Error estimation and adaptivity for stochastic collocation finite elements

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Joint work with

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Overview ... what is the talk about?

* Numerical solution of elliptic PDE problems with parametric or uncertain inputs
* Sparse grid stochastic collocation FEM (multilevel version)
* A posteriori error estimation (hierarchical error estimates)
* Adaptive algorithms for computing multilevel SC-FEM approximations
* 'Proof of concept' numerical experiments:
    - effectivity and robustness of the error estimation strategy
    - convergence rates of adaptive multilevel SC-FEM (optimality?)

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- A. Bespalov, D. Silvester, F. Xu, *Error estimation and adaptivity for stochastic collocation finite elements. Part I: single-level approximation*, Preprint, arXiv:2109.07320 (2021).
- A. Bespalov, D. Silvester, *Error estimation and adaptivity for stochastic collocation finite elements. Part II: multilevel approximation*, Preprint, arXiv:2202.08902 (2022).

Parametric model problem

Problem formulation: find $u : \overline{D} \times \Gamma \to \mathbb{R}$ satisfying

$$
\begin{aligned}
-\nabla_x \cdot (a(x, \mathbf{y}) \nabla_x u(x, \mathbf{y})) &= f(x, \mathbf{y}) \qquad && x \in D, \ \mathbf{y} \in \Gamma, \\
u(x, \mathbf{y}) &= 0 && x \in \partial D, \ \mathbf{y} \in \Gamma
\end{aligned}
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- Domains
  - $D \subset \mathbb{R}^2 \rightsquigarrow$ physical domain
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Remark: parameters $y_1, y_2, \ldots$ can be viewed as images (observations) of independent real-valued random variables with cumulative distribution functions $\pi_1(y_1), \pi_2(y_2), \ldots$. Then, the joint cumulative distribution function is defined as

$$\pi(\mathbf{y}) := \prod_{m=1}^{M} \pi_m(y_m), \quad \text{and} \quad \int_{-1}^{1} d\pi_m(y_m) = \int_{\Gamma} d\pi(\mathbf{y}) = 1.$$

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- Parametric diffusion coefficient $a(x, \mathbf{y})$

$$0 < a_{\min} \le \operatorname*{ess\,inf}_{x \in D} a(x, \mathbf{y}) \le \operatorname*{ess\,sup}_{x \in D} a(x, \mathbf{y}) \le a_{\max} < \infty \quad \pi\text{-a.e. on } \Gamma$$

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- Weak formulation: given $f \in L^2_\pi(\Gamma, L^2(D))$, find $u : \Gamma \to \mathbb{X} := H^1_0(D)$ s.t.

$$\int_D a(x, \mathbf{y}) \nabla u(x, \mathbf{y}) \cdot \nabla v(x)\, \mathrm{d}x = \int_D f(x, \mathbf{y}) v(x)\, \mathrm{d}x \quad \forall v \in \mathbb{X},\ \pi\text{-a.e. on } \Gamma$$

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- Well posed? [Babuška, Nobile, Tempone (2007)] : $\exists!\, u \in \mathbb{V} := L^2_\pi(\Gamma;\mathbb{X})$

Stochastic collocation FEM

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  - a family of nested sets of nodes on $[-1, 1]$ (e.g., Leja, Clenshaw–Curtis)
  - a monotone (or, downward-closed) finite index set $\Lambda_\bullet \subset \mathbb{N}^M$
    ($\boldsymbol{\nu} \in \Lambda_\bullet \Longrightarrow \boldsymbol{\nu} - \boldsymbol{\varepsilon}_m \in \Lambda_\bullet \quad \forall\, m = 1, \ldots, M$ such that $\nu_m > 1$,
    where $\boldsymbol{\varepsilon}_m$ is the m-th unit multi-index); $\mathbf{1} = (1, 1, \ldots, 1) \in \Lambda_\bullet$.
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  - sparse grid $\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$ of collocation points
  - sparse grid collocation operator

$$S_\bullet = S_{\Lambda_\bullet} := \sum_{\boldsymbol{\nu}\in\Lambda_\bullet} \Delta^{\boldsymbol{\kappa}(\boldsymbol{\nu})} = \sum_{\boldsymbol{\nu}\in\Lambda_\bullet} \bigotimes_{m=1}^{M} \Delta_m^{\kappa(\nu_m)} = \sum_{\boldsymbol{\nu}\in\Lambda_\bullet} \bigotimes_{m=1}^{M} \left(I_m^{\kappa(\nu_m)} - I_m^{\kappa(\nu_m-1)}\right)$$

  - interpolation property: $S_{\Lambda_\bullet} v(\mathbf{z}) = v(\mathbf{z}) \;\; \forall\, \mathbf{z} \in \mathcal{Y}_{\Lambda_\bullet}, \;\; \forall\, v \in C^0(\Gamma; \mathbb{X})$

