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A-posteriori-steered and adaptive p-robust multigrid solvers

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joint work with

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Outline

Setting

A-posteriori-steered multigrid

Adaptivity in a-posteriori-steered solvers

Adaptive finite element setting

Conclusion

Geometric multigrid solver with error control for high-order discretization:

- *polynomial degree p-robustness*
  Schöberl, Melenk, Pechstein, and Zaglmayr. *IMA J. Numer. Anal.* 2008
- *number of levels L-robustness*
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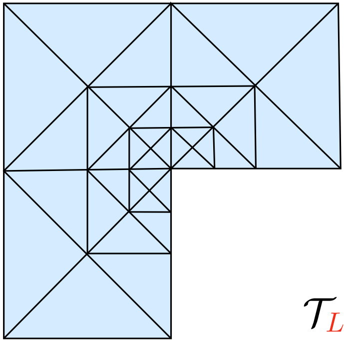
Model problem: Find $u \in H_0^1(\Omega)$ such that $\langle\!\langle u\,,\,v \rangle\!\rangle := (\mathbf{K}\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega)$

Fix $p \ge 1$, let $\mathbb{P}_p(\mathcal{T}_L) := \{v_L \in L^2(\Omega), v_L|_K \in \mathbb{P}_p(K)\ \forall K \in \mathcal{T}_L\}$

Define

$$\mathbb{V}_L^p := \mathbb{P}_p(\mathcal{T}_L) \cap H_0^1(\Omega)$$



$\mathcal{T}_L$

Discrete problem: Find $u_L \in \mathbb{V}_L^p$ such that

$$\langle\!\langle u_L\,,\,v_L \rangle\!\rangle = (f, v_L) \quad \forall v_L \in \mathbb{V}_L^p \qquad \text{(FE)}$$

By introducing a basis of $\mathbb{V}_L^p$: $\mathbb{A}_L \mathrm{U}_L = \mathrm{F}_L$

We work with the *basis-independent* functional formulation (FE)

Algebraic residual functional: $v_L \mapsto (f, v_L) - \langle\!\langle u_L^i\,,\,v_L \rangle\!\rangle \in \mathbb{R}, \quad v_L \in \mathbb{V}_L^p$

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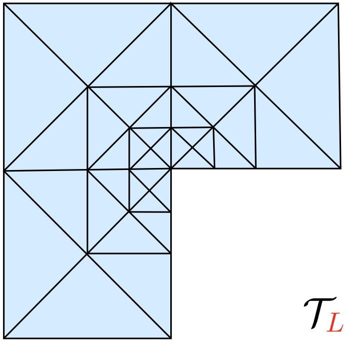

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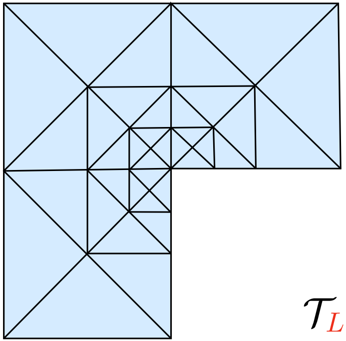

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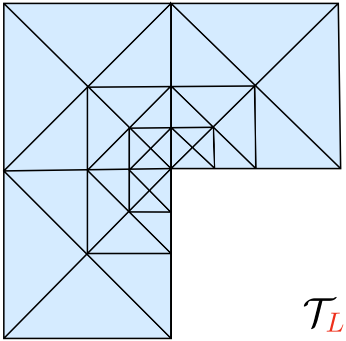

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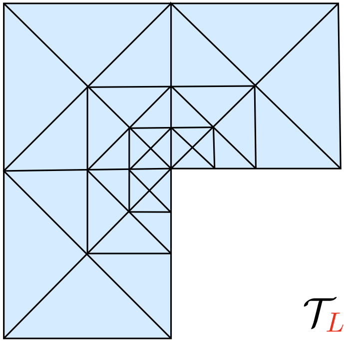

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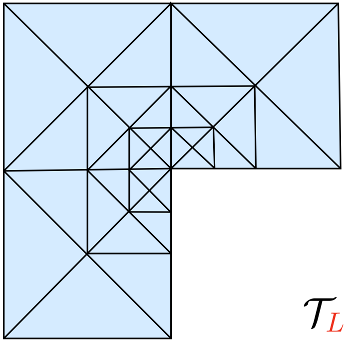

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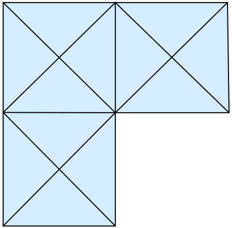
A hierarchy of meshes and spaces

Example: Two different hierarchies with $L = 3$ refinements



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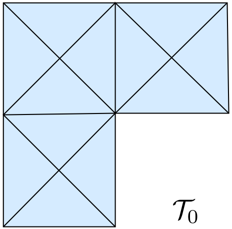
For given p and L, choose *increasing* polynomial degrees



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Economical choice: $p_0 = p_1 = \ldots = p_{L-1} = 1, \quad p_L = p$

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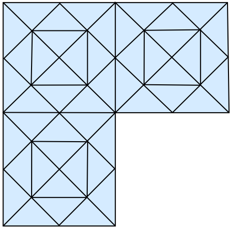

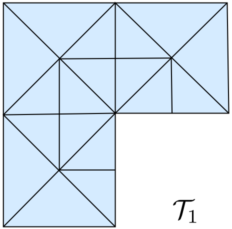

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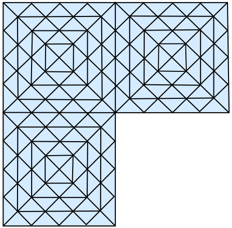

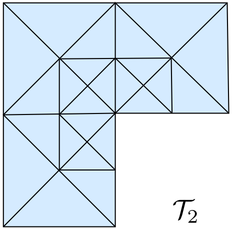

$\mathcal{T}_2$

A hierarchy of meshes and spaces

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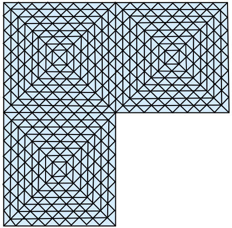

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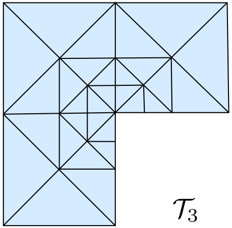

A hierarchy of meshes and spaces

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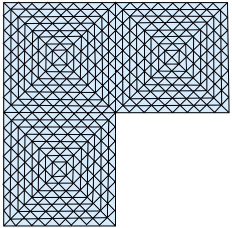

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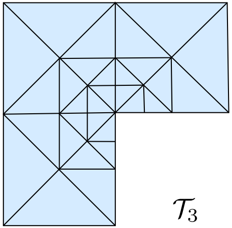

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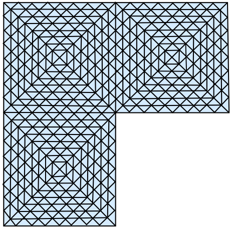

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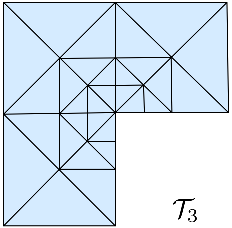

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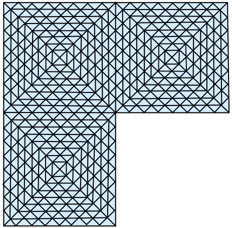

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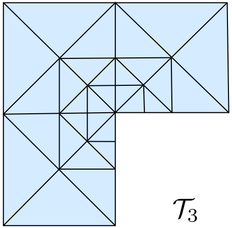

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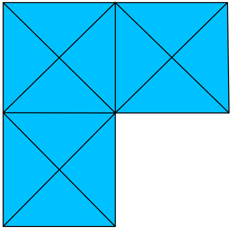

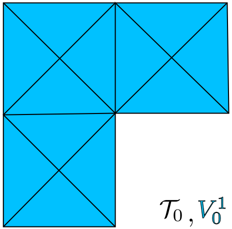

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A hierarchy of meshes and spaces

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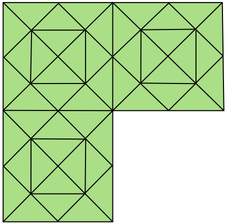

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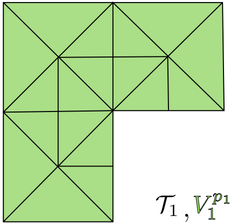

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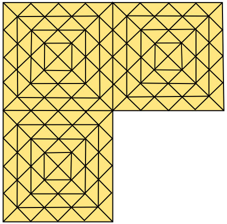

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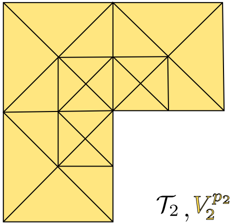

$\mathcal{T}_2\,, V_2^{p_2}$

Define the spaces

$$\mathbb{V}_\ell^{p_\ell} = \mathbb{P}_{p_\ell}(\mathcal{T}_\ell) \cap H_0^1(\Omega)$$

Economical choice: $p_0 = p_1 = \ldots = p_{L-1} = 1, \quad p_L = p$

A hierarchy of meshes and spaces

Example: Two different hierarchies with $L = 3$ refinements

Assumptions: The meshes $\{\mathcal{T}_\ell\}_{1\le\ell\le L}$ can be generated through *uniform* or *adaptive* refinement, satisfying

- $(C_{\mathrm{qu}}$-)quasi-uniform $\mathcal{T}_0$
- $(\kappa_{\mathcal{T}}$-)shape-regularity
- $(C_{\mathrm{ref}}$-)maximum strength of refinement

For given p and L, choose *increasing* polynomial degrees

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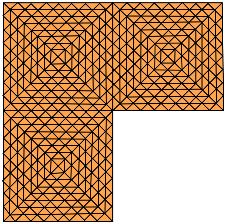

