Adaptive methods for fully nonlinear PDE

Ellya L. Kawecki   Iain Smears

Department of Mathematics
University College London

Engineering and Physical Sciences Research Council

Paris 2022

HJB and Isaacs equations

Isaacs equation with homogeneous Dirichlet condition (dimension $d \in \{2,3\}$)

$$F[u] := \inf_{\alpha\in\mathscr{A}} \sup_{\beta\in\mathscr{B}} \left[L^{\alpha\beta}u - f^{\alpha\beta}\right] = 0 \text{ in } \Omega, \qquad u = 0 \text{ on } \partial\Omega,$$

$$L^{\alpha\beta}u := a^{\alpha\beta} : \nabla^2 u + b^{\alpha\beta}\cdot\nabla u + c^{\alpha\beta}u$$

Hamilton–Jacobi–Bellman equation (HJB): when either $\mathscr{A}$ or $\mathscr{B}$ is a singleton set

$$\sup_{\beta\in\mathscr{B}} [L^{\beta}u - f^{\beta}] = 0, \quad \text{or} \quad \inf_{\alpha\in\mathscr{A}} [L^{\alpha}u - f^{\alpha}] = 0$$

- Notation: Hessian denoted by $\nabla^2 u \in \mathbb{R}^{d\times d}$
- This talk: construct convergent discretizations with adaptive mesh refinements for these problems
- For sake of presentation, we ignore lower order terms.

Framework

Assume data $a\colon (x,\alpha,\beta) \mapsto a^{\alpha\beta}(x) \in \mathbb{R}^{d\times d}$ and $f\colon (x,\alpha,\beta) \mapsto f^{\alpha\beta}(x) \in \mathbb{R}$ are

- uniformly continuous on $\Omega \times \mathscr{A} \times \mathscr{B}$
- a is symmetric and uniformly elliptic

Assume that Ω is convex and **Cordes condition** (Cordes 56): there exists a $\nu \in (0,1]$ such that

$$\frac{|a^{\alpha\beta}(x)|}{\mathrm{Tr}(a^{\alpha\beta}(x))} \leq \frac{1}{\sqrt{d-1+\nu}} \quad \forall x \in \Omega, \quad \forall (\alpha,\beta) \in \mathscr{A} \times \mathscr{B},$$

Frobenius norm $|\cdot|$ and trace $\mathrm{Tr}(\cdot)$ of $\mathbb{R}^{d\times d}$ matrices.

Existence and Uniqueness

Define the renormalized operator F_γ

$$\gamma^{\alpha\beta} := \frac{\operatorname{Tr} a^{\alpha\beta}}{|a^{\alpha\beta}|^2}, \qquad F_\gamma[v] := \inf_{\alpha\in\mathscr{A}} \sup_{\beta\in\mathscr{B}} \left[\gamma^{\alpha\beta}\left(L^{\alpha\beta}v - f^{\alpha\beta}\right)\right].$$

Theorem (S. & Süli 14, Kawecki & S. 21 M2AN)

There exists a unique $u \in H^2(\Omega) \cap H^1_0(\Omega)$ that solves $F[u] = 0$ and equivalently $F_\gamma[u] = 0$ pointwise a.e. in Ω.

Proof via formulation as monotone operator equation

$$A(u;v) := \int_\Omega F_\gamma[u]\Delta v = 0 \quad \forall v \in H^2(\Omega) \cap H^1_0(\Omega).$$

$$\|w - v\|^2_{H^2} \lesssim A(w; w - v) - A(v; w - v) \quad \forall w, v \in H^2(\Omega) \cap H^1_0(\Omega).$$

See also related works **S. & Süli 13+14+16, Gallistl 17, S. 18, Neilan & Wu 19, Gallistl & Süli 19, Blechschmidt, Herzog & Winkler 20, Kawecki & Brenner 20, Capdeboscq Sprekeler & Süli 20, Wu 21**

Notation for derivatives

- For $v \in BV(\Omega)$, with distributional derivative Dv:

$$\langle Dv, \phi \rangle_\Omega = \int_\Omega \nabla v \cdot \phi + \int_\Omega \phi \cdot d[Dv]_{\text{sing}} \quad \forall \phi \in C_0^\infty(\Omega; \mathbb{R}^d)$$

  So ∇v denotes density of **absolutely continuous part** (wrt Lebesgue measure) of distributional derivative Dv.