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$$\int_D a(x,\mathbf{z})\nabla u_{\bullet\mathbf{z}}(x)\cdot\nabla v(x)\,\mathrm{d}x = \int_D f(x)v(x)\,\mathrm{d}x \quad \forall v \in \mathbb{X}_{\bullet\mathbf{z}} := \mathcal{S}_0^1(\mathcal{T}_{\bullet\mathbf{z}}) \subset H_0^1(D)$$

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- We build a multivariable interpolant

$$u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y}),$$

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Key features of stochastic collocation FEM (SC-FEM)

- a sampling method that generates 'surrogate models'
- single-level ($\mathbb{X}_{\bullet\mathbf{z}} = \mathbb{X}_\bullet\ \forall\,\mathbf{z} \in \mathcal{Y}_\bullet$) *vs.* multilevel ($\mathbb{X}_{\bullet\mathbf{z}} \neq \mathbb{X}_{\bullet\mathbf{z}'}$ for $\mathbf{z} \neq \mathbf{z}'$)
- not a projection method $\rightsquigarrow$ no (global) Galerkin orthogonality

Multilevel stochastic collocation FEM

- Use a hierarchy of spatial and stochastic approximations to minimise cost
  [Teckentrup, Jantsch, Webster, Gunzburger (2015)]
  [Lang, Scheichl, Silvester (2020)]
  - inspired by the multigrid idea and multilevel Monte Carlo methods
  - makes use of the telescopic identity
  - based on (a priori) approximation properties of FEM spaces and polynomial interpolation

- Employ individually tailored spatial discretizations across collocation points
  [Feischl, Scaglioni (2021)]
  - Spatial and parametric refinements are driven by a posteriori error indicators $\rightsquigarrow$ a posteriori error analysis of SC-FEM approximations

A posteriori error estimation and adaptivity in SC-FEM

[Guignard, Nobile (2018)]

- Residual-based a posteriori error estimation strategy
- Affine parametric coefficient
- Single-level stochastic collocation FEM
- Adaptive sparse grid refinement algorithm

[Eigel, Ernst, Sprungk, Tamellini (2020)]

- Convergence analysis of adaptive sparse grid refinement algorithm
- Single-level stochastic collocation FEM

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- Multilevel construction of stochastic collocation FEM
- Convergence analysis of fully adaptive algorithm

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Our goal: a posteriori error estimation strategy for affine and nonaffine parametric coefficients together with multilevel adaptivity

Hierarchical a posteriori error estimation in FEM: main ideas

- Pythagoras theorem: $u \in \mathbb{X}$, $u_\bullet \in \mathbb{X}_\bullet \subset \mathbb{X}$, $\widehat{u}_\bullet \in \widehat{\mathbb{X}}_\bullet \supset \mathbb{X}_\bullet$ (enhanced approx.)

$$|||\, u - u_\bullet \,|||^2 = |||\, (u - \widehat{u}_\bullet) + (\widehat{u}_\bullet - u_\bullet) \,|||^2 = |||\, u - \widehat{u}_\bullet \,|||^2 + \underbrace{|||\, \widehat{u}_\bullet - u_\bullet \,|||^2}_{\text{computable}}$$

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  $\Longrightarrow$ hierarchical error estimation without computing enhanced approximations $\widehat{u}_\bullet$

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- By-product: reliable and efficient estimates of the error reduction $|||\, \widehat{u}_\bullet - u_\bullet \,|||$
  $\Longrightarrow$ key to adaptivity

Hierarchical error estimators in stochastic Galerkin FEM

- $\mathbb{V} := L^2_\pi(\Gamma; \mathbb{X}) \cong \mathbb{X} \otimes L^2_\pi(\Gamma)$

  $\mathbb{V}_\bullet := \mathbb{X}_\bullet \otimes \mathbb{P}_\bullet$ (single-level) *vs.* $\mathbb{V}_\bullet := \bigoplus_{\nu \in \mathcal{P}_\bullet} \left[\mathbb{X}_{\bullet\nu} \otimes \mathrm{span}\{P_\nu\} \right]$ (multilevel)

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  - Single-level stochastic Galerkin FEM
    [Bespalov, Silvester (2016)]

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  - Multilevel stochastic Galerkin FEM
    [Bespalov, Praetorius, Ruggeri; SIAM/ASA JUQ (2021)]

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Hierarchical a posteriori error estimation in SC-FEM (1/3)

[Bespalov, Silvester, Xu (2021)], [Bespalov, Silvester (2022)]

- $\mathbb{V} := L^2_\pi(\Gamma; H^1_0(D))$, $\|\cdot\| := \|\cdot\|_{\mathbb{V}}$
- An enhanced SC-FEM approximation $\widehat{u}_\bullet^{\mathrm{SC}}$ satisfying the saturation property

$$\|u - \widehat{u}_\bullet^{\mathrm{SC}}\| \le q_{\mathrm{sat}} \|u - u_\bullet^{\mathrm{SC}}\| \quad \text{with } q_{\mathrm{sat}} \in (0,1)$$

- This gives a reliable error estimate

$$\|u - u_\bullet^{\mathrm{SC}}\| \le \left(1 - q_{\mathrm{sat}}\right)^{-1} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\|$$

- How does one choose the enhanced approximation $\widehat{u}_\bullet^{\mathrm{SC}}$?

