A polynomial-degree-robust equilibrated estimator for the curl–curl problem

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Interplay of discretization and algebraic solvers:
a posteriori error estimates and adaptivity,
9 June 2022

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Outline

The Poisson problem

Poisson problem

Consider a Lipschitz polyhedral domain Ω and $f \in L^2(\Omega)$.

Our first model problem is to find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega$$

for all $v \in H_0^1(\Omega)$.

u_h is the Lagrange FEM approximation of u with degree $p + 1$.

For the sake of simplicity, I will assume that $f = f_h \in \mathcal{P}_p(\mathcal{T}_h)$.

The idea of flux equilibration

Assume that we have a field $\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that

$$\nabla \cdot \boldsymbol{\sigma}_h = f_h \text{ in } \Omega$$

at our disposal.

Then, we have

$$\begin{aligned}
(\nabla(u - u_h), \nabla v)_\Omega &= (f_h, v)_\Omega - (\nabla u_h, \nabla v) \\
&= (\nabla \cdot \boldsymbol{\sigma}_h, v)_\Omega - (\nabla u_h, \nabla v) \\
&= -(\boldsymbol{\sigma}_h + \nabla u_h, \nabla v),
\end{aligned}$$

for all $v \in H_0^1(\Omega)$ and in particular

$$\|\nabla(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \nabla u_h\|_\Omega.$$

Prager-Synge theorem

Equilibrated flux

$$\boldsymbol{\sigma}_h \in \boldsymbol{H}(\mathrm{div}, \Omega); \quad \boldsymbol{\nabla} \cdot \boldsymbol{\sigma}_h = f_h \text{ in } \Omega$$

Error estimate

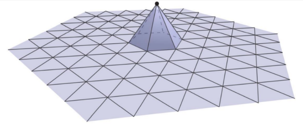
$$\|\boldsymbol{\nabla}(u - u_h)\|_\Omega \leq \|\boldsymbol{\sigma}_h + \boldsymbol{\nabla} u_h\|_\Omega.$$

W. Prager and J.L. Synge, 1947

The particular choice $\boldsymbol{\sigma} := -\boldsymbol{\nabla} u$ saturates the bound.

Local flux constructions

The idea flux would be $\boldsymbol{\sigma} := -\nabla u$.
Let's characterize it locally and cook up a discrete computable version.



Let us set

$$\boldsymbol{\sigma}^{\boldsymbol{a}} := \psi_{\boldsymbol{a}} \boldsymbol{\sigma}$$

where $\psi_{\boldsymbol{a}}$ is the "hat function" associated with the vertex $\boldsymbol{a}$, so that

$$\boldsymbol{\sigma} = \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}^{\boldsymbol{a}}.$$

Characterization of the local contributions

We know that $\boldsymbol{\sigma}^{\boldsymbol{a}} := -\psi_{\boldsymbol{a}} \nabla u$. So that in particular

$$\boldsymbol{\sigma}^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}})$$

where $\omega_{\boldsymbol{a}}$ is the set of tetrahedra $K \in \mathcal{T}_h$ sharing the vertex $\boldsymbol{a}$, and

$$\nabla \cdot \boldsymbol{\sigma}^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} f_h - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u.$$

As a result, we have the characterization

$$\boldsymbol{\sigma}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \psi_{\boldsymbol{a}} f_h - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u}} \|\boldsymbol{v} + \psi_{\boldsymbol{a}} \nabla u\|_{\omega_{\boldsymbol{a}}},$$

since the minimum is zero and achieved when $\boldsymbol{v} = -\psi_{\boldsymbol{a}} \nabla u$.