- If $\nabla v \in BV(\Omega; \mathbb{R}^d)$ then define

$$\nabla^2 v := \nabla(\nabla v), \quad \text{i.e.} \quad \nabla^2_{ij} v := \nabla_{x_j}(\nabla_{x_i} v)$$

Easy to check that

- If v is Sobolev regular e.g. $v \in H^2(\Omega)$ then ∇v and $\nabla^2 v$ coincide with weak gradient and Hessian respectively.
- If v is piecewise regular e.g. v in a (discontinuous) finite element space, then ∇v and $\nabla^2 v$ coincide with piecewise (broken) gradient and Hessian respectively.

Notation for derivatives

- For $v \in BV(\Omega)$, with distributional derivative Dv:

$$\langle Dv, \phi \rangle_\Omega = \int_\Omega \nabla v \cdot \phi + \int_\Omega \phi \cdot d[Dv]_{\text{sing}} \quad \forall \phi \in C_0^\infty(\Omega; \mathbb{R}^d)$$

So ∇v denotes density of **absolutely continuous part** (wrt Lebesgue measure) of distributional derivative Dv.

- If $\nabla v \in BV(\Omega; \mathbb{R}^d)$ then define

$$\nabla^2 v := \nabla(\nabla v), \quad \text{i.e.} \quad \nabla^2_{ij} v := \nabla_{x_j}(\nabla_{x_i} v)$$

Easy to check that

- If v is Sobolev regular e.g. $v \in H^2(\Omega)$ then ∇v and $\nabla^2 v$ coincide with weak gradient and Hessian respectively.
- If v is piecewise regular e.g. v in a (discontinuous) finite element space, then ∇v and $\nabla^2 v$ coincide with piecewise (broken) gradient and Hessian respectively.

Notation

- Shape-regular nested sequence of matching simplicial meshes $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$.
- Framework of **quasi-regular subdivisions** as in Morin, Siebert & Veeser 08.
- DG and C^0 FEM spaces: for fixed polynomial degree $p \geq 2$

$$V_k^0 := \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_p \ \forall K \in \mathcal{T}_k\},$$
$$V_k^1 := V_k^0 \cap H_0^1(\Omega).$$

Numerical approximations sought in V_k^s for a chosen fixed $s \in \{0, 1\}$.

- H^2-type mesh-dependent norms for $v \in V_k^s$:

$$\|v\|_k^2 := \int_\Omega \left[|\nabla^2 v|^2 + |\nabla v|^2 + |v|^2\right] + \int_{\mathcal{F}_k^I} h_k^{-1} |[\![\nabla v]\!]|^2 + \int_{\mathcal{F}_k} h_k^{-3} |[\![v]\!]|^2.$$

with jumps

$$[\![\phi]\!]_F(x) = \lim_{\epsilon \searrow 0} [\phi(x + \epsilon \boldsymbol{n}_F) - \phi(x - \epsilon \boldsymbol{n}_F)] \quad \forall x \in F$$

with $\boldsymbol{n}_F$ fixed unit normal to F.

Notation

- Shape-regular nested sequence of matching simplicial meshes $\{\mathcal{T}_k\}_{k\in\mathbb{N}}$.
- Framework of **quasi-regular subdivisions** as in **Morin, Siebert & Veeser 08**.
- DG and C^0 FEM spaces: for fixed polynomial degree $p \geq 2$

$$V_k^0 := \{v \in L^2(\Omega) : v|_K \in \mathbb{P}_p \ \forall K \in \mathcal{T}_k\},$$
$$V_k^1 := V_k^0 \cap H_0^1(\Omega).$$

Numerical approximations sought in V_k^s for a chosen fixed $s \in \{0, 1\}$.