Hierarchical a posteriori error estimation in SC-FEM (2/3)

Recall that $u_\bullet^{\mathrm{SC}}(x, \mathbf{y}) = \sum_{\mathbf{z} \in \mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

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Single-level SC-FEM

- $$\widehat{u}_\bullet^{\mathrm{SC}}(x,\mathbf{y}) := \underbrace{\sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \widehat{u}_{\bullet\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})}_{\text{spatial enhancement}} + \underbrace{\left(\sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet} \widetilde{u}_{\bullet\mathbf{z}'}(x)\, \widehat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) - u_\bullet^{\mathrm{SC}}(x,\mathbf{y})\right)}_{\text{parametric enhancement}}$$
  - $\widehat{u}_{\bullet\mathbf{z}} \in \widehat{\mathbb{X}}_{\bullet\mathbf{z}} = \widehat{\mathbb{X}}_\bullet$ (uniform mesh-refinement) $\forall\,\mathbf{z}\in\mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$
  - $\widehat{\mathcal{Y}}_\bullet = \mathcal{Y}_{\widehat{\Lambda}_\bullet}$ with $\widehat{\Lambda}_\bullet := \Lambda_\bullet \cup \mathrm{R}(\Lambda_\bullet)$ $\rightsquigarrow$ $\widehat{\Lambda}_\bullet$ is monotone!
  - $\widetilde{u}_{\bullet\mathbf{z}'} \in \mathbb{X}_{\bullet\mathbf{z}'} = \mathbb{X}_\bullet$ $\forall\,\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet$

Hierarchical a posteriori error estimation in SC-FEM (2/3)

Recall that $u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

Single-level SC-FEM

- $$\widehat{u}_\bullet^{\mathrm{SC}}(x,\mathbf{y}) := \underbrace{\sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \widehat{u}_{\bullet\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})}_{\text{spatial enhancement}} + \underbrace{\left(\sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet} \widetilde{u}_{\bullet\mathbf{z}'}(x)\, \widehat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) - u_\bullet^{\mathrm{SC}}(x,\mathbf{y})\right)}_{\text{parametric enhancement}}$$

- A posteriori error estimate

$$\|u - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\|$$

$$\le \frac{1}{1-q_{\mathrm{sat}}} \left(\underbrace{\left\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \right\|}_{\text{spatial estimate}} + \underbrace{\left\| \sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet\setminus\mathcal{Y}_\bullet} (\widetilde{u}_{\bullet\mathbf{z}'} - u_\bullet^{\mathrm{SC}}(\cdot,\mathbf{z}'))\widehat{L}_{\bullet\mathbf{z}'} \right\|}_{\text{parametric estimate}} \right)$$

Hierarchical a posteriori error estimation in SC-FEM (3/3)

Recall that $u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

Multilevel SC-FEM

- $$\widehat{u}_\bullet^{\mathrm{SC}}(x,\mathbf{y}) := \underbrace{\sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \widehat{u}_{\bullet\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y})}_{\text{spatial enhancement}} + \underbrace{\left(\sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet} u_{0\mathbf{z}'}(x)\, \widehat{L}_{\bullet\mathbf{z}'}(\mathbf{y}) - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}}(x)\, L_{\bullet\mathbf{z}}(\mathbf{y}) \right)}_{\text{parametric enhancement}}$$
  - $\widehat{u}_{\bullet\mathbf{z}} \in \widehat{\mathbb{X}}_{\bullet\mathbf{z}}$ (uniform mesh-refinement) $\forall\, \mathbf{z} \in \mathcal{Y}_\bullet = \mathcal{Y}_{\Lambda_\bullet}$
  - $\widehat{\mathcal{Y}}_\bullet = \mathcal{Y}_{\widehat{\Lambda}_\bullet}$ with $\widehat{\Lambda}_\bullet := \Lambda_\bullet \cup \mathrm{R}(\Lambda_\bullet)$ $\rightsquigarrow$ $\widehat{\Lambda}_\bullet$ is monotone!
  - $u_{0\mathbf{z}}, u_{0\mathbf{z}'} \in \mathbb{X}_0 := \mathcal{S}_0^1(\mathcal{T}_0)$ $\forall\, \mathbf{z} \in \mathcal{Y}_\bullet$ and $\forall\, \mathbf{z}' \in \widehat{\mathcal{Y}}_\bullet$

Recall that $u_\bullet^{\mathrm{SC}}(x,\mathbf{y}) = \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{\bullet\mathbf{z}}(x) L_{\bullet\mathbf{z}}(\mathbf{y})$

