & three notable extensions

Conclusion

- new: abstract framework for full linear convergence
  - applies to (strongly monotone) energy minimization problems
  - exploits only usual properties of residual error estimators
  - relies only on contractive linearization / solver
- implication: adaptive algorithm with quasi-optimal cost (instead of only optimal rate)
  - if rate-optimal, then even cost-optimal

Extension 1: Full linear convergence for arbitrary λ!

- stop linearization if $|\mathcal{J}(u_\ell^K) - \mathcal{J}(u_\ell^{K-1})|^{1/2} \le \lambda\, \eta_\ell(u_\ell^K)$
  - instead of $|||u_\ell^K - u_\ell^{K-1}||| \le \lambda\, \eta_\ell(u_\ell^K)$

$\Longrightarrow$ [HPW21] full linear convergence holds for arbitrary $\lambda > 0$
and holds for Zarantonello iteration & Kacanov iteration & damped Newton method

- Heid, Wihler: Calcolo, 57 (2020)
- Heid, Praetorius, Wihler: Comput. Methods Appl. Math., 21 (2021)
- Chen, Nochetto, Xu: Numer. Math., 120 (2012)
- Wu, Zheng: Appl. Numer. Math., 113 (2017)

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$\Longrightarrow$ [HPW21] full linear convergence holds for arbitrary $\lambda > 0$
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- consequence: rates with respect to costs and dofs always coincide!

- applies to linear PDEs with contractive PCG [CNX12] or multigrid [WZ17] solvers

Heid, Wihler: Calcolo, 57 (2020)
Heid, Praetorius, Wihler: Comput. Methods Appl. Math., 21 (2021)
Chen, Nochetto, Xu: Numer. Math., 120 (2012)
Wu, Zheng: Appl. Numer. Math., 113 (2017)

Extension 2: Including an algebraic solver!

- employ contractive algebraic solver for the linear systems, i.e., $u^\star \approx u_\ell^\star \approx u_\ell^k \approx u_\ell^{k,j}$

$\Longrightarrow$ index set $(\ell, k, j) \in \mathcal{Q} \subset \mathbb{N}_0^3$

- stop algebraic solver if $\|u_\ell^{k,J} - u_\ell^{k,J-1}\| \le \mu\,[\,\eta_\ell(u_\ell^{k,J}) + \|u_\ell^{k,J} - u_\ell^{k-1,J}\|\,]$
- stop Zarantonello linearization if $\|u_\ell^{K,J} - u_\ell^{K-1,J}\| \le \lambda\,\eta_\ell(u_\ell^{K,J})$

$\Longrightarrow$ full linear convergence for sufficiently small λ and μ [HPSV21]

Haberl, Praetorius, Schimanko, Vohralik: Numer. Math., 147 (2021)

Miraci, Praetorius, Vohralik: in progress (2022+)

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- full linear convergence for arbitrary λ and sufficiently small μ [MPV22+] and extension to Zarantonello iteration & Kacanov iteration & damped Newton method

- ongoing: new algebraic stopping criterion $C\,\|u_\ell^{k,J} - u_\ell^{k-1,J}\|^2 \le \mathcal{J}(u_\ell^{k-1,J}) - \mathcal{J}(u_\ell^{k,J})$?!

Haberl, Praetorius, Schimanko, Vohralik: Numer. Math., 147 (2021)
Miraci, Praetorius, Vohralik: in progress (2022+)

Extension 3: Application to semilinear PDEs

Model problem

$$-\mathrm{div}\,(A\nabla u^\star) + b(u^\star) = f \quad \text{in } \Omega$$
$$u^\star = 0 \quad \text{on } \Gamma = \partial\Omega$$

- monotonicity: A SPD and $b(\cdot)$ smooth with $b'(\xi) \geq 0$
- polynomial growth: $b^{(k)}(\xi) \leq C\,(1 + |\xi|^{n-k})$ for some $n \in \mathbb{N}$ and all $0 \leq k \leq n$

$\Longrightarrow$ full linear convergence for sufficiently small λ

- difficulty: resulting operator $\boldsymbol{A}$ is strongly monotone, but only locally Lipschitz

Becker, Brunner, Innerberger, Melenk, Praetorius: Comput. Math. Appl., 118 (2022)

Becker, Brunner, Innerberger, Praetorius: in progress (2022+)

Thank you for your attention!

- Gregor Gantner, Alexander Haberl, Dirk Praetorius, Stefan Schimanko:
  Rate optimality of adaptive finite element methods with respect to the overall computational costs
  Math. Comp., 90 (2021), 2011–2040
- Alexander Haberl, Dirk Praetorius, Stefan Schimanko, Martin Vohralik:
  Convergence and quasi-optimal cost of adaptive algorithms for nonlinear operators including iterative linearization and algebraic solver
  Numer. Math., 147 (2021), 679–725
- Pascal Heid, Dirk Praetorius, Thomas Wihler:
  Energy contraction and optimal convergence of adaptive iterative linearized finite element methods
  Comput. Meth. Appl. Math., 21 (2021), 407–422

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