- H^2-type mesh-dependent norms for $v \in V_k^s$:

$$\|v\|_k^2 := \int_\Omega \left[|\nabla^2 v|^2 + |\nabla v|^2 + |v|^2\right] + \int_{\mathcal{F}_k^I} h_k^{-1} |[\![\nabla v]\!]|^2 + \int_{\mathcal{F}_k} h_k^{-3} |[\![v]\!]|^2.$$

with jumps

$$[\![\phi]\!]_F(x) = \lim_{\epsilon \searrow 0} [\phi(x + \epsilon \boldsymbol{n}_F) - \phi(x - \epsilon \boldsymbol{n}_F)] \quad \forall x \in F$$

with $\boldsymbol{n}_F$ fixed unit normal to F.

Family of numerical schemes: find $u_k \in V_k^s$ such that

$$A_k(u_k; v_k) = 0 \qquad \forall v_k \in V_k^s,$$

where $$A_k(w_k; v_k) := \underbrace{\int_\Omega F_\gamma[w_k]\Delta_k v_k}_{PDE} + \underbrace{\theta S_k(w_k, v_k)}_{Stabilization} + \underbrace{J_k^{\sigma,\rho}(w_k, v_k)}_{Penalization}$$

$\theta \in [0, 1]$, $\sigma, \rho > 0$.

Numerical schemes

Family of numerical schemes: find $u_k \in V_k^s$ such that

$$A_k(u_k; v_k) = 0 \qquad \forall v_k \in V_k^s,$$

where $$A_k(w_k; v_k) := \underbrace{\int_\Omega F_\gamma[w_k]\Delta_k v_k}_{PDE} + \underbrace{\theta S_k(w_k, v_k)}_{Stabilization} + \underbrace{J_k^{\sigma,\rho}(w_k, v_k)}_{Penalization}$$

$\theta \in [0,1]$, $\sigma, \rho > 0$.

Lifted Laplacian $\Delta_k v_k := \Delta v_k - r_k([\![\nabla v_k \cdot \boldsymbol{n}]\!])$ where $r_k([\![\nabla v_k \cdot \boldsymbol{n}]\!]) \in V_{k,q}^0$ defined by

$$\int_\Omega r([\![\nabla v_k \cdot \boldsymbol{n}]\!])\varphi_k = \int_{\mathcal{F}_k^I} [\![\nabla v_k \cdot \boldsymbol{n}]\!] \{\varphi_k\} \quad \forall \varphi_k \in V_{k,q}^0$$

with $V_{k,q}^0$ piecewise poly. of degree $q \geq p - 2$ on $\mathcal{T}_k$.

Family of numerical schemes: find $u_k \in V_k^s$ such that

$$A_k(u_k; v_k) = 0 \qquad \forall v_k \in V_k^s,$$

where $$A_k(w_k; v_k) := \underbrace{\int_\Omega F_\gamma[w_k] \Delta_k v_k}_{PDE} + \underbrace{\theta S_k(w_k, v_k)}_{Stabilization} + \underbrace{J_k^{\sigma,\rho}(w_k, v_k)}_{Penalization}$$

$\theta \in [0,1]$, $\sigma, \rho > 0$.

$$\begin{aligned} S_k(w_k, v_k) := & \int_\Omega \left[\nabla^2 w_k : \nabla^2 v_k - \Delta w_k \Delta v_k\right] \\ & + \int_{\mathcal{F}_k^I} \left[\{\Delta_T w_k\} [\![\nabla v_k \cdot \boldsymbol{n}]\!] + \{\Delta_T v_k\} [\![\nabla w_k \cdot \boldsymbol{n}]\!]\right] \\ & - \int_{\mathcal{F}_k} \left[\nabla_T \{\nabla w_k \cdot \boldsymbol{n}\} \cdot [\![\nabla_T v_k]\!] + \nabla_T \{\nabla v_k \cdot \boldsymbol{n}\} \cdot [\![\nabla_T w_k]\!]\right], \end{aligned}$$

with tangential operators ∇_T and Δ_T.
C.f. **S. & Süli 13** for derivation.