Multilevel SC-FEM

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- A posteriori error estimate

$$\|u - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\|$$

$$\le \frac{1}{1-q_{\mathrm{sat}}} \left(\underbrace{\left\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \right\|}_{\text{spatial estimate}} + \underbrace{\left\| \sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet\setminus\mathcal{Y}_\bullet} \Big(u_{0\mathbf{z}'} - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \Big) \widehat{L}_{\bullet\mathbf{z}'} \right\|}_{\text{parametric estimate}} \right)$$

Error indicators for adaptive SC-FEM

- A posteriori error estimate

$$\|u - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \eta_\bullet$$

$$\eta_\bullet := \Bigg(\underbrace{\Big\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \Big\|}_{\text{spatial estimate } \mu_\bullet} + \underbrace{\Big\| \sum_{\mathbf{z}'\in\widehat{\mathcal{Y}}_\bullet\setminus\mathcal{Y}_\bullet} \Big(u_{0\mathbf{z}'} - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \Big) \widehat{L}_{\bullet\mathbf{z}'} \Big\|}_{\text{parametric estimate } \tau_\bullet} \Bigg)$$

Error indicators for adaptive SC-FEM

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- Spatial error indicators

$$\mu_\bullet := \Big\| \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} (\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}})\, L_{\bullet\mathbf{z}} \Big\| \le \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \|\widehat{u}_{\bullet\mathbf{z}} - u_{\bullet\mathbf{z}}\|_{\mathbb{X}} \, \|L_{\bullet\mathbf{z}}\|_{L^2_\pi(\Gamma)} \lesssim \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} \mu_{\bullet\mathbf{z}} \, \|L_{\bullet\mathbf{z}}\|_{L^2_\pi(\Gamma)}$$

Error indicators for adaptive SC-FEM

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$$\|u - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \|\widehat{u}_\bullet^{\mathrm{SC}} - u_\bullet^{\mathrm{SC}}\| \le \frac{1}{1-q_{\mathrm{sat}}} \eta_\bullet$$

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- Parametric error indicators

$$\tau_\bullet \le \sum_{\boldsymbol{\nu}\in\mathrm{R}(\Lambda_\bullet)} \underbrace{\sum_{\mathbf{z}'\in\widetilde{\mathcal{Y}}_{\bullet\boldsymbol{\nu}}} \Big\| u_{0\mathbf{z}'} - \sum_{\mathbf{z}\in\mathcal{Y}_\bullet} u_{0\mathbf{z}} L_{\bullet\mathbf{z}}(\mathbf{z}') \Big\|_{\mathbb{X}} \|\widehat{L}_{\bullet\mathbf{z}'}\|_{L^2_\pi(\Gamma)}}_{:= \tau_{\bullet\boldsymbol{\nu}}} = \sum_{\boldsymbol{\nu}\in\mathrm{R}(\Lambda_\bullet)} \tau_{\bullet\boldsymbol{\nu}}$$

Adaptive SC-FEM algorithm

INPUT: $\Lambda_0 = \{\mathbf{1}\}$; $\mathcal{T}_{0\mathbf{z}} := \mathcal{T}_0\ \forall\, \mathbf{z} \in \widehat{\mathcal{Y}}_0 = \mathcal{Y}_{\Lambda_0 \cup \mathrm{R}(\Lambda_0)}$; output counter k; tolerance tol

FOR $\ell = 0, 1, 2, 3, \ldots$ DO:

- SOLVE: compute $u_{\ell\mathbf{z}} \in \mathbb{X}_{\ell\mathbf{z}}$ for all $\mathbf{z} \in \widehat{\mathcal{Y}}_\ell = \mathcal{Y}_{\widehat{\Lambda}_\ell} = \mathcal{Y}_{\Lambda_\ell \cup \mathrm{R}(\Lambda_\ell)}$
- ESTIMATE: compute error indicators
  - spatial indicators $\mu_{\ell\mathbf{z}}$ for all $\mathbf{z} \in \mathcal{Y}_\ell$
  - parametric indicators $\tau_{\ell\boldsymbol{\nu}}$ for all $\boldsymbol{\nu} \in \mathrm{R}(\Lambda_\ell)$
  - If $\ell = jk, j \in \mathbb{N}$, compute the total error estimate η_ℓ and exit if $\eta_\ell <$ tol
- MARK: mark certain edges/elements $\mathcal{M}_{\ell\mathbf{z}}$ ($\mathbf{z} \in \mathcal{Y}_\ell$) and indices $\Upsilon_\ell \subseteq \mathrm{R}(\Lambda_\ell)$
- REFINE: enhance approximations
  - mesh refinement (NVB) $\rightsquigarrow$ $\mathcal{T}_{(\ell+1)\mathbf{z}} :=$ **refine**$(\mathcal{T}_{\ell\mathbf{z}}, \mathcal{M}_{\ell\mathbf{z}})$ for all $\mathbf{z} \in \mathcal{Y}_\ell$
  - parametric enrichment $\rightsquigarrow$ $\Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$, construct meshes $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ for each $\mathbf{z}' \in \bigcup_{\boldsymbol{\nu} \in \Upsilon_\ell} \widetilde{\mathcal{Y}}_{\ell\boldsymbol{\nu}}$ and initialise $\mathcal{T}_{(\ell+1)\mathbf{z}} := \mathcal{T}_0$ for all $\mathbf{z} \in \widehat{\mathcal{Y}}_{\ell+1} \setminus \mathcal{Y}_{\ell+1}$