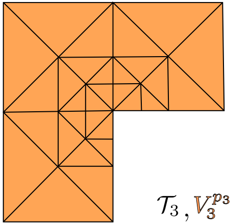

$\mathcal{T}_3\,, V_3^{p_3}$

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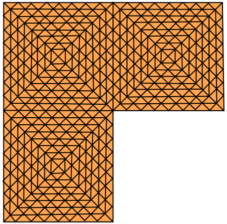

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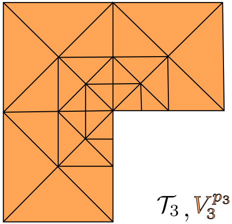


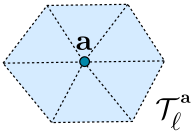
$\mathcal{T}_3, V_3^{p_3}$

Patches

Let $\mathcal{V}_\ell$ be the set of vertices of $\mathcal{T}_\ell$

Given a vertex $\mathbf{a} \in \mathcal{V}_\ell$, we denote

- $\mathcal{T}_\ell^{\mathbf{a}}$ the patch of elements sharing vertex $\mathbf{a}$
- $\omega_\ell^{\mathbf{a}}$ the corresponding patch subdomain
- $\mathbb{V}_\ell^{\mathbf{a}} = \mathbb{P}_{p_\ell}(\mathcal{T}_\ell) \cap H_0^1(\omega_\ell^{\mathbf{a}})$ the associated local space



patch subdomain $\omega_\ell^{\mathbf{a}}$
for a vertex $\mathbf{a} \in \mathcal{V}_\ell$

A-posteriori-steered multigrid

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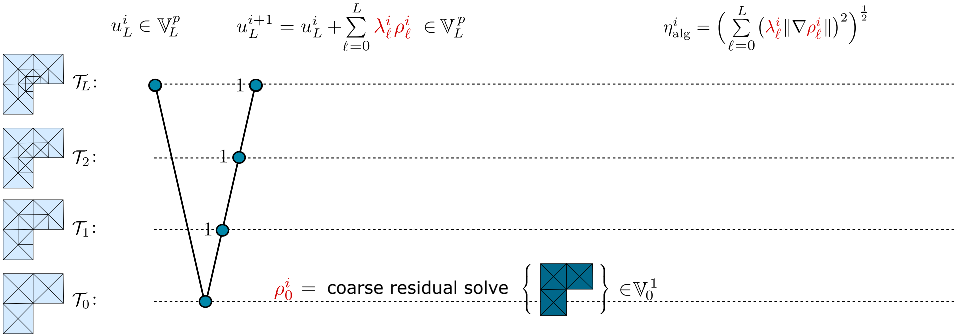

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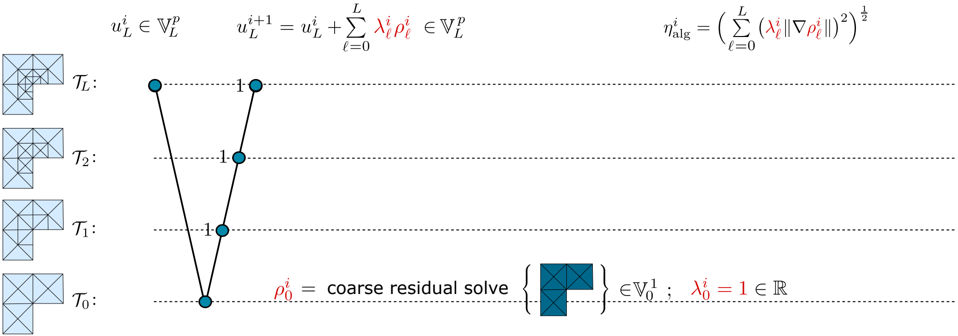

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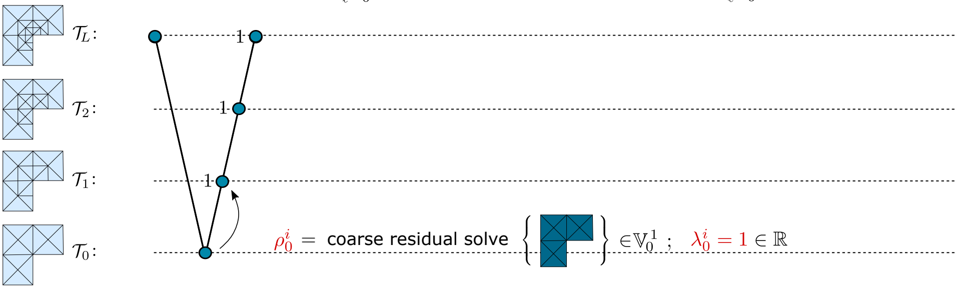

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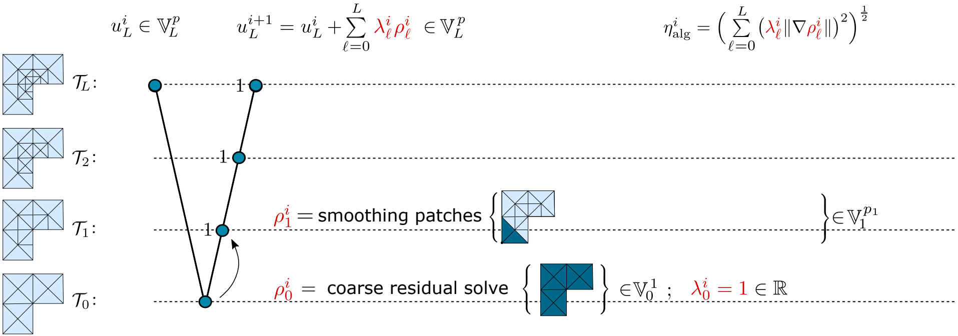

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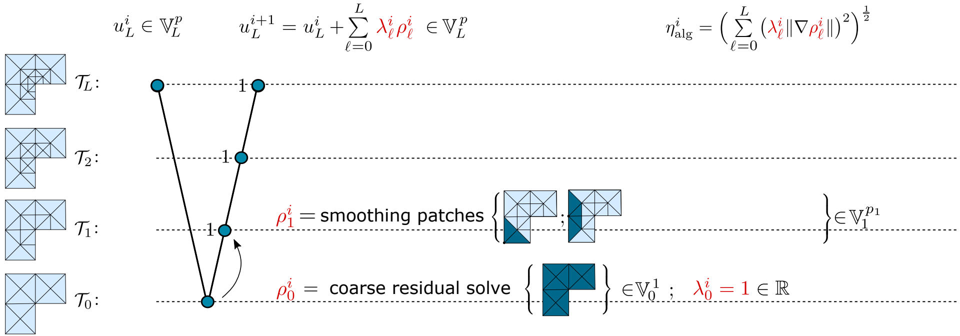

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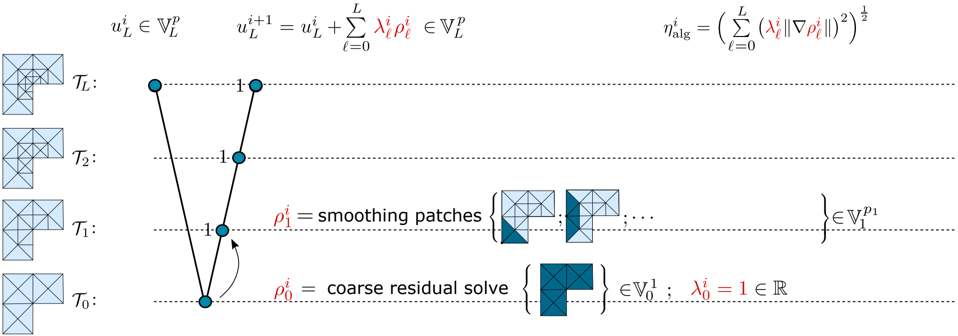
NumPDEs ASC TUWIEN



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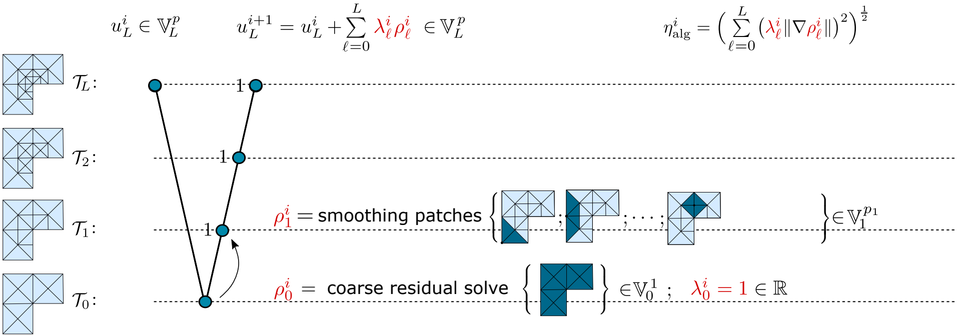

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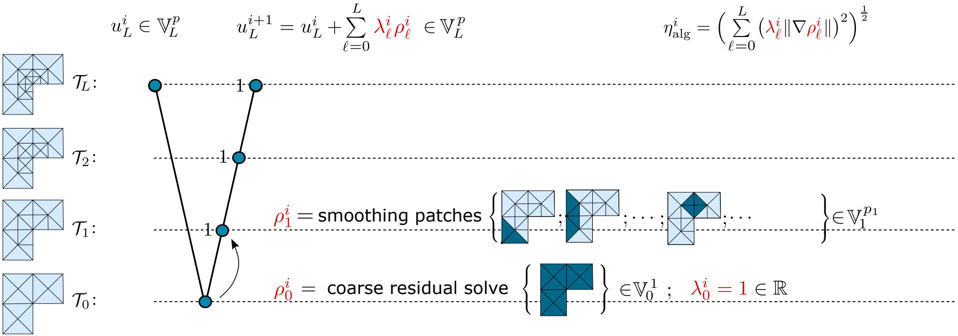

