Adaptive multilevel stochastic Galerkin FEM for parametric PDE problems

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Workshop
"Interplay of discretization and algebraic solvers: a posteriori error estimates and adaptivity"

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What is this talk about...

* Elliptic PDE problems with random inputs
  (or, more generally, high-dimensional parametric elliptic PDEs)
* Stochastic Galerkin finite element method (SGFEM)
* Multilevel adaptivity for stochastic Galerkin FEM
* Theoretical convergence and rate optimality analysis
* Implementation and numerical results

Parametric model problem (1/2)

Problem formulation: find $u : D \times \Gamma \to \mathbb{R}$ satisfying

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\begin{aligned}
-\nabla_x \cdot (a(x,\mathbf{y})\nabla_x u(x,\mathbf{y})) &= f(x) \qquad && x \in D,\ \mathbf{y} \in \Gamma, \\
u(x,\mathbf{y}) &= 0 && x \in \partial D,\ \mathbf{y} \in \Gamma
\end{aligned}
$$

- Domains
  - $D \subset \mathbb{R}^2 \rightsquigarrow$ physical domain
  - $\Gamma := [-1,1]^{\mathbb{N}} \rightsquigarrow$ parameter domain

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Remark: parameters $y_1, y_2, \ldots$ can be seen as images (observations) of independent real-valued random variables with cumulative distribution functions $\pi_1(y_1), \pi_2(y_2), \ldots$. Then, the joint cumulative distribution function is defined as

$$\pi(\mathbf{y}) := \prod_{m=1}^{\infty} \pi_m(y_m), \quad \text{and} \quad \int_{-1}^{1} d\pi_m(y_m) = \int_{\Gamma} d\pi(\mathbf{y}) = 1.$$

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- Affine-parametric diffusion coefficient
  - $\Gamma := [-1,1]^{\mathbb{N}} \rightsquigarrow$ parameter domain
  - $a(x,\mathbf{y}) = a_0(x) + \sum_{m \in \mathbb{N}} y_m a_m(x)$ for $x \in D$, $\mathbf{y} = (y_m)_{m \in \mathbb{N}} \in \Gamma$

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  - $0 < a_0^{\min} \le a_0(x) \le a_0^{\max} < \infty$ for almost all $x \in D$
  - $\tau := \dfrac{1}{a_0^{\min}} \left\| \sum_{m \in \mathbb{N}} |a_m| \right\|_{L^\infty(D)} < 1$ & $\sum_{m \in \mathbb{N}} \|a_m\|_{L^\infty(D)} < \infty$

Remark: $a_0(x)$ typically represents the mean field, i.e., $a_0(x) = \int_\Gamma a(x, \mathbf{y}) \mathrm{d}\pi(\mathbf{y})$.

Parametric model problem (2/2)

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- Bilinear forms on $\mathbb{V}$
  - $B_0(u,v) := \displaystyle\int_\Gamma \int_D a_0(x) \nabla u(x,\mathbf{y}) \cdot \nabla v(x,\mathbf{y})\, \mathrm{d}x\, \mathrm{d}\pi(\mathbf{y})$
  - $B(u,v) := B_0(u,v) + \displaystyle\sum_{m\in\mathbb{N}} \int_\Gamma y_m \int_D a_m(x) \nabla u(x,\mathbf{y}) \cdot \nabla v(x,\mathbf{y})\, \mathrm{d}x\, \mathrm{d}\pi(\mathbf{y})$

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Weak formulation: given $f \in L^2(D)$, find $u \in \mathbb{V}$ such that

$$
B(u,v) = F(v) := \int_\Gamma \int_D f(x) v(x,\mathbf{y})\,\mathrm{d}x\,\mathrm{d}\pi(\mathbf{y}) \quad \text{for all } v \in \mathbb{V} \qquad (\star)
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[Schwab, Gittelson '11]: the assumptions on $a(x,\mathbf{y})$ ensure the wellposedness of $(\star)$.