Numerical schemes

Family of numerical schemes: find $u_k \in V_k^s$ such that

$$A_k(u_k; v_k) = 0 \qquad \forall v_k \in V_k^s,$$

where $$A_k(w_k; v_k) := \underbrace{\int_\Omega F_\gamma[w_k]\Delta_k v_k}_{PDE} + \underbrace{\theta S_k(w_k, v_k)}_{Stabilization} + \underbrace{J_k^{\sigma,\rho}(w_k, v_k)}_{Penalization}$$

$\theta \in [0,1]$, $\sigma, \rho > 0$.

$$J_k^{\sigma,\rho}(w_k, v_k) := \int_{\mathcal{F}_k^I} \sigma h_k^{-1} [\![\nabla w_k]\!] \cdot [\![\nabla v_k]\!] + \int_{\mathcal{F}_k} \rho h_k^{-3} [\![w_k]\!][\![v_k]\!]$$

Numerical method: main properties

Summary of results from unified analysis in **Kawecki & S. 21 (M2AN)**:

- Stability, existence and uniqueness of numerical solutions
- Near-best approximation

$$\|u - u_k\|_k \lesssim \inf_{v_k \in V_k^s} \|u - v_k\|_k .$$

- **Reliable and efficient a posteriori error estimators:**

$$\|u - v_k\|_k \eqsim \eta_k(v_k) \quad \forall v_k \in V_k^s,$$

where $[\eta_k(v_k)]^2 \coloneqq \sum_{K \in \mathcal{T}_k} [\eta_k(v_k, K)]^2$ and

$$[\eta_k(v_k, K)]^2 \coloneqq \int_K |F_\gamma[v_k]|^2 + \sum_{\substack{F \in \mathcal{F}_k^I \\ F \subset \partial K}} \int_F h_k^{-1} |[\![\nabla v_k]\!]|^2 + \sum_{\substack{F \in \mathcal{F}_k \\ F \subset \partial K}} \int_F h_k^{-3} |[\![v_k]\!]|^2$$

In fact efficiency is also shown to be **local**.

No reduction property: No positive powers h_k in estimators.

Adaptive refinement

Adaptive loop: Solve. Estimate. Mark. Refine.

Sufficient condition for marking: *refine at least one element with maximum estimator.*
Suppose that marked set $\mathcal{M}_k \subset \mathcal{T}_k$ satisfies

$$\max_{K\in\mathcal{T}_k} \eta_k(u_k, K) = \max_{K\in\mathcal{M}_k} \eta_k(u_k, K).$$

Compatible with common strategies, e.g. bulk-chasing or maximum marking strategy.

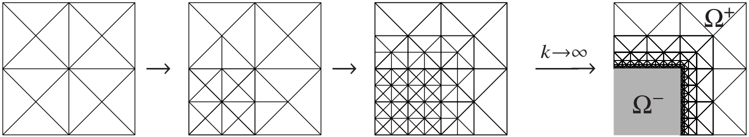
Main result: Convergence — Kawecki & S. 21 (FoCM)

The sequence of numerical solutions $\{u_k\}_{k\in\mathbb{N}}$ converges to the solution u with

$$\lim_{k\to\infty} \|u - u_k\|_k = 0, \qquad \lim_{k\to\infty} \eta_k(u_k) = 0.$$

Convergence holds for all stable choices of penalty parameters σ, ρ.

Concept of limit space



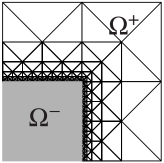
Main ingredient in proof: construction of **limit spaces** V_∞^s of the sequence $\{V_k^s\}_{k\in\mathbb{N}}$

Key properties required

- every $v \in V_\infty^s$ is a strong limit of a sequence of functions $v_k \in V_k^s$ (in suitable norms)
- every bounded sequence $\{v_k\}_{k\in\mathbb{N}}$ with $v_k \in V_k^s$ has a subsequence with weak limit $v \in V_\infty^s$ (with weak convergence in a suitable sense)

How to define and analyse these nonstandard function spaces?