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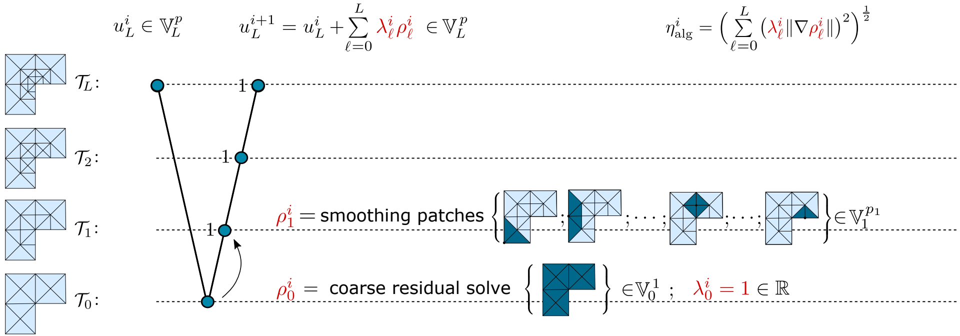

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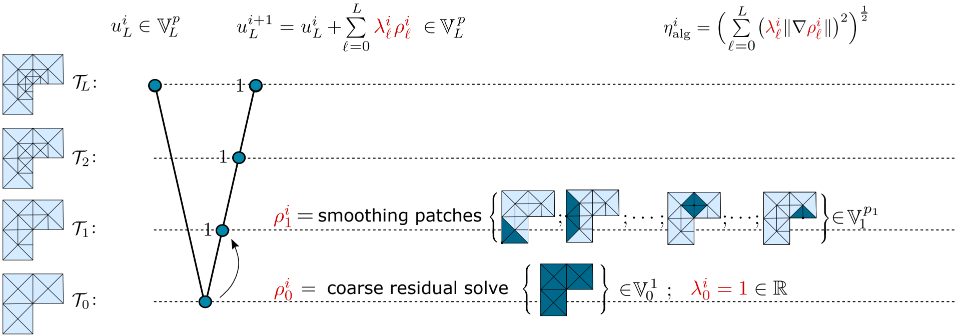

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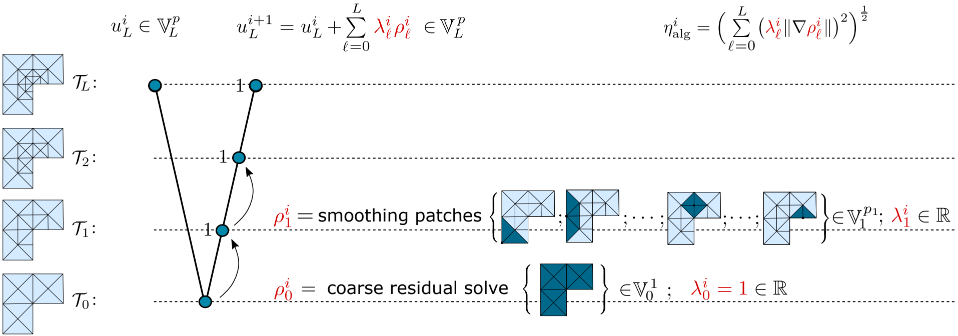

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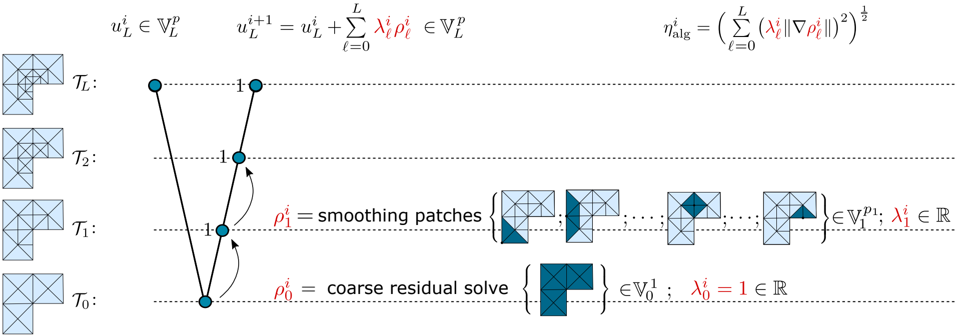

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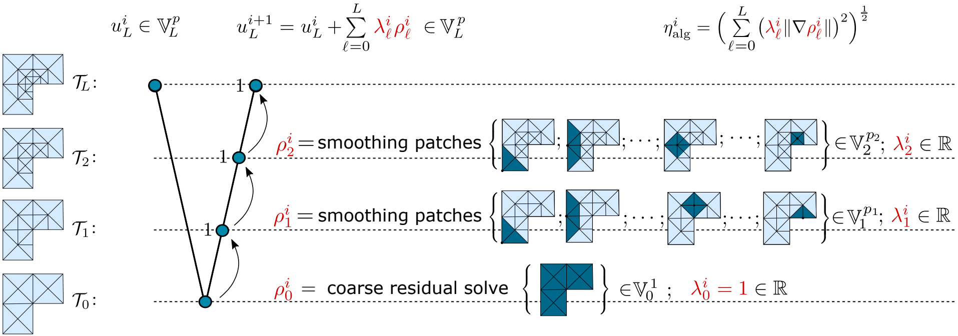

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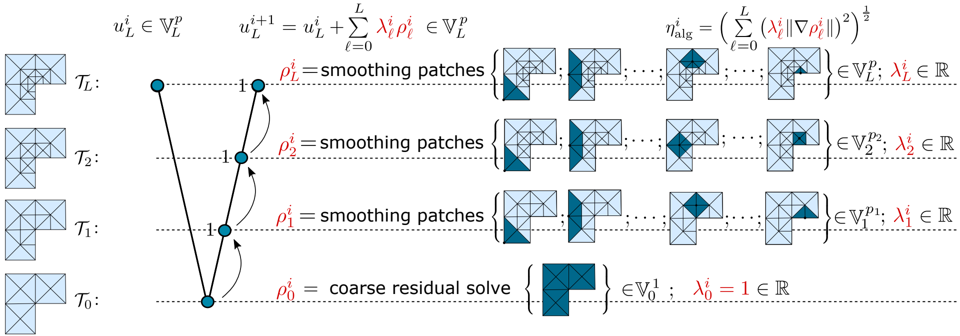

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Multilevel lifting of the algebraic residual

Let $u_L^i \in \mathbb{V}_L^p$ be *arbitrary*.

Coarse solve: Define $\rho_0^i \in \mathbb{V}_0^1$ by: $\underbrace{\langle\!\langle \rho_0^i , v_0 \rangle\!\rangle}_{\text{global lifting}} = \underbrace{(f, v_0) - \langle\!\langle u_L^i , v_0 \rangle\!\rangle}_{\text{global algebraic residual}}, \quad \forall v_0 \in \mathbb{V}_0^1$ and set $\lambda_0^i := 1$

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NumPDEs ASC ▸ TUWIEN

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Proposition (Pythagorean error representation of one solver step)

For $u_L^i \in \mathbb{V}_L^p$, let $u_L^{i+1} \in \mathbb{V}_L^p$ be the next iterate constructed from u_L^i by our solver.

$$\Longrightarrow \qquad \underbrace{\|u_L - u_L^{i+1}\|^2}_{\text{new error}} = \underbrace{\|u_J - u_J^i\|^2}_{\text{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2}_{=\left(\eta_{\mathrm{alg}}^i\right)^2 \text{ computable error decrease}}.$$

Proof: From finest to coarsest level and by the optimal step-sizes $\lambda_\ell^i := \frac{(f,\rho_\ell^i) - \langle\!\langle u_L^i + \sum_{k=0}^{\ell-1} \lambda_k^i \rho_k^i, \rho_\ell^i \rangle\!\rangle}{\|\rho_\ell^i\|^2}$:

$$\begin{aligned}
\|u_L - u_L^{i+1}\|^2 &= \Big\| u_L - \big(u_L^i + \sum_{\ell=0}^{L} \lambda_\ell^i \rho_\ell^i\big) \Big\|^2 \\
&= \Big\| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i \Big\|^2 - 2\lambda_L^i \\
&= \Big\| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i \Big\|^2 - \left(\lambda_L^i \|\rho_L^i\|\right)^2 = \ldots = \|u_L - u_L^i\|^2 - \sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2 \\
&=
\end{aligned}$$

Proposition (Pythagorean error representation of one solver step)

For $u_L^i \in \mathbb{V}_L^p$, let $u_L^{i+1} \in \mathbb{V}_L^p$ be the next iterate constructed from u_L^i by our solver.

$$\Longrightarrow \qquad \underbrace{\|u_L - u_L^{i+1}\|^2}_{\textit{new error}} = \underbrace{\|u_J - u_J^i\|^2}_{\textit{old error}} - \underbrace{\sum_{j=0}^{J} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2}_{=\left(\eta_{\mathrm{alg}}^i\right)^2 \ \textit{computable error decrease}}.$$

Proof: From finest to coarsest level and by the optimal step-sizes $\lambda_\ell^i := \frac{(f,\rho_\ell^i) - \langle\!\langle u_L^i + \sum_{k=0}^{\ell-1} \lambda_k^i \rho_k^i, \rho_\ell^i \rangle\!\rangle}{\|\rho_\ell^i\|^2}$:

$$\begin{aligned}
\|u_L - u_L^{i+1}\|^2 &= \Big\|u_L - \big(u_L^i + \sum_{\ell=0}^{L} \lambda_\ell^i \rho_\ell^i\big)\Big\|^2 \\
&= \Big\|u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i\Big\|^2 - 2\lambda_L^i \\
&= \Big\|u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i\Big\|^2 - \left(\lambda_L^i \|\rho_L^i\|\right)^2 = \ldots = \|u_L - u_L^i\|^2 - \sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2 \\
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$$= \left\| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i \right\|^2 - 2\lambda_L^i \left(\langle\!\langle u_L \, , \rho_L^i \rangle\!\rangle - \left\langle\!\!\left\langle u_L^i + \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i \, , \rho_L^i \right\rangle\!\!\right\rangle \right) + \left(\lambda_L^i \| \rho_L^i \| \right)^2$$

$$= \left\| u_L - u_L^i - \sum_{\ell=0}^{L-1} \lambda_\ell^i \rho_\ell^i \right\|^2 - \left(\lambda_L^i \| \rho_L^i \| \right)^2 = \ldots = \| u_L - u_L^i \|^2 - \sum_{\ell=0}^{L} \left(\lambda_\ell^i \| \rho_\ell^i \| \right)^2$$

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Theorem (p-robust reliable and efficient bound on the algebraic error)

Let $u_L^i \in \mathbb{V}_L^p$ be arbitrary. Let η_{alg}^i be the associated estimator of the algebraic error.