Stochastic Galerkin FEM (1/2)

- Finite dimensional subspace

$$\mathbb{V}_\bullet \subset \mathbb{V} \cong \mathbb{X} \otimes \mathbb{P}$$

Galerkin projection:

$$\text{find} \quad u_\bullet \in \mathbb{V}_\bullet \text{ such that } \quad B(u_\bullet, v_\bullet) = F(v_\bullet) \quad \text{for all } v \in \mathbb{V}_\bullet$$

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- Galerkin orthogonality

$$B(u - u_\bullet, v_\bullet) = 0 \quad \text{for all } v_\bullet \in \mathbb{V}_\bullet$$

- Best approximation property

$$|||\, u - u_\bullet \,||| = \min_{v_\bullet \in \mathbb{V}_\bullet} |||\, u - v_\bullet \,|||, \quad \text{where} \quad |||\, \cdot \,||| := B(\cdot, \cdot)^{1/2}$$

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- Main question: how to choose $\mathbb{V}_\bullet$?

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- $\{P_\nu : \nu \in \mathfrak{I}\}$ is a countable orthonormal polynomial basis of $\mathbb{P} = L^2_\pi(\Gamma)$
- $\mathfrak{I} := \{\nu \in \mathbb{N}_0^{\mathbb{N}} : \#\mathrm{supp}(\nu) < \infty\}$ where $\mathrm{supp}(\nu) = \{m \in \mathbb{N} : \nu_m \neq 0\}$

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- Discretisation in the parameter domain
  - finite index set $\mathcal{P}_\bullet \subset \mathcal{I} \implies \mathbb{P}_\bullet = \mathrm{span}\{P_\nu : \nu \in \mathcal{P}_\bullet\} \subset \mathbb{P} = L^2_\pi(\Gamma)$

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- Discretisation in the physical domain (multilevel SGFEM)
  - a hierarchy of meshes $\rightsquigarrow \{\mathcal{T}_{\bullet\nu}\}_{\nu \in \mathcal{P}_\bullet}$
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- $\mathbb{V}_\bullet := \bigoplus_{\nu\in\mathcal{P}_\bullet} \left[\mathbb{X}_{\bullet\nu} \otimes \mathrm{span}\{P_\nu\}\right]$, $\dim \mathbb{V}_\bullet = \sum_{\nu\in\mathcal{P}_\bullet} \dim \mathbb{X}_{\bullet\nu}$

Multilevel methods

- Multilevel methods in UQ
  - Multilevel Monte Carlo
    [Giles '08], [Cliffe, Giles, Scheichl, Teckentrup '11], [Giles '15]
  - Multilevel quasi-Monte Carlo
    [Kuo, Schwab, Sloan '15]
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    - theoretical benchmarks for convergence analysis
      [Cohen, DeVore, Schwab '10, '11], [Gittelson '13]
    - practical realisations
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    - optimal convergence rates for practical realisations $\rightsquigarrow$ open problem...

Hierarchical a posteriori error estimates(1/3)

- Why hierarchical error estimates?
  - provide effective error estimation in the energy norm (eff. indices ≈ 1)
  - provide reliable and efficient estimates of the error reduction;
    error reduction indicators $\rightsquigarrow$ key to adaptivity
  - facilitate convergence analysis of adaptive algorithms

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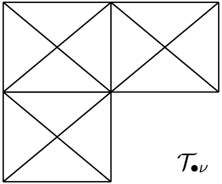
- Main ideas
  - use an enhanced approximation $\widehat{u}_\bullet \in \widehat{V}_\bullet \supset V_\bullet$ to estimate

$$\left|\!\left|\!\left| u - u_\bullet \right|\!\right|\!\right| \approx \underbrace{\left|\!\left|\!\left| \widehat{u}_\bullet - u_\bullet \right|\!\right|\!\right|}_{\text{computable}}$$

  - the enhanced approximation is based on ‘uniform refinement’
  - avoiding the computation of $\widehat{u}_\bullet$ via a decomposition of $\widehat{V}_\bullet$
    in the spirit of [Bank, Weiser ’85]

Hierarchical a posteriori error estimates (2/3)