Discrete construction

Recall that

$$\boldsymbol{\sigma}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v} = \psi_{\boldsymbol{a}} f_h - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u}} \|\boldsymbol{v} + \psi_{\boldsymbol{a}} \nabla u\|_{\omega_{\boldsymbol{a}}}.$$

As a discrete counterpart, we set

$$\boldsymbol{\sigma}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \nabla \cdot \boldsymbol{v}_h = \psi_{\boldsymbol{a}} f_h - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h}} \|\boldsymbol{v}_h + \psi_{\boldsymbol{a}} \nabla u_h\|_{\omega_{\boldsymbol{a}}}.$$

This is indeed well-defined since

$$(\psi_{\boldsymbol{a}} f - \nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h, 1)_{\omega_{\boldsymbol{a}}} = (f, \psi_{\boldsymbol{a}})_\Omega - (\nabla u_h, \nabla \psi_{\boldsymbol{a}})_\Omega = 0$$

whenever $\boldsymbol{a} \notin \partial\Omega$.

The presence of the "correction" $\boldsymbol{\theta}_h^{\boldsymbol{a}} := -\nabla \psi_{\boldsymbol{a}} \cdot \nabla u_h$ is crucial!

Reliability and efficiency

Summation provides an equilibrated flux:

$$\boldsymbol{\sigma}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{\sigma}_h^{\boldsymbol{a}},$$

and reliability follows from the Prager–Synge theorem.

Efficiency

$$\|\boldsymbol{\sigma}_h + \nabla u_h\|_K \lesssim \|\nabla(u - u_h)\|_{\widetilde{K}}$$

The hidden constant does not depend on p.

D. Braess, V. Pillwein and J. Schöberl, 2009

A. Ern and M. Vohralík, 2020

The curl–curl problem

The curl–curl problem

For the sake of simplicity, assume that Ω is simply connected, and consider a divergence-free right-hand side $\boldsymbol{J}_h \in \boldsymbol{RT}_p(\mathcal{T}_h)$.

Our model problem is then to find $\boldsymbol{A} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ such that

$$(\nabla \times \boldsymbol{A}, \nabla \times \boldsymbol{v})_\Omega = (\boldsymbol{J}_h, \boldsymbol{v})_\Omega, \qquad (\boldsymbol{A}, \nabla q)_\Omega = 0,$$

for all $\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ and $q \in H^1_0(\Omega)$.

$\boldsymbol{A}_h \in \boldsymbol{N}_p(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathbf{curl}, \Omega)$ is the Nédélec approximation of $\boldsymbol{A}$.

Prager-Synge theorem

Equilibrated flux

$$\boldsymbol{B}_h \in \boldsymbol{H}(\mathbf{curl}, \Omega); \quad \boldsymbol{\nabla} \times \boldsymbol{B}_h = \boldsymbol{J}_h$$

Error estimate

$$\|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_\Omega \leq \|\boldsymbol{B}_h - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_\Omega$$

The "ideal" flux $\boldsymbol{B} := \boldsymbol{\nabla} \times \boldsymbol{A}$ saturates the bound.

The issue with localization (1/2)

We follow the same steps than for the Poisson problem.

The ideal flux is $\boldsymbol{B} := \boldsymbol{\nabla} \times \boldsymbol{A}$. If we set $\boldsymbol{B}^{\boldsymbol{a}} := \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}$, then

$$\boldsymbol{\nabla} \times \boldsymbol{B}^{\boldsymbol{a}} = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A},$$

so that

$$\boldsymbol{B}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v} = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}}} \|\boldsymbol{v} - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}\|_{\omega_{\boldsymbol{a}}}.$$

The issue with localization (2/2)

Recall that

$$\boldsymbol{B}^{\boldsymbol{a}} = \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v} = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}}} \|\boldsymbol{v} - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}\|_{\omega_{\boldsymbol{a}}}.$$

Unfortunately, we can not set

$$\boldsymbol{B}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_h = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h}} \|\boldsymbol{v}_h - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}}$$

as the minimization set is empty: the field

$$\psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h$$

is not divergence-free!

Possible solutions

An initial idea for lowest-order elements:

- D. Braess and J. Schöberl, 2008

Recently developped extensions:

- J. Gedicke, S. Geevers and I. Perugia, 2019
- J. Gedicke, S. Geevers, I. Perugia and J. Schöberl, 2020
- T. Chaumont-Frelet and M. Vohralík, 2021

Here, I will detail the last construction.