$\Longrightarrow \quad \|u_L - u_L^i\| \geq \eta_{\mathrm{alg}}^i$ *and* $\eta_{\mathrm{alg}}^i \geq \beta \|u_L - u_L^i\|$ *with* $0 < \beta(\kappa_{\mathcal{T}}, L, d, \mathbf{K}) < 1$

Theorem (p-robust error contraction of the multilevel solver)

For $u_L^i \in \mathbb{V}_L^p$, let $u_\ell^{i+1} \in \mathbb{V}_L^p$ be constructed from u_L^i using one step of the solver.

$\Longrightarrow \quad \|u_L - u_\ell^{i+1}\| \leq \alpha \|u_L - u_L^i\|$ *with* $\alpha = \sqrt{1-\beta^2}$

Remark:

- β is independent of the polynomial degree p
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Main results

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p-robustness: iteration numbers for uniform meshes

Stopping criterion:
$$\frac{\|\mathrm{F}_L - A_L \mathrm{U}_L^{i_s}\|}{\|\mathrm{F}_L\|} \leq 10^{-5} \frac{\|\mathrm{F}_L - A_L \mathrm{U}_L^{0}\|}{\|\mathrm{F}_L\|}.$$

The mesh hierarchies here are obtained from L *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$

			H^2-regular			H^1-regular			
			Sine	Peak	L-shape	Checkerboard		Skyscraper	
			$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathcal{J}(\mathbf{K})=O(10^6)$	$\mathcal{J}(\mathbf{K})=O(1)$	$\mathcal{J}(\mathbf{K})=O(10^7)$
			$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$
L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	21	18	18	19	19
	3	$1e^5$	29	28	29	27	28	31	31
	6	$6e^5$	30	30	26	24	25	28	28
	9	$1e^6$	31	30	23	23	23	26	26
4	1	$6e^4$	21	20	21	19	19	19	19
	3	$6e^5$	29	29	28	26	27	30	30
	6	$2e^6$	31	30	25	24	24	27	27
	9	$5e^6$	32	31	23	22	23	25	25

Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases

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L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s
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	9	$1e^6$	31	30	23	23	23	26	26
4	1	$6e^4$	21	20	21	19	19	19	19
	3	$6e^5$	29	29	28	26	27	30	30
	6	$2e^6$	31	30	25	24	24	27	27
	9	$5e^6$	32	31	23	22	23	25	25

Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases

p-robustness: iteration numbers for uniform meshes

Stopping criterion:
$$\frac{\|\mathrm{F}_L - A_L \mathrm{U}_L^{i_s}\|}{\|\mathrm{F}_L\|} \leq 10^{-5} \frac{\|\mathrm{F}_L - A_L \mathrm{U}_L^{0}\|}{\|\mathrm{F}_L\|}.$$

The mesh hierarchies here are obtained from L *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$

			H^2-regular			H^1-regular			
			Sine	Peak	L-shape	Checkerboard		Skyscraper	
			$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathcal{J}(\mathbf{K})=O(10^6)$	$\mathcal{J}(\mathbf{K})=O(1)$	$\mathcal{J}(\mathbf{K})=O(10^7)$
			$1\to 1,p$	$1\to 1,p$	$1\to 1,p$	$1\to 1,p$	$1\to 1,p$	$1\to 1,p$	$1\to 1,p$
L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	21	18	18	19	19
	3	$1e^5$	29	28	29	27	28	31	31
	6	$6e^5$	30	30	26	24	25	28	28
	9	$1e^6$	31	30	23	23	23	26	26
4	1	$6e^4$	21	20	21	19	19	19	19
	3	$6e^5$	29	29	28	26	27	30	30
	6	$2e^6$	31	30	25	24	24	27	27
	9	$5e^6$	32	31	23	22	23	25	25

Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases

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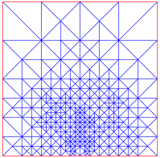
			$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathbf{K}=I$	$\mathcal{J}(\mathbf{K})=O(10^6)$	$\mathcal{J}(\mathbf{K})=O(1)$	$\mathcal{J}(\mathbf{K})=O(10^7)$
			$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$	$1\to 1, p$
L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	21	18	18	19	19
	3	$1e^5$	29	28	29	27	28	31	31
	6	$6e^5$	30	30	26	24	25	28	28
	9	$1e^6$	31	30	23	23	23	26	26
4	1	$6e^4$	21	20	21	19	19	19	19
	3	$6e^5$	29	29	28	26	27	30	30
	6	$2e^6$	31	30	25	24	24	27	27
	9	$5e^6$	32	31	23	22	23	25	25

Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases

p-robustness: iteration numbers for graded meshes

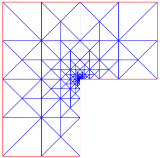
Peak, $1, p \to p$

L	p	i_{s}	L	p	i_{s}	L	p	i_{s}
4	1	14	8	1	16	16	1	16
	3	11		3	9		3	9
	6	9		6	8		6	8
	9	8		9	8		9	9



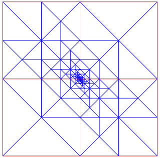
L-shape, $\mathbf{K} = I$, $1, p \to p$

L	p	i_{s}	L	p	i_{s}	L	p	i_{s}
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4



Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

L	p	i_{s}	L	p	i_{s}	L	p	i_{s}
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



Low-regularity tests: indicate *linear L-dependence* in accordance with the theory

p-robustness: iteration numbers for graded meshes

Peak, $1, p \to p$

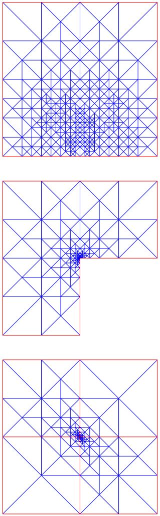
L	p	i_s	L	p	i_s	L	p	i_s
4	1	14	8	1	16	16	1	16
	3	11		3	9		3	9
	6	9		6	8		6	8
	9	8		9	8		9	9

L-shape, $\mathbf{K} = I$, $1, p \to p$

L	p	i_s	L	p	i_s	L	p	i_s
5	1	16	10	1	15	15	1	17
	3	7		3	6		3	11
	6	6		6	5		6	5
	9	5		9	5		9	4

Checkerboard, $\mathcal{J}(\mathbf{K}) = O(10^6)$, $1, p \to p$

L	p	i_s	L	p	i_s	L	p	i_s
5	1	33	10	1	57	15	1	97
	3	15		3	23		3	32
	6	12		6	15		6	20
	9	11		9	12		9	15



Low-regularity tests: indicate *linear L-dependence* in accordance with the theory

Comparison with other multilevel solvers

We compare our methods with solvers from literature in terms of the number of iterations (and CPU times).

Antonietti et al. *J. Sci. Comput.* 2017
Botti et al. *J. Comput. Phys.* 2017
Schöberl. *Tech. report.* 2014

Comparison with other multilevel solvers

We compare our methods with solvers from literature in terms of the number of iterations (and CPU times).

		∼MG(0,1) -bJ $1, p \to p$		∼MG(0,adapt) -bJ (wRAS) $1 \nearrow p$		PCG(MG (3,3)-bJ) $p \to p$		MG(1,1)- PCG(iChol) $1 \nearrow p$		MG(0,1)- bGS $1 \to 1, p$		MG(3,3)- GS $1 \nearrow p$	
L	p	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time	i_s	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m

Antonietti et al. *J. Sci. Comput.* 2017

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		∼MG(0,1) -bJ $1,p\to p$		∼MG(0,adapt) -bJ (wRAS) $1\nearrow p$		PCG(MG (3,3)-bJ) $p\to p$		MG(1,1)-PCG(iChol) $1\nearrow p$		MG(0,1)-bGS $1\to 1,p$		MG(3,3)-GS $1\nearrow p$	
L	p	i_{s}	time	i_{s}	time	i_{s}	time	i_{s}	time	i_{s}	time	i_{s}	time
4	1	19	0.12 s	9	0.11 s	11	0.20 s	16	0.74 s	11	0.06 s	4	0.05 s
	3	11	2.07 s	7	1.62 s	3	2.34 s	44	27.48 s	10	9.64 s	5	1.37 s
	6	9	20.19 s	4	12.54 s	3	38.40 s	>80	>6.87m	9	34.78 s	6	14.44 s
	9	9	2.13m	3	49.84 s	2	2.24m	>80	>23.08m	8	1.72m	9	1.21m
								not p-robust				not p-robust	

Antonietti et al. *J. Sci. Comput.* 2017

Botti et al. *J. Comput. Phys.* 2017

Schöberl. *Tech. report.* 2014

Adaptivity in a-posteriori-steered solvers

Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|u_L - u_L^i\|^2 \approx \left(\eta_{\mathrm{alg}}^i\right)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2}_{\text{① localization by levels}} \qquad \underbrace{\|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\rho_{\ell,\mathbf{a}}\|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

① Adaptive number of post-smoothing steps

② Adaptive local smoothing

Adaptivity in a-posteriori-steered solvers

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$$\|\!|u_L - u_L^i|\!\|^2 \approx \left(\eta_{\mathrm{alg}}^i\right)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\!|\rho_\ell^i|\!\|\right)^2}_{\text{① localization by levels}} = \underbrace{\|\!|\rho_0^i|\!\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\!|\rho_{\ell,\mathbf{a}}|\!\|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

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Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