- Enhancement of approximations in physical domain
  - initial mesh $\mathcal{T}_0$
  - add new vertices to $\mathcal{T}_{\bullet\nu}$ $\rightsquigarrow$ mesh refinement
  - mesh refinement by newest vertex bisection (NVB)
  - $\widehat{\mathcal{T}}_{\bullet\nu}$ $\rightsquigarrow$ uniform refinement of $\mathcal{T}_{\bullet\nu}$



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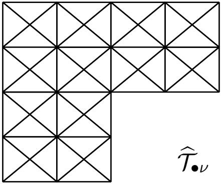

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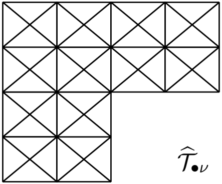

$\widehat{\mathcal{T}}_{\bullet\nu}$

- Enhancement of approximations in the parameter domain
  - add new indices to $\mathcal{P}_\bullet$
  - finite set $\mathcal{Q}_\bullet \subset \mathcal{I} \setminus \mathcal{P}_\bullet$ $\rightsquigarrow$ detail index set ('boundary' of $\mathcal{P}_\bullet$)
  - $\widehat{\mathcal{P}}_\bullet = \mathcal{P}_\bullet \cup \mathcal{Q}_\bullet$ $\rightsquigarrow$ uniform enrichment of $\mathcal{P}_\bullet$
  - $\widehat{\mathbb{P}}_\bullet = \mathrm{span}\{P_\nu : \nu \in \widehat{\mathcal{P}}_\bullet\} \supset \mathbb{P}_\bullet$

Example

- $\mathcal{P}_\bullet = \{(0,0,\dots);(1,0,\dots);(0,1,0,\dots)\}$

  $\Longrightarrow \mathcal{Q}_\bullet = \{(2,0,\dots);(1,1,0,\dots);(0,2,0,\dots);(0,0,1,0,\dots)\}$

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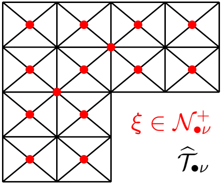

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  - $\widehat{\mathbb{P}}_\bullet = \mathrm{span}\{P_\nu : \nu \in \widehat{\mathcal{P}}_\bullet\} \supset \mathbb{P}_\bullet$

- Local error indicators
  - local spatial indicators $\eta_\bullet(\nu, \xi)$ for each $\nu \in \mathcal{P}_\bullet$ and for each $\xi \in \mathcal{N}_{\bullet\nu}^{+}$
  - individual parametric indicators $\eta_\bullet(\nu)$ for each $\nu \in \mathcal{Q}_\bullet$

Hierarchical a posteriori error estimates (3/3)

- Total error estimate (for multilevel SGFEM)

$$\eta_\bullet^2 = (\text{estim. } \mathbb{X}\text{-error})^2 + (\text{estim. } \mathbb{P}\text{-error})^2 = \sum_{\nu \in \mathcal{P}_\bullet} \sum_{\xi \in \mathcal{N}_{\bullet,\nu}^+} \eta_\bullet^2(\nu, \xi) + \sum_{\nu \in \mathcal{Q}_\bullet} \eta_\bullet^2(\nu)$$

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Theorem 1 [B., Praetorius, Ruggeri; SIAM/ASA JUQ '21]

- $\eta_\bullet \simeq |||\, \widehat{u}_\bullet - u_\bullet \,|||$
- $|||\, u - u_\bullet \,||| \gtrsim \eta_\bullet$ (efficiency)
- saturation assumption $\Longrightarrow$ $|||\, u - u_\bullet \,||| \lesssim \eta_\bullet$ (reliability)

Adaptive SGFEM algorithm

INPUT: initial mesh $\mathcal{T}_0$, initial index set $\mathcal{P}_0 = \{(0, 0, \dots)\}$, tolerance tol

FOR $\ell = 0, 1, 2, 3, \dots$ DO:

- SOLVE: compute $u_\ell \in \mathbb{V}_\ell$ for index set $\mathcal{P}_\ell$ and meshes $\mathcal{T}_{\ell,\nu}$ $(\nu \in \mathcal{P}_\ell)$
- ESTIMATE: compute *local* error indicators and the *total* error estimate
  - spatial & parametric indicators
    $\{\eta_\ell(\nu, \xi);\ \xi \in \mathcal{N}^+_{\ell,\nu},\ \nu \in \mathcal{P}_\ell\}$ & $\{\eta_\ell(\nu);\ \nu \in \mathcal{Q}_\ell\}$
  - energy error estimate η_ℓ
  - IF $\eta_\ell <$ tol THEN STOP
- MARK: mark certain vertices $\mathcal{M}_{\ell,\nu} \subseteq \mathcal{N}^+_{\ell,\nu}$ $(\nu \in \mathcal{P}_\ell)$ and indices $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$
- REFINE: enhance approximation space
  - mesh refinement (NVB) $\rightsquigarrow \mathcal{T}_{\ell+1,\nu} = \textbf{refine}(\mathcal{T}_{\ell,\nu}, \mathcal{M}_{\ell,\nu})\ \ \forall\, \nu \in \mathcal{P}_\ell$
  - parametric enrichment $\rightsquigarrow \mathcal{P}_{\ell+1} = \mathcal{P}_\ell \cup \mathcal{R}_\ell,\ \ \mathcal{T}_{\ell,\nu} = \mathcal{T}_0\ \ \forall\, \nu \in \mathcal{R}_\ell$

OUTPUT: stochastic Galerkin approximations $\{u_\ell\}$ and error estimates $\{\eta_\ell\}$

Marking strategy

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- Separate marking/enrichment of spatial and parametric components
  - two independent marking thresholds $\theta_{\mathbb{X}} \in (0, 1]$ and $\theta_{\mathbb{P}} \in (0, 1]$

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- Combined marking/enrichment of spatial and parametric components
  - use Dörfler marking for

$$\{\eta_\ell(\nu, \xi);\ \nu \in \mathcal{P}_\ell,\ \xi \in \mathcal{N}_{\ell,\nu}^{+}\} \cup \{\eta_\ell(\nu);\ \nu \in \mathcal{Q}_\ell\}$$

    with $\theta \in (0, 1]$ yields $\mathcal{M}_{\ell,\nu} \subseteq \mathcal{N}_{\ell,\nu}^{+}$ $(\nu \in \mathcal{P}_\ell)$ and $\mathcal{R}_\ell \subseteq \mathcal{Q}_\ell$ satisfying

$$\theta\, \eta_\ell^2 \le \sum_{\nu \in \mathcal{P}_\ell} \sum_{\xi \in \mathcal{M}_{\ell,\nu}} \eta_\ell^2(\nu, \xi) + \sum_{\nu \in \mathcal{R}_\ell} \eta_\ell^2(\nu)$$

  - too costly for the single-level SGFEM

Convergence of adaptive algorithm (1/2)

Single-level SGFEM: [B., Praetorius, Rocchi, Ruggeri; SINUM '19]

Multilevel SGFEM: [B., Praetorius, Ruggeri; IMANUM (appeared online)]

Theorem 2 (plain convergence)

For any $\theta_{\mathbb{X}}$, $\theta_{\mathbb{P}} \in (0, 1]$ (resp., $\theta \in (0, 1]$) $\Longrightarrow \lim_{\ell \to \infty} \eta_\ell = 0$.

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Remarks

- Theorem 2 provides a theoretical guarantee that the adaptive algorithm terminates after a finite number of iterations
- No assumptions on the Galerkin approximations generated by the algorithm
- The result extends to more general marking criteria and to $D \subset \mathbb{R}^3$

Proof of Theorem 2

Energy error estimate: $\eta_\ell^2 = \sum_{\nu \in \mathcal{P}_\ell} \sum_{\xi \in \mathcal{N}^+_{\ell,\nu}} \eta_\ell^2(\nu, \xi) + \sum_{\nu \in \mathcal{Q}_\ell} \eta_\ell^2(\nu)$

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Proposition 1 (Extension of the results in [Morin, Siebert, Veeser '08])

The multilevel spatial refinement that follows Dörfler marking strategy *along a subsequence* $(\ell_k)_{k \in \mathcal{N}_0}$ guarantees convergence of spatial error estimates *along this subsequence*, i.e., $\sum_{\nu \in \mathcal{P}_{\ell_k}} \sum_{\xi \in \mathcal{N}_{\ell_k}^+} \eta_{\ell_k}(\nu, \xi)^2 \to 0$ as $k \to \infty$.