The idea (1/2)

Let $\boldsymbol{\theta}^{\boldsymbol{a}} := \nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}$. There are two important properties:

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}^{\boldsymbol{a}} = \mathbf{0}, \qquad \nabla \cdot \boldsymbol{\theta}^{\boldsymbol{a}} = -\nabla\psi_{\boldsymbol{a}} \cdot \boldsymbol{J}_h.$$

At the discrete level, we have

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} (\nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h) = \mathbf{0}, \qquad \nabla \cdot (\nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h) \neq -\nabla\psi_{\boldsymbol{a}} \cdot \boldsymbol{J}_h.$$

It is tempting to set

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}\in\boldsymbol{RT}_p(\mathcal{T}_h^{\boldsymbol{a}})\cap\boldsymbol{H}_0(\mathrm{div},\omega_{\boldsymbol{a}}) \\ \nabla\cdot\boldsymbol{v}=-\nabla\psi_{\boldsymbol{a}}\cdot\boldsymbol{J}_h}} \|\boldsymbol{v} - \nabla\psi_{\boldsymbol{a}} \times \nabla \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}},$$

but it does not sum up to zero.

The idea (2/2)

If we set

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{RT}_p(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\boldsymbol{\nabla} \psi_{\boldsymbol{a}} \cdot \boldsymbol{J}_h}} \|\boldsymbol{v} - \boldsymbol{\nabla} \psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}},$$

then

$$\widehat{\boldsymbol{\theta}}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \neq \mathbf{0} \qquad \underline{\text{but}} \qquad \boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\theta}}_h = 0.$$

If we had an alternative decomposition

$$\widehat{\boldsymbol{\theta}}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \qquad \underline{\text{with}} \qquad \boldsymbol{\nabla} \cdot \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = 0,$$

then the correction $\boldsymbol{\theta}_h^{\boldsymbol{a}} := \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ would work!

Unfortunately, divergence-free Raviart-Thomas basis are complicated.

Trick #1: Over-constrained minimization

We instead set

$$\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v} \in \boldsymbol{RT}_p(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}) \\ (\boldsymbol{v} - \boldsymbol{\nabla}\psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h, \boldsymbol{r})_{\omega_{\boldsymbol{a}}} = 0 \ \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h^{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v} = -\boldsymbol{\nabla}\psi_{\boldsymbol{a}} \cdot \boldsymbol{J}_h}} \|\boldsymbol{v} - \boldsymbol{\nabla}\psi_{\boldsymbol{a}} \times \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}},$$

where the mean-value constrain works because $\boldsymbol{A}_h$ solves the discrete problem. We then set

$$\widehat{\boldsymbol{\theta}}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}}.$$

Importantly, we have

$$(\widehat{\boldsymbol{\theta}}_h, \boldsymbol{r})_\Omega = 0 \quad \forall \boldsymbol{r} \in \boldsymbol{\mathcal{P}}_0(\mathcal{T}_h).$$

Trick #2: Divergence-free decomposition of $\widehat{\theta}$ (1/2)

The $\widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$ have the correct divergence, but does not sum up to zero.
We need a divergence-free decomposition of $\widehat{\boldsymbol{\theta}}_h$ into contributions $\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}$.