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Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|\nabla(u_L - u_L^i)\|^2 \approx (\eta_{\mathrm{alg}}^i)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\nabla\rho_\ell^i\|\right)^2}_{\text{① localization by levels}} = \underbrace{\|\nabla\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\nabla\rho_{\ell,\mathbf{a}}\|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

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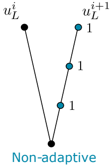

② Adaptive local smoothing

Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

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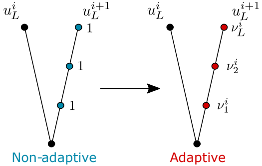

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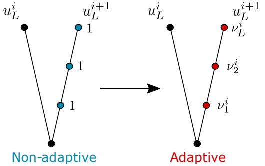

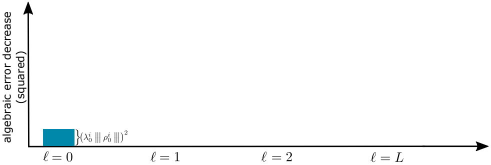

② Adaptive local smoothing

Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|\!|u_L - u_L^i\|\!|^2 \approx \left(\eta_{\mathrm{alg}}^i\right)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\!|\rho_\ell^i\|\!|\right)^2}_{\text{① localization by levels}} = \underbrace{\|\!|\rho_0^i\|\!|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\!|\rho_{\ell,\mathbf{a}}\|\!|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

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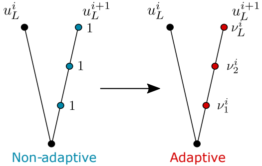

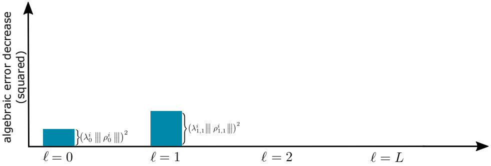

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Adaptivity in a-posteriori-steered solvers

NumPDEs ASC TUWIEN

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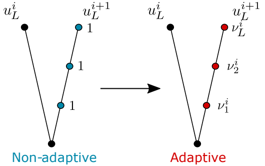

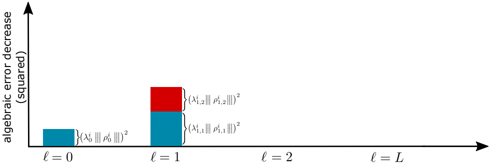

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$$\|u_L - u_L^i\|^2 \approx \left(\eta_{\mathrm{alg}}^i\right)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2}_{\text{① localization by levels}} = \underbrace{\|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\rho_{\ell,\mathbf{a}}\|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

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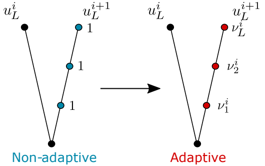

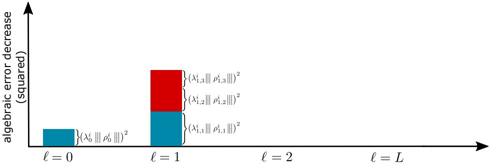

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Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

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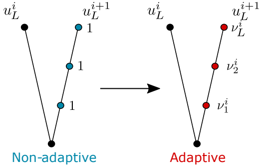
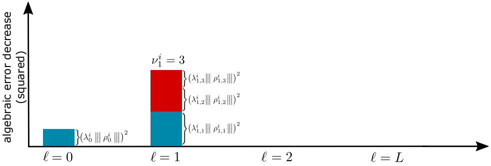
① Adaptive number of post-smoothing steps





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$$\|u_L - u_L^i\|^2 \approx \left(\eta_{\mathrm{alg}}^i\right)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2}_{\text{① localization by levels}} = \underbrace{\|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\rho_{\ell,\mathbf{a}}\|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

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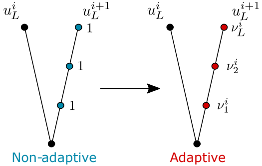
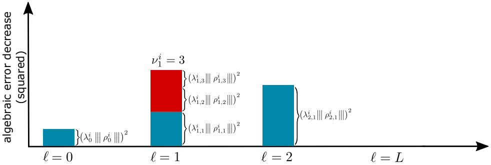




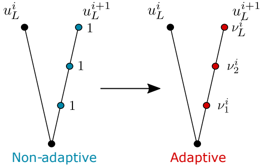
② Adaptive local smoothing

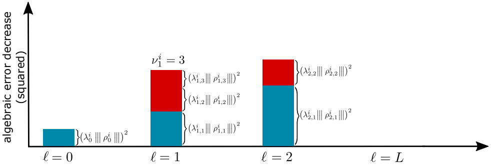
Adaptivity in a-posteriori-steered solvers

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$|||u_L - u_L^i|||^2 \approx (\eta_{\mathrm{alg}}^i)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i |||\rho_\ell^i|||\right)^2}_{\text{① localization by levels}} = \underbrace{|||\rho_0^i|||^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} |||\rho_{\ell,\mathbf{a}}|||^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

① Adaptive number of post-smoothing steps





② Adaptive local smoothing

Starting point: **equivalence** of the algebraic error with a **localized** a posteriori estimate

$$\|u_L - u_L^i\|^2 \approx (\eta_{\mathrm{alg}}^i)^2 = \underbrace{\sum_{\ell=0}^{L} \left(\lambda_\ell^i \|\rho_\ell^i\|\right)^2}_{\text{① localization by levels}} = \underbrace{\|\rho_0^i\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\rho_{\ell,\mathbf{a}}\|^2_{\omega_\ell^{\mathbf{a}}}}_{\text{② localization by patches}}$$

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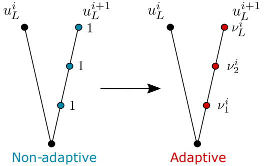

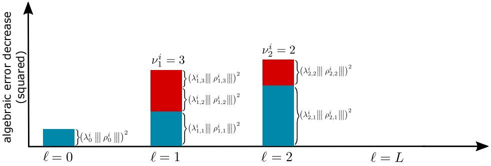

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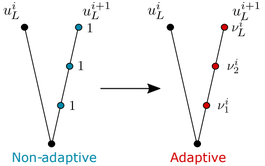

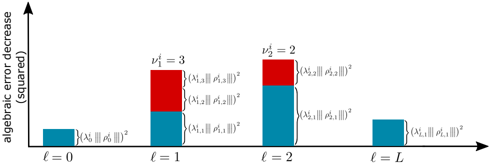

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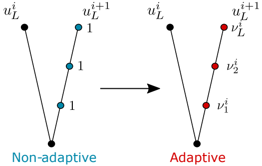

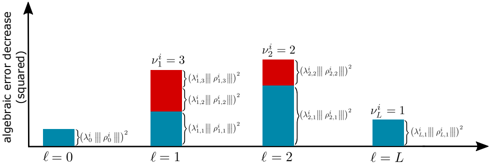

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Adaptivity in a-posteriori-steered solvers

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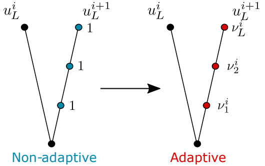

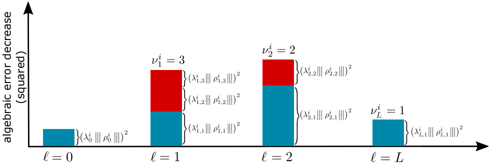

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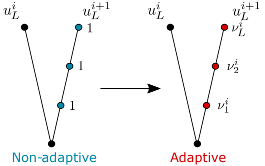

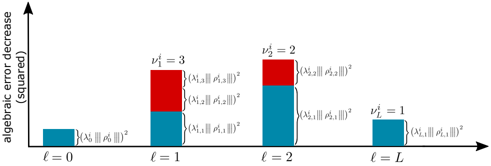

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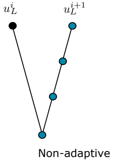

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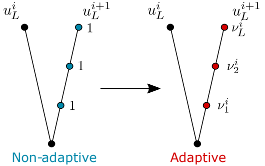

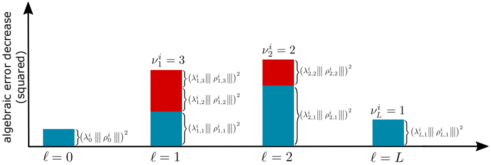

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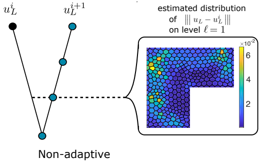

Adaptivity in a-posteriori-steered solvers

NumPDEs
ASC ▸ TUWIEN

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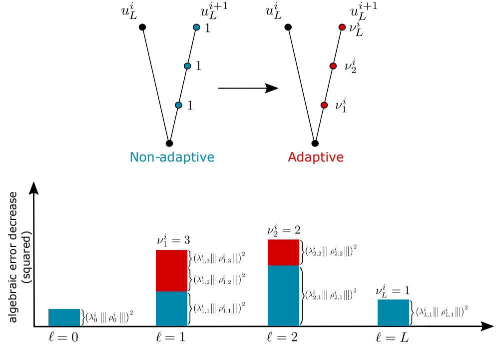

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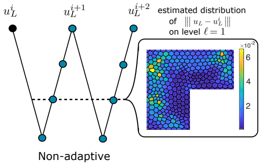

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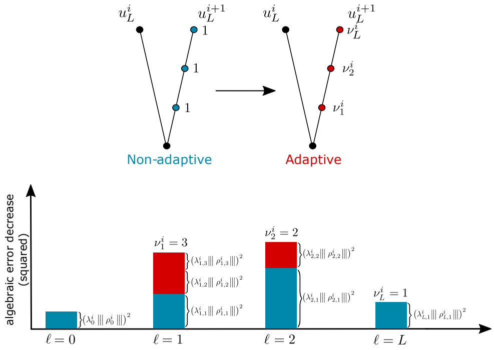

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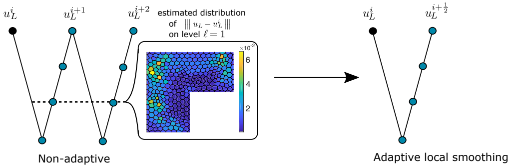