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Proposition 2

The parametric enrichment that follows the Dörfler marking strategy *along a subsequence* $(\mathcal{P}_{\ell_k})_{k \in \mathcal{N}_0} \subset (\mathcal{P}_\ell)_{\ell \in \mathcal{N}_0}$ guarantees convergence of the *whole sequence* of parametric error estimates, i.e., $\sum_{\nu \in \mathcal{Q}_\ell} \eta_\ell(\nu)^2 \to 0$ as $\ell \to \infty$.

Convergence of adaptive algorithm (2/2)

Single-level SGFEM: [B., Praetorius, Rocchi, Ruggeri; SINUM '19]

Multilevel SGFEM: [B., Praetorius, Ruggeri; IMANUM (appeared online)]

Theorem 3 (linear convergence)

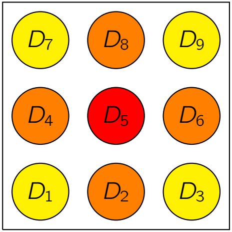
Saturation assumption $\Longrightarrow$ linear converg. for any $\theta_{\mathbb{X}}$, $\theta_{\mathbb{P}} \in (0,1]$ (resp., $\theta \in (0,1]$):

$$\exists\, q_{\mathrm{lin}} \in (0,1) \text{ s.t. } |||\, u - u_{\ell+1} \,||| \le q_{\mathrm{lin}} \,|||\, u - u_\ell \,||| \quad \forall\, \ell \in \mathbb{N}_0$$

Numerical results: cookie problem

[B., Praetorius, Ruggeri; IMANUM (appeared online)]

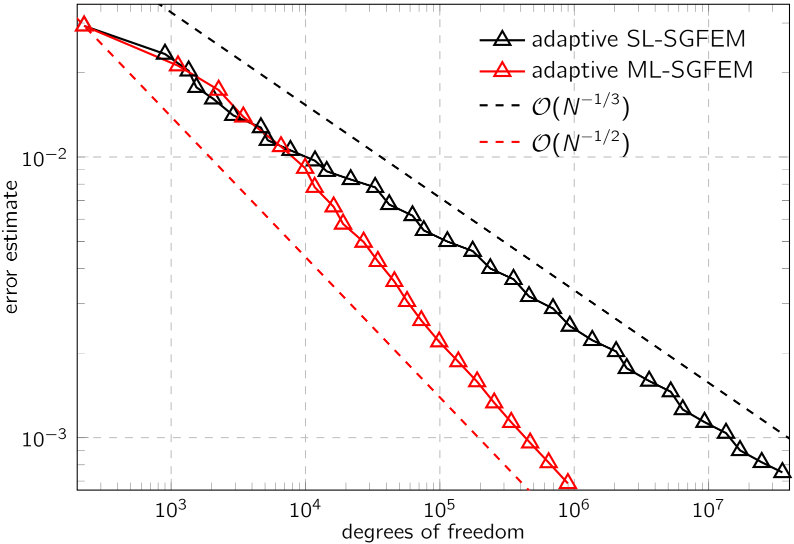
- $-\nabla \cdot (a \nabla u) = f$ in $D \times \Gamma$, $\quad u = 0$ on $\partial D \times \Gamma$
- $a(x, \mathbf{y}) = a_0(x) + \sum_{m \in \mathbb{N}} y_m a_m(x)$
- $D = (0,1)^2 \rightsquigarrow$ square domain
  - nine circular inclusions $D_m \subset D$ $(m = 1, \ldots, 9)$
- Expansion coefficients $\{a_m\}_{m \in \mathbb{N}_0}$
  - $a_0 \equiv 1$
  - $a_m = 0.5\, \chi_{D_m}$ for $m = 1, 3, 7, 9$
  - $a_m = 0.7\, \chi_{D_m}$ for $m = 2, 4, 6, 8$
  - $a_m = 0.9\, \chi_{D_m}$ for $m = 5$
  - $a_m \equiv 0$ for $m > 9$