See also different approach by **Kreuzer & Georgoulis 18, Dominicus, Gaspoz & Kreuzer 20**



$$\mathcal{T}^+ := \bigcup_{m\geq 0} \bigcap_{k\geq m} \mathcal{T}_k = \text{never-refined elements}, \qquad \Omega^+ := \bigcup_{K\in\mathcal{T}^+} K, \qquad \Omega^- := \overline{\Omega} \setminus \Omega^+$$

$\mathcal{F}^+ :=$ faces of all elements in $\mathcal{T}^+$, $\quad \mathcal{F}^{I+} :=$ interior faces of $\mathcal{F}^+$

Note skeletons of $\mathcal{F}^+$ and $\mathcal{F}^{I+}$ are (countably) rectifiable sets

Define mesh-size function on Ω^+

$$h_+|_K := |K|^{1/d} \quad \forall K \in \mathcal{T}^+, \qquad h_+|_F := |F|_{\mathcal{H}^{d-1}}^{1/(d-1)} \quad \forall F \in \mathcal{F}^+$$

Definitions: First-order spaces

Intrinsic definition of limit space requires introducing some original function spaces:

- First order spaces $H^1(\Omega;\mathcal{T}^+)$ and $H^1_D(\Omega;\mathcal{T}^+)$
- Second-order space $H^2_D(\Omega;\mathcal{T}^+)$
- finally $V^s_\infty \subset H^2_D(\Omega;\mathcal{T}^+)$

Definition: $H^1(\Omega;\mathcal{T}^+)$ and $H^1_D(\Omega;\mathcal{T}^+)$ Kawecki & S. 21 (FoCM)

Let $H^1(\Omega;\mathcal{T}^+)$ be the space of functions v s.t.

- $v \in BV(\Omega) \cap L^2(\Omega)$
- **Distributional derivative:** For all test functions $\phi \in C_0^\infty(\Omega;\mathbb{R}^d)$

$$\langle Dv, \phi\rangle_{\mathbb{R}^d} = \int_\Omega \nabla v \cdot \phi - \int_{\mathcal{F}^{I+}} [\![v]\!](\phi\cdot \boldsymbol{n}),$$

- **Finite norm:** $\|v\|^2_{H^1(\Omega;\mathcal{T}^+)} := \int_\Omega \left[|\nabla v|^2 + |v|^2\right] + \int_{\mathcal{F}^{I+}} h_+^{-1}|[\![v]\!]|^2 < \infty.$

Definitions: First-order spaces

Intrinsic definition of limit space requires introducing some original function spaces:

- First order spaces $H^1(\Omega;\mathcal{T}^+)$ and $H^1_D(\Omega;\mathcal{T}^+)$
- Second-order space $H^2_D(\Omega;\mathcal{T}^+)$
- finally $V^s_\infty \subset H^2_D(\Omega;\mathcal{T}^+)$

Definition: $H^1(\Omega;\mathcal{T}^+)$ and $H^1_D(\Omega;\mathcal{T}^+)$ — Kawecki & S. 21 (FoCM)

Let $H^1(\Omega;\mathcal{T}^+)$ be the space of functions v s.t.

- $v \in BV(\Omega) \cap L^2(\Omega)$
- **Distributional derivative:** For all test functions $\phi \in C_0^\infty(\Omega;\mathbb{R}^d)$

$$\langle Dv, \phi\rangle_{\mathbb{R}^d} = \int_\Omega \nabla v \cdot \phi - \int_{\mathcal{F}^{I+}} [\![v]\!](\phi{\cdot}\boldsymbol{n}),$$

- **Finite norm:** $\|v\|^2_{H^1(\Omega;\mathcal{T}^+)} := \int_\Omega \left[|\nabla v|^2 + |v|^2\right] + \int_{\mathcal{F}^{I+}} h_+^{-1}|[\![v]\!]|^2 < \infty.$

Definitions: First-order spaces

Intrinsic definition of limit space requires introducing some original function spaces:

- First order spaces $H^1(\Omega;\mathcal{T}^+)$ and $H^1_D(\Omega;\mathcal{T}^+)$
- Second-order space $H^2_D(\Omega;\mathcal{T}^+)$
- finally $V^s_\infty \subset H^2_D(\Omega;\mathcal{T}^+)$

Definition: $H^1(\Omega;\mathcal{T}^+)$ and $H^1_D(\Omega;\mathcal{T}^+)$ — Kawecki & S. 21 (FoCM)

Let $H^1_D(\Omega;\mathcal{T}^+)$ be the space of functions v s.t.