OUTPUT: SC-FEM approximation $u_{\ell^*}^{\mathrm{SC}}$ and the error estimate η_{ℓ^*} for some $\ell^* = jk$

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- If $\widetilde{\mu}_\ell \geq \widetilde{\tau}_\ell$, then proceed as follows:
  - set $\Upsilon_\ell := \emptyset$
  - for each $\mathbf{z} \in \mathcal{Y}_\ell$, determine $\mathcal{M}_{\ell\mathbf{z}} \subseteq \mathcal{T}_{\ell\mathbf{z}}$ such that

$$\theta_{\mathbb{X}} \sum_{\mathbf{z}\in\mathcal{Y}_\ell} \sum_{T\in\mathcal{T}_{\ell\mathbf{z}}} \mu_{\ell\mathbf{z}}(T)\|L_{\ell\mathbf{z}}\|_{L^2_\pi(\Gamma)} \leq \sum_{\mathbf{z}\in\mathcal{Y}_\ell} \sum_{T\in\mathcal{M}_{\ell\mathbf{z}}} \mu_{\ell\mathbf{z}}(T)\|L_{\ell\mathbf{z}}\|_{L^2_\pi(\Gamma)}$$

    with minimal cumulative cardinality $\sum_{\mathbf{z}\in\mathcal{Y}_\ell} \#\mathcal{M}_{\ell\mathbf{z}}$

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    with minimal cumulative cardinality $\sum_{\mathbf{z}\in\mathcal{Y}_\ell} \#\mathcal{M}_{\ell\mathbf{z}}$

- Otherwise, if $\widetilde{\mu}_\ell < \widetilde{\tau}_\ell$, then proceed as follows:
  - set $\mathcal{M}_{\ell\mathbf{z}} := \emptyset$ for all $\mathbf{z} \in \mathcal{Y}_\ell$
  - determine $\Upsilon_\ell \subseteq \mathrm{R}(\Lambda_\ell)$ of minimal cardinality such that

$$\theta_{\mathcal{Y}} \sum_{\boldsymbol{\nu}\in\mathrm{R}(\Lambda_\ell)} \tau_{\ell\boldsymbol{\nu}} \leq \sum_{\boldsymbol{\nu}\in\Upsilon_\ell} \tau_{\ell\boldsymbol{\nu}}.$$

Parametric enrichment

- Parametric enrichment $\rightsquigarrow \Lambda_{\ell+1} := \Lambda_\ell \cup \Upsilon_\ell$

  Key idea: allocation of meshes $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ for new collocation points $\mathbf{z}' \in \bigcup_{\boldsymbol{\nu}\in\Upsilon_\ell} \widetilde{\mathcal{Y}}_{\ell\boldsymbol{\nu}}$

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- Set $\widetilde{\texttt{tol}} := (\#\mathcal{Y}_\ell)^{-1} \sum_{\mathbf{z}\in\mathcal{Y}_\ell} \mu_{\ell\mathbf{z}} \| L_{(\ell+1)\mathbf{z}} \|_{L^2_\pi(\Gamma)}$

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- For each new collocation point $\mathbf{z}' \in \bigcup_{\boldsymbol{\nu}\in\Upsilon_\ell} \widetilde{\mathcal{Y}}_{\ell\boldsymbol{\nu}}$
  - Initialise the mesh $\mathcal{T}_{(\ell+1)\mathbf{z}'} := \mathcal{T}_0$
  - Iterate the standard adaptive loop

    $$\text{SOLVE} \rightarrow \text{ESTIMATE} \rightarrow \text{MARK} \rightarrow \text{REFINE}$$

    until the resolution of the mesh $\mathcal{T}_{(\ell+1)\mathbf{z}'}$ is such that

    $$\mu_{(\ell+1)\mathbf{z}'} \| L_{(\ell+1)\mathbf{z}'} \|_{L^2_\pi(\Gamma)} < \widetilde{\texttt{tol}}$$

Experiment I: affine parametric coefficient

- $-\nabla \cdot (a \nabla u) = 1$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $D := (-1, 1)^2$

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$$a(x, \mathbf{y}) = 1 + \sum_{m=1}^{M} y_m \left(A m^{-2} \cos(2\pi \beta_1(m) x_1) \cos(2\pi \beta_2(m) x_2) \right),$$

$$A = 0.547, \quad \beta_1(m) + \beta_2(m) =: k_m \in \{1, 1;\ 2, 2, 2;\ 3, 3, 3, 3;\ 4, 4, 4, 4, 4;\ \ldots\}$$

- $\{y_m\}_{m \in \mathbb{N}}$ are images of $U(-1, 1)$ iid mean-zero r.v. $\implies \mathrm{d}\pi_m(y_m) = \frac{1}{2}\,\mathrm{d}y_m$