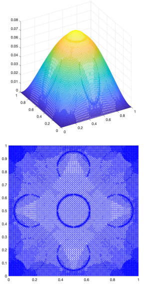


D

- $f \equiv 1$
- $\mathrm{d}\pi_m(y_m) = \frac{1}{2}\, \mathrm{d}y_m \rightsquigarrow$ uniform probability measure on $[-1, 1]$

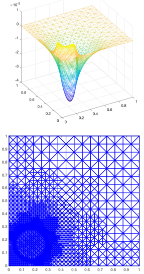
Numerical results: single-level vs. multilevel adaptive SGFEM





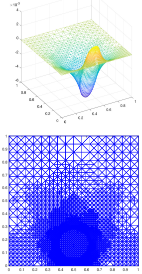
$\nu = \mathbf{0}$

$\#\mathcal{T}_{\ell,\nu} = 84050$



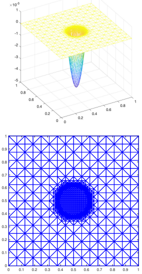
$\nu = (1\ 0)$

$\#\mathcal{T}_{\ell,\nu} = 10994$



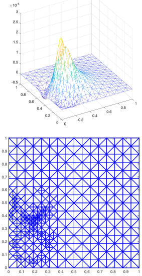
$\nu = (0\ 1)$

$\#\mathcal{T}_{\ell,\nu} = 16420$



$\nu = (0\ 0\ 0\ 0\ 1)$

$\#\mathcal{T}_{\ell,\nu} = 9528$



$\nu = (1\ 0\ 0\ 1)$

$\#\mathcal{T}_{\ell,\nu} = 839$

Optimal convergence of adaptive multilevel SGFEM

[B., Praetorius, Ruggeri; IMANUM (appeared online)]

- Concept of 'multilevel structure' $\rightsquigarrow \mathbb{P}_\bullet = [\mathcal{P}_\bullet, (\mathcal{T}_{\bullet\nu})_{\nu\in\mathcal{P}_\bullet}]$, $\#\mathbb{P}_\bullet \simeq \dim \mathbb{V}_\bullet$

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- Concept of optimality $\rightsquigarrow$ approximation class $\mathbb{A}_s$ $(s > 0)$

$$w \in \mathbb{A}_s \quad \Longleftrightarrow \quad \exists \left\{\mathbb{P}^\star_\ell\right\}_{\ell\in\mathbb{N}_0} \text{ such that } |||\, w - w^\star_\ell \,||| = \mathcal{O}\big((\dim \mathbb{V}^\star_\ell)^{-s}\big)$$

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Theorem 4 (rate optimality)

Strong saturation assumption $\Longrightarrow$ optimal convergence for sufficiently small θ, i.e., if $s > 0$ and $u \in \mathbb{A}_s$, then

$$\sup_{\ell\in\mathbb{N}_0} (\#\mathbb{P}_\ell - \#\mathbb{P}_0 + 1)^s |||\, u - u_\ell \,||| \le C\, \|u\|_{\mathbb{A}_s},$$

where $\|u\|_{\mathbb{A}_s} := \sup\limits_{N\in\mathbb{N}_0} (N+1)^s \min\limits_{\substack{\mathbb{P}_{\rm opt}\in\mathbb{REFINE}(\mathbb{P}_0)\\ \#\mathbb{P}_{\rm opt}-\#\mathbb{P}_0\le N}} \min\limits_{v_{\rm opt}\in\mathbb{V}_{\rm opt}} |||\, u - v_{\rm opt} \,|||$