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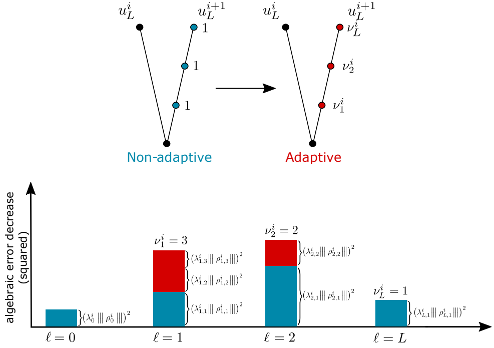

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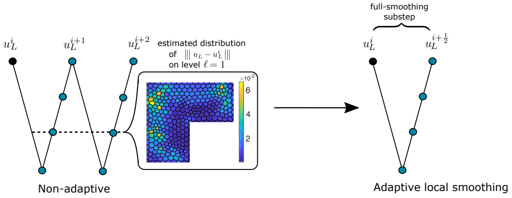

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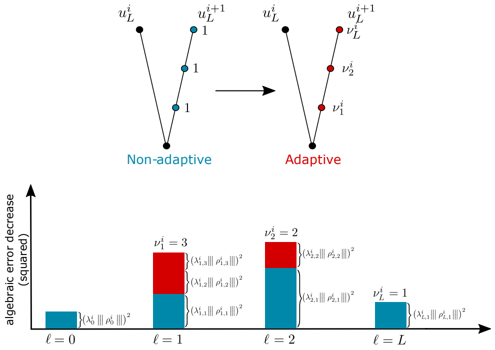

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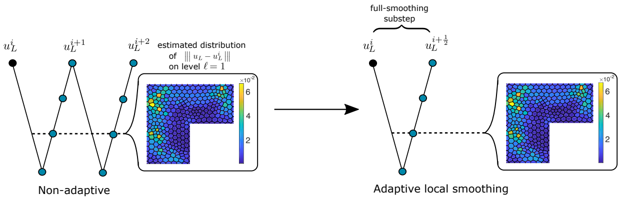

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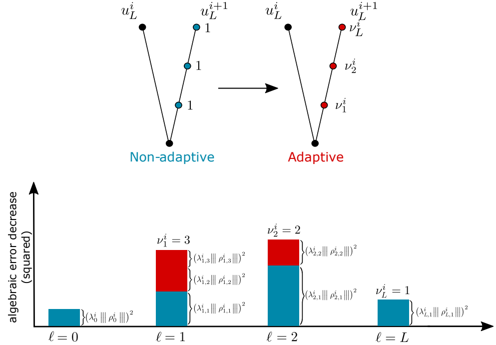

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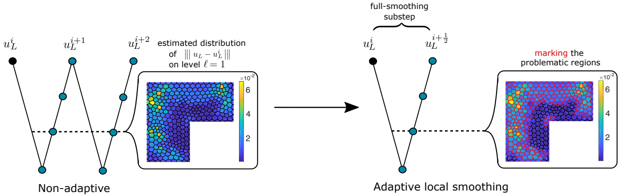

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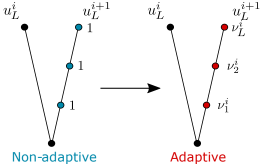

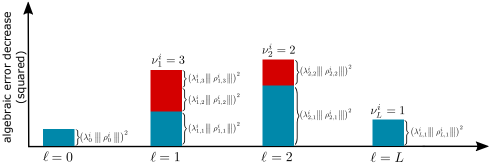

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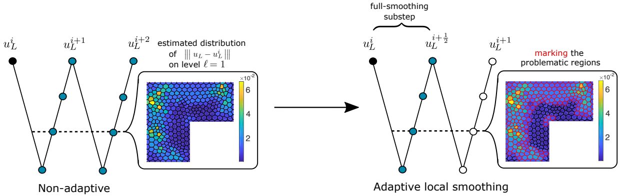

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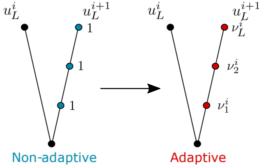

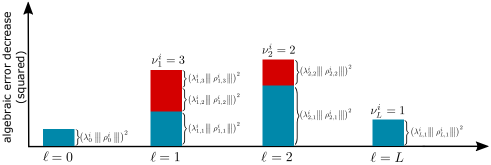

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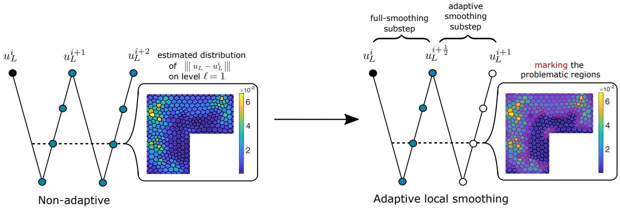

Adaptive finite element setting

Fitting into AFEM setting

Optimal convergence rates wrt to overall computational cost for contractive solvers.

Algorithm

Input $\mathcal{T}_0$, u_0^0, $0 < \theta \leq 1$

For each $L = 0, 1, 2, \ldots$ *do*

- **SOLVE & ESTIMATE** *For* $i = 1, 2, \ldots, i_s$, *repeat*
  - *compute* u_L^i, $\eta_{\mathrm{alg}}^i =: \eta_{\mathrm{alg}}(u_L^i)$
  - *compute* $\eta_{\mathrm{disc}}(T, u_L^i)$ *for all* $T \in \mathcal{T}_\ell$

  until $\eta_{\mathrm{alg}}(u_L^{i_s}) \leq \mu \eta_{\mathrm{disc}}(u_L^{i_s})$ $\longrightarrow$ *idea: equilibrate algebraic and discretization errors*
- **MARK** *choose* $\mathcal{M}_L \subseteq \mathcal{T}_L$ *such that* $\theta \sum_{T \in \mathcal{T}_L} \eta_{\mathrm{disc}}(T, u_L^{i_s})^2 \leq \sum_{T \in \mathcal{M}_L} \eta_{\mathrm{disc}}(T, u_L^{i_s})^2$
- **REFINE** $\mathcal{T}_{L+1} := \mathtt{refine}(\mathcal{T}_L, \mathcal{M}_L)$, $u_{L+1}^0 := u_L^{i_s}$ $\longrightarrow$ *nested iterations with error control on all* u_L *except* u_0^0

Output *Discrete solutions* $u_L^{i_s}$ *and corresponding estimators* $\eta_{\mathrm{alg}}(u_L^{i_s}), \eta_{\mathrm{disc}}(u_L^{i_s})$

Stevenson. *Found. Comput. Math.* 2007
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Key to obtaining L-robustness

Remark: From now on, consider $p_0 = \ldots = p_{\ell-1} = 1$ and $p_L = p$.

previously | improvement

For intermediate levels $\ell \in \{1, \ldots, L-1\}$:

smoothing on *all* patches | smoothing *locally*

For the finest level L: smoothing on all patches *when* $p > 1$.

Takeaway message:

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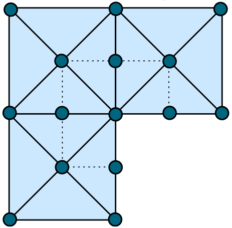

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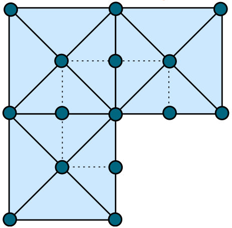

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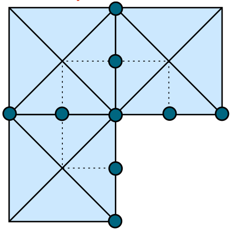

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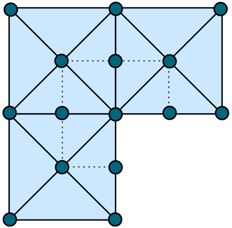

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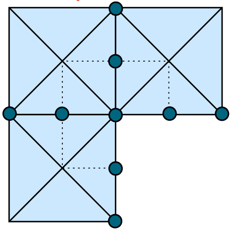

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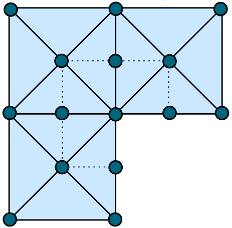

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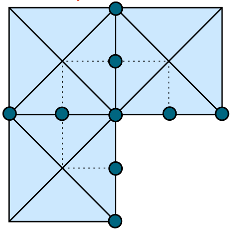

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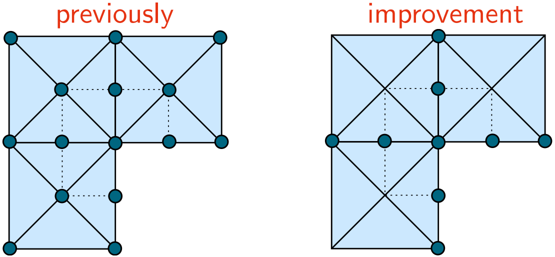

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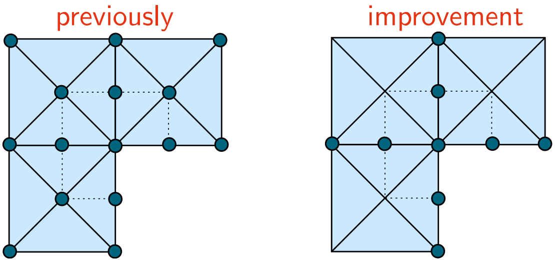

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Main results

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Theorem (h- and p-robust reliable and efficient bound on the algebraic error)

Let $u_L^i \in \mathbb{V}_L^p$ *be arbitrary. Let* η_{alg}^i *be the associated estimator on the algebraic error.*

$\Longrightarrow$ $\quad \|u_L - u_L^i\| \geq \eta_{\mathrm{alg}}^i$ *and* $\eta_{\mathrm{alg}}^i \geq \beta \|u_L - u_L^i\|$ *with* $0 < \beta(\kappa_{\mathcal{T}}, d, \mathbf{K}) < 1$