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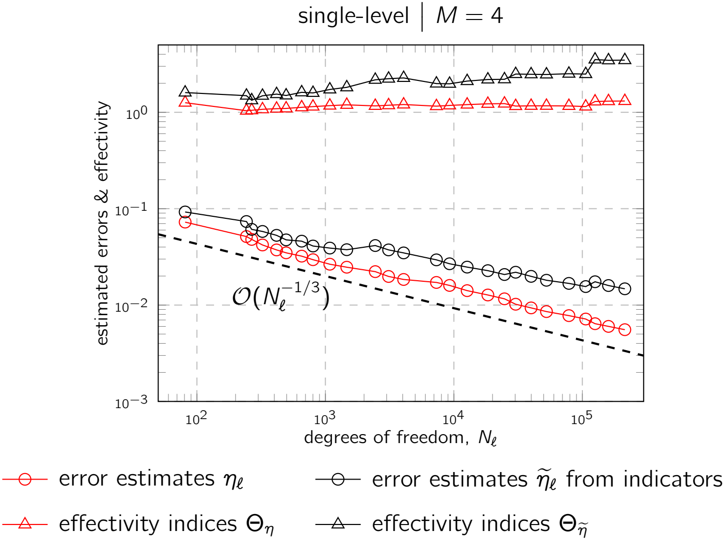
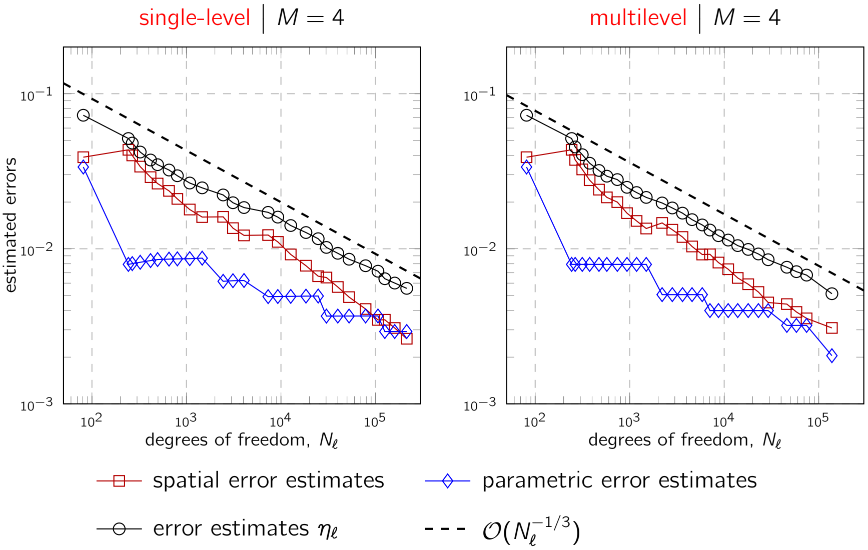
- Adaptive stochastic collocation FEM (Clenshaw–Curtis collocation points)
    - marking parameters $\theta_{\mathbb{X}} = \theta_{\mathcal{Y}} = 0.3$
    - error tolerance $\texttt{tol} = 6\mathrm{e}{-3}$
    - ('expensive') direct error estimates $\eta_\bullet$ vs. ('cheap') error estimates from indicators
    - single-level *vs.* multilevel refinement

Experiment I: effectivity of two error estimates

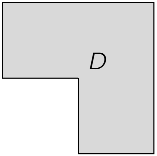


Experiment I: single-level vs. multilevel refinement



Experiment II: nonaffine parametric coefficient

- $-\nabla \cdot (a \nabla u) = 1$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $D := (-1,1)^2 \setminus (-1,0]^2 \rightsquigarrow$ L-shaped domain



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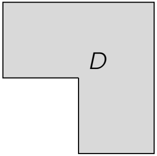

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- Diffusion coefficient $\;a(x,\mathbf{y}) = exp(h(x,\mathbf{y}))$
  - $h(x,\mathbf{y}) = 1 + \sum_{m=1}^{M} \sqrt{\lambda_m}\, \varphi_m(x)\, y_m$
  - $\{(\lambda_m, \varphi_m(x))\}_{m=1}^{\infty}$ are the eigenpairs of $\displaystyle\int_{D \cup (-1,0]^2} \mathrm{Cov}[h](x,x')\,\varphi(x')\,\mathrm{d}x'$
  - $\mathrm{Cov}[h](x,x') = \sigma^2 \exp\left(-|x_1 - x_1'| - |x_2 - x_2'|\right)$
  - $\{y_m\}_{m \in \{1,\dots,M\}}$ are images of $U(-1,1)$ iid mean-zero r.v., $\mathrm{d}\pi_m(y_m) = \frac{1}{2}\,\mathrm{d}y_m$

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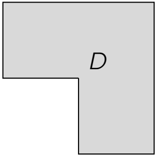