Saturation assumption vs. strong saturation assumption

- Saturation assumption
  - SGFEM solution $u_\bullet \in \mathbb{V}_\bullet$
  - Enhanced (uniformly refined) SGFEM solution $\widehat{u}_\bullet \in \widehat{\mathbb{V}}_\bullet$
  - There exist a constant $q_{\rm sat} \in (0,1)$ s.t. $|||\, u - \widehat{u}_\bullet \,||| \le q_{\rm sat} |||\, u - u_\bullet \,|||$
  - $\widehat{\mathbb{V}}_\bullet = \widehat{\mathbb{X}}_\bullet \otimes \widehat{\mathbb{P}}_\bullet \Longrightarrow q_{\rm sat} = q_{{\rm sat},\mathbb{X}} \cdot q_{{\rm sat},\mathbb{P}} \in (0,1)$ if $q_{{\rm sat},\mathbb{X}} < 1$ or $q_{{\rm sat},\mathbb{P}} < 1$

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- Strong saturation assumption
  - $\mathbb{P}_\bullet \in \mathbb{REFINE}(\mathbb{P}_0) \rightsquigarrow$ any multilevel structure obtained from $\mathbb{P}_0$
  - $\mathbb{P}_\star \in \mathbb{REFINE}(\mathbb{P}_\bullet) \rightsquigarrow$ a refined multilevel structure obtained from $\mathbb{P}_\bullet$
  - $\mathbb{P}_\circ := \mathbb{REFINE}(\mathbb{P}_\bullet, \mathbb{M}_\bullet) \rightsquigarrow$ a multilevel structure obtained from $\mathbb{P}_\bullet$ by one step of multilevel refinement towards $\mathbb{P}_\star$
  - There exist constants $0 < \kappa_{\rm sat} \le q_{\rm sat} < 1$ such that

$$|||\, u - u_\star \,||| \le \kappa_{\rm sat} \,|||\, u - u_\bullet \,||| \quad \Longrightarrow \quad |||\, u - u_\circ \,||| \le q_{\rm sat} \,|||\, u - u_\bullet \,|||$$

Multilevel SGFEM: implementation challenges

- linear algebra

$$\mathbf{A} = (\mathbf{A}_{\nu\mu})_{\nu,\mu\in\mathcal{P}_\bullet}, \quad \mathbf{A}_{\nu\mu} = \sum_{m=0}^{\infty} [G_m]_{\nu\mu} K_m^{\nu\mu} = \sum_{m=0}^{M} [G_m]_{\nu\mu} K_m^{\nu\mu}$$

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$$[G_m]_{\nu\mu} = \begin{cases} \delta_{\nu\mu} & \text{if } m = 0, \\ \displaystyle\int_\Gamma y_m P_\mu(\mathbf{y}) P_\nu(\mathbf{y})\, d\pi(\mathbf{y}) & \text{if } m \in \mathbb{N}, \end{cases}$$

$$[K_m^{\nu\mu}]_{ij} = \int_D a_m(x) \nabla\varphi_{\bullet\mu,z_j}(x) \cdot \nabla\varphi_{\bullet\nu,z_i}(x)\, dx$$

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- the binary search tree algorithm $\rightsquigarrow$ $\mathcal{O}(\#\mathcal{T}_{\bullet\nu} + \#\mathcal{T}_{\bullet\mu})$ complexity (tbc...)

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- solver: preconditioned MINRES or CG
- preconditioner: mean-based preconditioner with diagonal blocks $K_0^{\nu\nu}$

Concluding remarks

What have we achieved?

- Design of adaptive algorithms for single-level and multilevel SGFEM
- Theoretical analysis
  - convergence of the adaptive algorithm
  - linear convergence (under the saturation assumption)
  - rate-optimality of adaptive multilevel SGFEM (under the strong saturation assumption)
- Implementation: Stochastic T-IFISS
  `http://web.mat.bham.ac.uk/A.Bespalov/software/`

Concluding remarks

What have we achieved?

- Design of adaptive algorithms for single-level and multilevel SGFEM
- Theoretical analysis
  - convergence of the adaptive algorithm
  - linear convergence (under the saturation assumption)
  - rate-optimality of adaptive multilevel SGFEM (under the strong saturation assumption)
- Implementation: Stochastic T-IFISS
  `http://web.mat.bham.ac.uk/A.Bespalov/software/`

References

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