We set for each $\boldsymbol{a} \in \mathcal{V}_h$ and $K \in \mathcal{T}_h^{\boldsymbol{a}}$

$$\widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}}|_K := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{RT}_{p+1}(K) \\ \boldsymbol{\nabla} \cdot \boldsymbol{v}_h = 0 \\ \boldsymbol{v}_h \cdot \boldsymbol{n}_K = \psi_{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h \cdot \boldsymbol{n}_K \text{ on } \partial K}} \|\boldsymbol{v}_h - \psi_{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h\|_K$$

This is indeed well-posed since

$$\begin{aligned} (\psi_{\boldsymbol{a}} \widehat{\boldsymbol{\theta}}_h \cdot \boldsymbol{n}_K, 1)_{\partial K} &= (\psi_{\boldsymbol{a}}, \widehat{\boldsymbol{\theta}}_h \cdot \boldsymbol{n}_K)_{\partial K} \\ &= (\boldsymbol{\nabla} \psi_{\boldsymbol{a}}, \widehat{\boldsymbol{\theta}}_h)_K + (\boldsymbol{\nabla} \cdot \widehat{\boldsymbol{\theta}}_h, \psi_{\boldsymbol{a}})_K \\ &= 0. \end{aligned}$$

Trick #2: Divergence-free decomposition of $\widehat{\theta}$ (2/2)

We can then actually show that

$$\sum_{\boldsymbol{a}\in\mathcal{V}_h} \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} = \widehat{\boldsymbol{\theta}}_h,$$

so that setting

$$\boldsymbol{\theta}_h^{\boldsymbol{a}} := \widehat{\boldsymbol{\theta}}_h^{\boldsymbol{a}} - \widetilde{\boldsymbol{\theta}}_h^{\boldsymbol{a}} \in \boldsymbol{H}_0(\mathrm{div}, \omega_{\boldsymbol{a}}),$$

we have

$$\boldsymbol{\nabla}\cdot\boldsymbol{\theta}_h^{\boldsymbol{a}} = -\boldsymbol{\nabla}\psi_{\boldsymbol{a}}\cdot\boldsymbol{J}_h, \qquad \sum_{\boldsymbol{a}\in\mathcal{V}_h} \boldsymbol{\theta}_h^{\boldsymbol{a}} = \boldsymbol{0}.$$

Local flux contruction and efficiency

Since the $\boldsymbol{\theta}_h^{\boldsymbol{a}}$ have all the required properties, we can now set

$$\boldsymbol{B}_h^{\boldsymbol{a}} := \arg \min_{\substack{\boldsymbol{v}_h \in \boldsymbol{N}_{p+1}(\mathcal{T}_h^{\boldsymbol{a}}) \cap \boldsymbol{H}_0(\mathbf{curl}, \omega_{\boldsymbol{a}}) \\ \boldsymbol{\nabla} \times \boldsymbol{v}_h = \psi_{\boldsymbol{a}} \boldsymbol{J}_h + \boldsymbol{\theta}_h^{\boldsymbol{a}}}} \|\boldsymbol{v}_h - \psi_{\boldsymbol{a}} \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_{\omega_{\boldsymbol{a}}},$$

and we construct an equilibrated flux as

$$\boldsymbol{B}_h := \sum_{\boldsymbol{a} \in \mathcal{V}_h} \boldsymbol{B}_h^{\boldsymbol{a}}.$$

Efficiency

$$\|\boldsymbol{B}_h - \boldsymbol{\nabla} \times \boldsymbol{A}_h\|_K \lesssim \|\boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\widetilde{K}}$$

The hidden constant does not depend on p.

T. Chaumont-Frelet, A. Ern and M. Vohralík, 2021

Concluding remarks

A quick landscape

p-robustness in $\boldsymbol{H}(\mathbf{curl})$

- T. Chaumont-Frelet, A. Ern and M. Vohralík, 2020: single element
- T. Chaumont-Frelet, A. Ern and M. Vohralík, 2021: edge patch
- T. Chaumont-Frelet, M. Vohralík, in preparation: vertex patch

Equilibration strategies for $\boldsymbol{H}(\mathbf{curl})$

- T. Chaumont-Frelet, A. Ern and M. Vohralík, 2021: broken patchwise
- T. Chaumont-Frelet, M. Vohralík, 2021: Prager-Synge
- T. Chaumont-Frelet, 2021: alternative equilibration

Commuting quasi-interpolation under minimal regularity

- T. Chaumont-Frelet, M. Vohralík, in preparation