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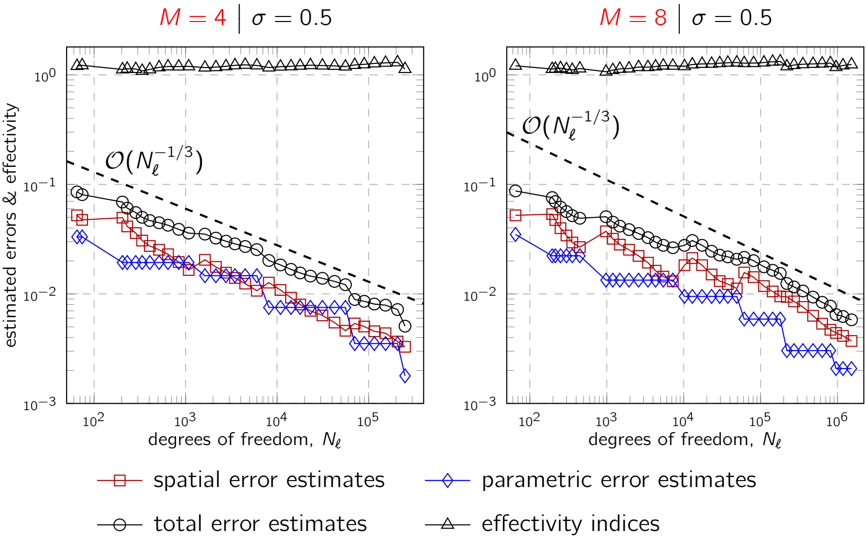
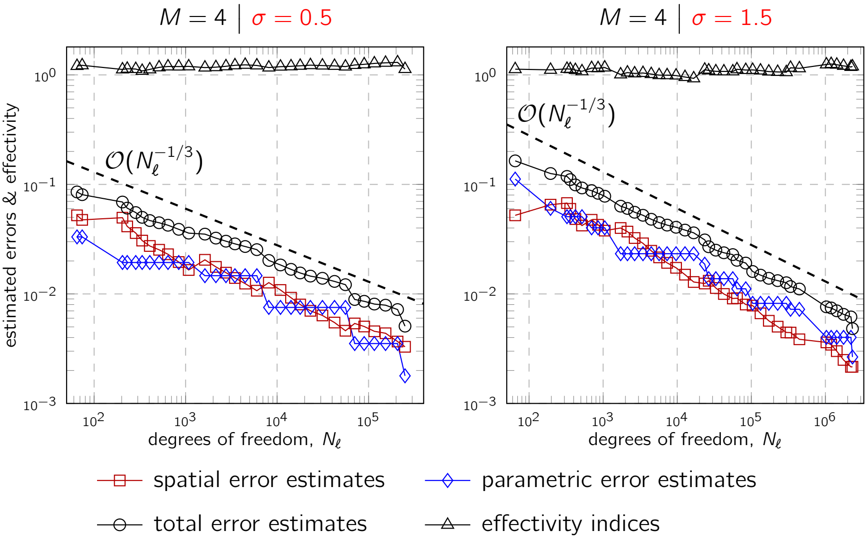
  - effectivity and robustness of the error estimates $\eta_\bullet$
  - single-level *vs.* multilevel refinement

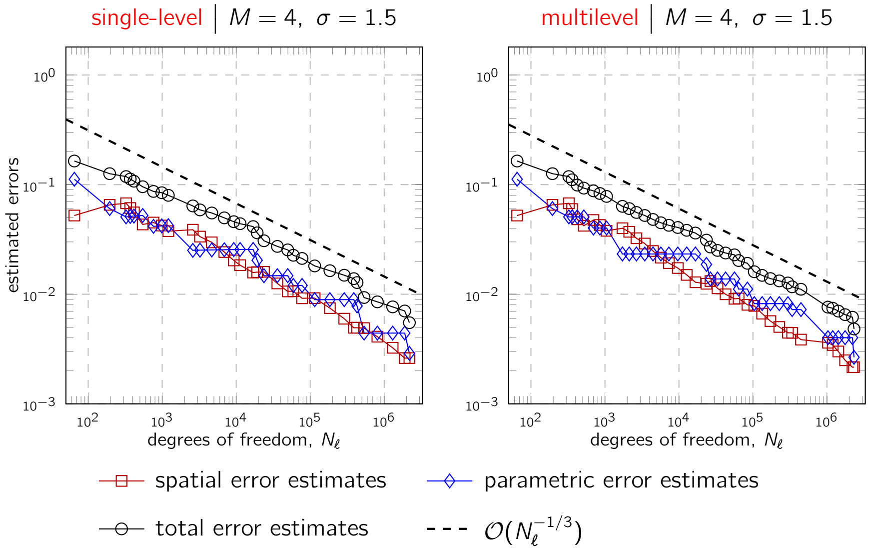
Experiment II: effectivity and robustness of error estimates (1/2)



Experiment II: effectivity and robustness of error estimates (2/2)