Theorem (h- and p-robust error contraction of the multilevel solver)

For $u_L^i \in \mathbb{V}_L^p$, *let* $u_L^{i+1} \in \mathbb{V}_L^p$ *be constructed from* u_L^i *using one step of the solver.*

$\Longrightarrow$ $\quad \|u_L - u_L^{i+1}\| \leq \alpha \|u_L - u_L^i\|$ *with* $\alpha = \sqrt{1-\beta^2}$

Remark: Complete *independence* from L is obtained even under minimal H^1-regularity

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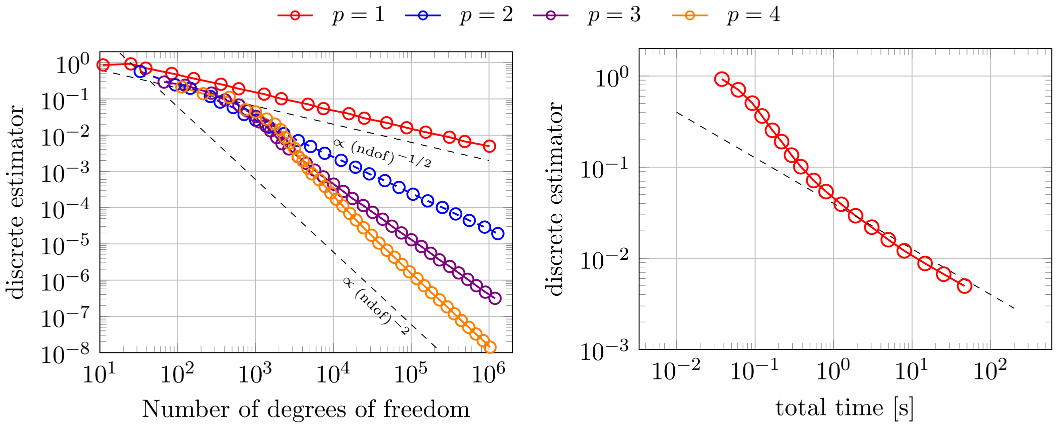
L-shape problem



Innerberger, Praetorius. MooAFEM: An object oriented Matlab code for higher-order (nonlinear) adaptive FEM. 2022+

Conclusion

Contributions and perspectives

We presented:

- A p-**robustly efficient** a posteriori algebraic error estimator
- A p-**robust contractive** multigrid solver steered by the a posteriori estimator
- Optimal level-wise **step-sizes** in the error correction stage
- Two **adaptive** multigrid variants:
  - Approach 1: **adaptive number of smoothing steps** per level
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Future work would explore:

- Extension of the theory to cover variable p elements of the finest level
- Extension of the approach to fractional diffusion problem and BEM
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Thank you for your attention!

- Miraçi, Papež, and Vohralík. A multilevel algebraic error estimator and the corresponding iterative solver with p-robust behavior. *SIAM J. Numer. Anal.* (2020)
- Miraçi, Papež, and Vohralík. A-posteriori-steered p-robust multigrid with optimal step-sizes and adaptive number of smoothing steps. *SIAM J. Sci. Comput.* (2021)
- Miraçi, Praetorius, and Streitberger. Optimal local p-robust multigrid for FEM on graded bisection grids. *In preparation.*

Dr. Ani Miraçi

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Additional results

Corollary (Equivalence of the two main results)

Proving the efficiency of the a posteriori estimator η^i_{alg} is equivalent to proving the solver contraction.

Proof: By using the *link between solver and estimator* given by the Pythagorean formula, there holds:

$$\begin{aligned} &\left(\eta^i_{\mathrm{alg}}\right)^2 \geq \beta^2 \|u_L - u^i_L\|^2 \quad \text{(estimator efficiency)} \\ \Leftrightarrow\ & \|u_L - u^i_L\|^2 - \|u_L - u^{i+1}_L\|^2 \geq \beta^2 \|u_L - u^i_L\|^2 \\ \Leftrightarrow\ & \|u_L - u^{i+1}_L\|^2 \leq (1-\beta^2) \|u_L - u^i_L\|^2 \quad \text{(solver contraction).} \end{aligned}$$

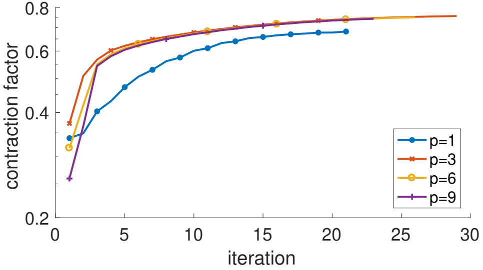
Corollary (Equivalence of error–global estimator–local estimators)

Let the assumptions of Theorem 2 hold. Then

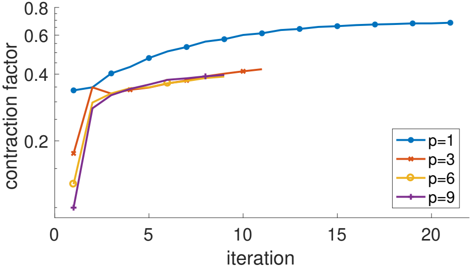
$$\|u_L - u^i_L\|^2 \approx \left(\eta^i_{\mathrm{alg}}\right)^2 = \sum_{\ell=0}^{L} \left(\lambda^i_\ell \|\rho^i_\ell\|\right)^2 = \|\rho^i_0\|^2 + \sum_{\ell=1}^{L} \lambda^i_\ell \sum_{\mathbf{a}\in\mathcal{V}_\ell} \|\rho_{\ell,\mathbf{a}}\|^2_{\omega^{\mathbf{a}}_\ell}.$$

p-robustness: contraction factors

L-shape problem, $L = 3$, and mesh hierarchy $p_\ell = 1$ (left) and $p_\ell = p$ (right), $\ell \in \{1, \ldots, L-1\}$



$1 \to 1, p$



$1, p \to p$

p-robustness: iteration numbers for uniform refinements

Stopping criterion:
$$\frac{\|\mathrm{F}_L - \mathbb{A}_L \mathrm{U}_L^{i_\mathrm{s}}\|}{\|\mathrm{F}_L\|} \leq 10^{-5} \frac{\|\mathrm{F}_L - \mathbb{A}_L \mathrm{U}_L^{0}\|}{\|\mathrm{F}_L\|}.$$

The mesh hierarchies here are obtained from L *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

H^2-regular: Sine, Peak; H^1-regular: L-shape, Checkerboard, Skyscraper

			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$
L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2\mathrm{e}^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1\mathrm{e}^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6\mathrm{e}^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1\mathrm{e}^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6\mathrm{e}^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6\mathrm{e}^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2\mathrm{e}^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5\mathrm{e}^6$	**32**	14	31	15	23	**9**	22	9	23	9	25	9	25	9

Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases.

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L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2e^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
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	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
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L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
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	9	$1e^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6e^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6e^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2e^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5e^6$	**32**	14	31	15	23	**9**	22	9	23	9	25	9	25	9

Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases.

Stopping criterion: $$\frac{\|\mathrm{F}_L - \mathbb{A}_L \mathrm{U}_L^{i_\mathrm{s}}\|}{\|\mathrm{F}_L\|} \leq 10^{-5} \frac{\|\mathrm{F}_L - \mathbb{A}_L \mathrm{U}_L^{0}\|}{\|\mathrm{F}_L\|}.$$

The mesh hierarchies here are obtained from L *uniform refinements* of an initial Delaunay mesh $\mathcal{T}_0$.

			H^2-regular				H^1-regular									
			Sine		Peak		L-shape		Checkerboard				Skyscraper			
			$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathbf{K}=I$		$\mathcal{J}(\mathbf{K})=O(10^6)$		$\mathcal{J}(\mathbf{K})=O(1)$		$\mathcal{J}(\mathbf{K})=O(10^7)$	
			$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$	$1\to1,p$	$1,p\to p$
L	p	DoF	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s	i_s
3	1	$2\mathrm{e}^4$	19	19	19	19	21	21	18	18	18	18	19	19	19	19
	3	$1\mathrm{e}^5$	29	13	28	14	29	11	27	11	28	11	31	13	31	13
	6	$6\mathrm{e}^5$	30	13	30	14	26	9	24	9	25	10	28	11	28	11
	9	$1\mathrm{e}^6$	31	14	30	14	23	9	23	9	23	9	26	10	26	10
4	1	$6\mathrm{e}^4$	21	21	20	20	21	21	19	19	19	19	19	19	19	19
	3	$6\mathrm{e}^5$	29	13	29	14	28	11	26	11	27	11	30	11	30	11
	6	$2\mathrm{e}^6$	31	13	30	14	25	9	24	9	24	9	27	10	27	10
	9	$5\mathrm{e}^6$	**32**	14	31	15	23	**9**	22	9	23	9	25	9	25	9

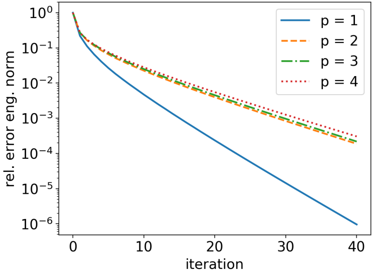
Numerical $\mathbf{K}$- and L-robustness is observed even in low-regularity cases.

Numerical tests in three space dimensions

Test cases: exact solution u when available; $\mathbf{K} = I$ except where explicitly specified, uniform mesh refinement, $p_\ell = 1$, $\ell \in \{1, \ldots, L\}$, and $L = 4$.

Cube: $\Omega := (0,1)^3$,

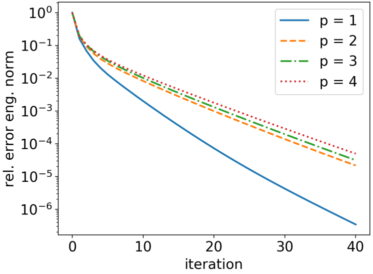
$u(x,y,z) = x(x-1)y(y-1)z(z-1)$.