Experiment II: single-level vs. multilevel refinement



Experiment III: one peak problem

[Kornhuber, Youett (2018)], [Lang, Scheichl, Silvester (2020)]

- $-\nabla^2 u = f(x, \mathbf{y})$ in $D \times \Gamma$, $\quad u = g$ on $\partial D \times \Gamma$
- $D := (-4, 4)^2$, $\Gamma = [-1, 1]^2$, $\mathbf{y} = (y_1, y_2)$, $y_1, y_2 \sim U[-1, 1]$
- $u(x, \mathbf{y}) = \exp\big(-\frac{50}{16}\{\alpha(y_1)(x_1 - y_1)^2 + (x_2 - y_2)^2\}\big)$ with $\alpha(y_1) = (9y_1 + 11)/2$

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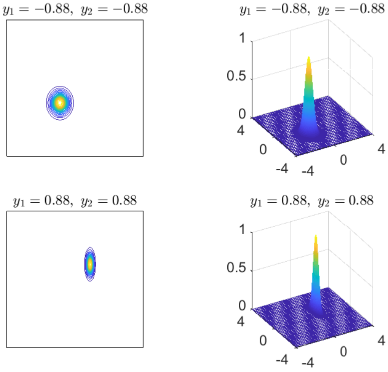

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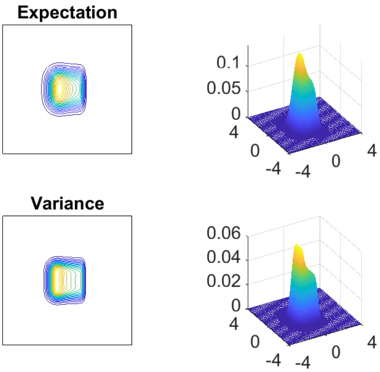

Experiment III: one peak problem

[Kornhuber, Youett '18], [Lang, Scheichl, Silvester '20]

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$$Q := \int_\Gamma \int_D \big(u(x, \mathbf{y})\big)^2 \, \mathrm{d}x \, \mathrm{d}\pi(\mathbf{y}) = 0.24152872\ldots$$

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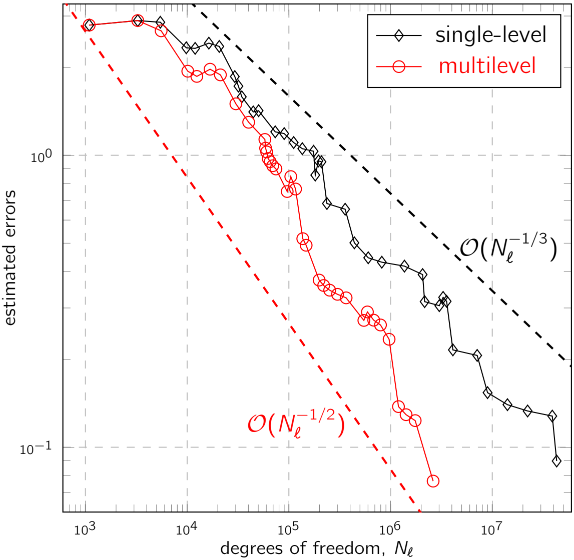
- Adaptive stochastic collocation FEM
  - marking parameters $\theta_{\mathbb{X}} = \theta_{\mathcal{Y}} = 0.3$
  - error tolerance $\texttt{tol} = 1\mathrm{e}{-1}$
  - single-level *vs.* multilevel refinement

Experiment III: single-level vs. multilevel refinement (1/2)



Experiment 3: single-level vs. multilevel refinement (2/2)

	single-level SC-FEM	multilevel SC-FEM
# iterations	38	34
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final #dof	42,961,659	2,620,343
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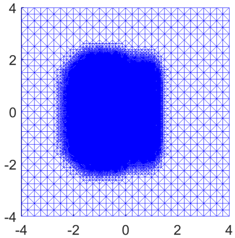
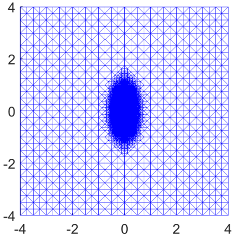
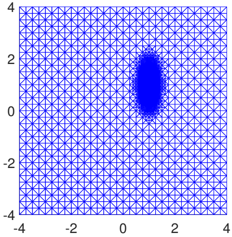


single-level
mesh $\mathcal{T}_\ell$



mesh $\mathcal{T}_{\ell \mathbf{z}}$
$\mathbf{z} = (0, 0)$



mesh $\mathcal{T}_{\ell \mathbf{z}}$
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Concluding remarks

What have we achieved?

- An a posteriori error estimation strategy for stochastic collocation FEM
  - Reliable, effective and robust a posteriori error estimates
  - Applicable to problems with affine and nonaffine parametric inputs
  - Practical error indicators for multilevel adaptivity
- Adaptive multilevel SC-FEM algorithm
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What's next?

- Convergence analysis
- Application to the goal-oriented error estimation and adaptivity . . .

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