Nested cubes: $\Omega := (-1,1)^3$,

unknown analytic solution,

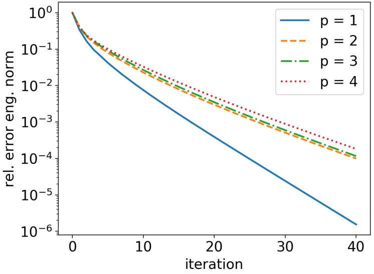
$\mathbf{K} = 10^5 * I$ in $(-0.5, 0.5)^3$.



Checkers cubes: $\Omega := (0,1)^3$,

unknown analytic solution,

$\mathbf{K} = 10^6 * I$ in $(0, 0.5)^3 \cup (0.5, 1)^3$.



Test cases: exact solution u when available; $\mathbf{K} = I$ except where explicitly specified, uniform mesh refinement, $p_\ell = 1$, $\ell \in \{1, \ldots, L\}$, and $L = 4$.

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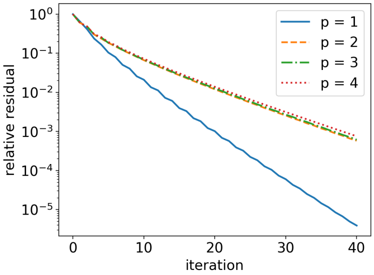

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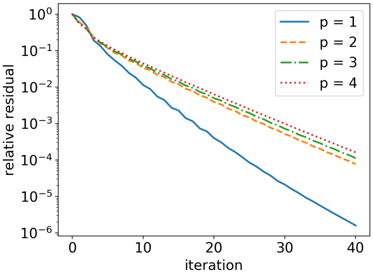
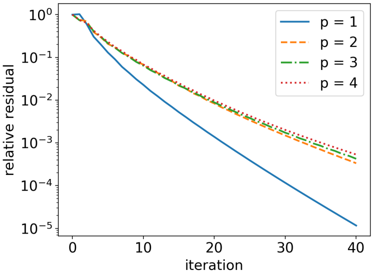


Checkers cubes: $\Omega := (0,1)^3$,

unknown analytic solution,

$\mathbf{K} = 10^6 * I$ in $(0, 0.5)^3 \cup (0.5, 1)^3$.



Numerical advantages of optimal step-sizes

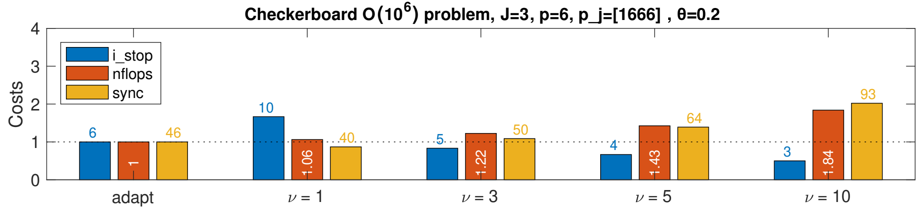
Level-wise optimal step-sizes determined by line search:

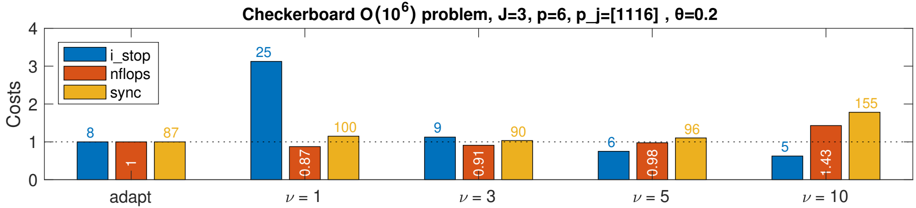
- *analytically*: **Pythagorean formula** for the algebraic error
- *numerically*: advantages of using even a single global step-size on level L

		Sine		Peak		L-shape	
L	p	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J	wRAS	MG(0,1)-J
3	1	21	-	19	68	17	44
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	10	-
4	1	23	-	20	-	18	-
	3	15	-	15	-	12	-
	6	13	-	14	-	10	-
	9	13	-	14	-	9	-
5	1	22	-	20	-	17	-
	3	15	-	15	-	12	-
	6	13	-	14	-	9	-
	9	13	-	13	-	8	-

For $p = 1$: **wRAS** and **MG(0,1)-J** only differ by the use of the global optimal step-size.

Number of post-smoothing steps: adaptive vs fixed





$$\textbf{nflops} := \frac{|\mathcal{V}_0|^3}{3} + \sum_{\ell=1}^{L} \sum_{\mathbf{a}\in\mathcal{V}_L} \frac{\mathsf{ndof}(V_\ell^{\mathbf{a}})^3}{3} + \sum_{i=1}^{i_s} \Big[2|\mathcal{V}_0|^2 + \sum_{\ell=1}^{L} \nu_\ell^i \sum_{\mathbf{a}\in\mathcal{V}_L} 2\mathsf{ndof}(V_\ell^{\mathbf{a}})^2 \Big] + \sum_{i=1}^{i_s} \sum_{\ell=1}^{L} \Big[2\,\mathsf{nnz}(\mathcal{I}_{\ell-1}^L) + 2\,\mathsf{nnz}(\mathcal{I}_L^{\ell-1}) + 2\nu_\ell^i\,\mathsf{nnz}(A_L) + 3\nu_\ell^i (2\,\mathsf{size}(A_L)) \Big];$$

$$\textbf{sync} := i_s + \sum_{i=1}^{i_s} \sum_{\ell=1}^{L} \nu_\ell^i.$$

Dörfler's bulk-chasing criterion: $\theta^2 \left(\left\| \mathbf{K}^{\frac{1}{2}} \nabla \rho_0^i \right\|^2 + \sum_{\ell=1}^{L} \lambda_\ell^i \sum_{\mathbf{a} \in \mathcal{V}_L} \left\| \mathbf{K}^{\frac{1}{2}} \nabla \rho_{\ell,\mathbf{a}} \right\|^2_{\omega_\ell^{\mathbf{a}}} \right) \leq \sum_{\ell \in \mathcal{M}} \lambda_\ell^i \sum_{\mathbf{a} \in \mathcal{M}_L} \left\| \mathbf{K}^{\frac{1}{2}} \nabla \rho_{\ell,\mathbf{a}} \right\|^2_{\omega_\ell^{\mathbf{a}}}.$

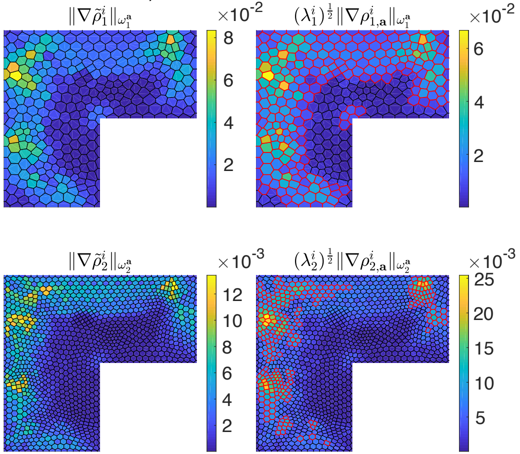
Hierarchy: uniform refinement, $L = 2$, $p_1 = p_2 = 3$.

- local algebraic error indicators $\|\rho_{\ell,\mathbf{a}}\|_{\omega_\ell^{\mathbf{a}}}$
- local algebraic error distribution $\|\tilde{\rho}_\ell^i\|_{\omega_\ell^{\mathbf{a}}}$

with $\tilde{\rho}_0^i = \rho_0^i$ and $\tilde{\rho}_\ell^i \in \mathbb{V}_\ell^{p_\ell}$, for $\ell \in \{1, \dots, L\}$, given by

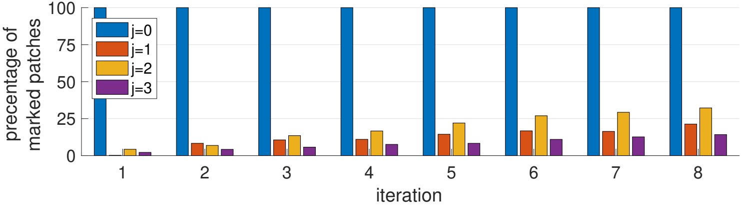
$$\langle\!\langle \tilde{\rho}_\ell^i , v_\ell \rangle\!\rangle = (f, v_\ell) - \langle\!\langle u_L^i , v_\ell \rangle\!\rangle - \sum_{k=0}^{\ell-1} \langle\!\langle \tilde{\rho}_k^i , v_\ell \rangle\!\rangle \quad \forall v_\ell \in \mathbb{V}_\ell^{p_\ell},$$

so that $\sum_{\ell=0}^{L} \tilde{\rho}_\ell^i = u_L - u_L^i$.

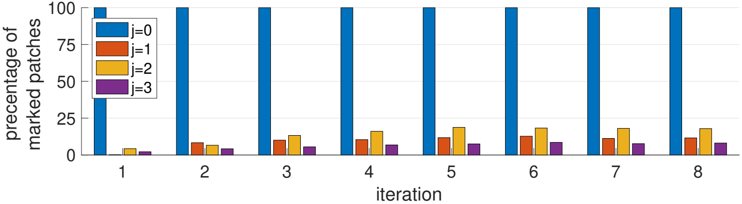


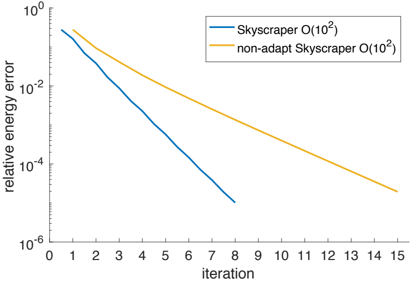
Does the adaptivity pay off?

Skyscraper $O(10^2)$ test case (non-adaptive 15 iterations)



Skyscraper $O(10^5)$ test case (non-adaptive 15 iterations)





Hierarchy: $L = 3$, $p_0 = 1$, $p_1 = 1$, $p_2 = 2$, $p_3 = 3$, $\theta = 0.95$