- $v \in L^2(\Omega)$ and **zero-extension** of v to $\mathbb{R}^d$ belongs to $BV(\mathbb{R}^d)$,
- **Distributional derivative:** For all test functions $\phi \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$,

$$\langle Dv, \phi\rangle_{\mathbb{R}^d} = \int_\Omega \nabla v \cdot \phi - \int_{\mathcal{F}^+} [\![v]\!](\phi\cdot\boldsymbol{n}),$$

- **Finite norm:** $\|v\|^2_{H^1_D(\Omega;\mathcal{T}^+)} := \int_\Omega \left[|\nabla v|^2 + |v|^2\right] + \int_{\mathcal{F}^+} h_+^{-1}|[\![v]\!]|^2 < \infty.$

Definitions: Second-order space and limit space

Definition: $H^2_D(\Omega;\mathcal{T}^+)$ — Kawecki & S. 21 (FoCM)

Let $H^2_D(\Omega;\mathcal{T}^+)$ denote the space of functions v such that

- $v \in H^1_D(\Omega;\mathcal{T}^+)$
- $\nabla_{x_i} v \in H^1(\Omega;\mathcal{T}^+)$ for all $i = 1,\ldots,d$, where $\nabla v = (\nabla_{x_1} v,\ldots,\nabla_{x_d} v)$,
- **Finite norm** $\|v\|_{H^2_D(\Omega;\mathcal{T}^+)} < \infty$ where

$$\|v\|^2_{H^2_D(\Omega;\mathcal{T}^+)} := \int_\Omega \left[|\nabla^2 v|^2 + |\nabla v|^2 + |v|^2\right] + \int_{\mathcal{F}^{I+}} h_+^{-1}|[\![\nabla v]\!]|^2 + \int_{\mathcal{F}^+} h_+^{-3}|[\![v]\!]|^2$$

Definition: Limit space V^s_∞, $s \in \{0,1\}$

$$V^0_\infty := \{v \in H^2_D(\Omega;\mathcal{T}^+) : v|_K \in \mathbb{P}_p \ \forall K \in \mathcal{T}^+\}, \quad V^1_\infty := V^0_\infty \cap H^1_0(\Omega).$$

Same norm as $H^2_D(\Omega;\mathcal{T}^+)$.

Recall $\nabla^2 v := \nabla(\nabla v)$ absolutely continuous part of $D(\nabla v)$

Limit space: elementary properties

Elementary properties

- Spaces $H^1(\Omega;\mathcal{T}^+)$, $H^1_D(\Omega;\mathcal{T}^+)$, $H^2_D(\Omega;\mathcal{T}^+)$ and V^s_∞ all Hilbert spaces under respective inner-products.
- $H^1(\Omega)$ closed subspace of $H^1(\Omega;\mathcal{T}^+)$.
- $H^1_0(\Omega)$ closed subspace of $H^1_D(\Omega;\mathcal{T}^+)$.
- $H^2(\Omega)\cap H^1_0(\Omega)$ closed subspace of $H^2_D(\Omega;\mathcal{T}^+)$.
- Functions in $H^1(\Omega;\mathcal{T}^+)$ have Poincaré and L^2-trace inequalities $\implies \|v\|_k$ well-defined for all $k\in\mathbb{N}$ and $v\in V^s_\infty$.
- Piecewise regularity on Ω^+:
  - If $v\in H^1(\Omega;\mathcal{T}^+)$ then $v|_K\in H^1(K)$ for all $K\in\mathcal{T}^+$.
  - If $v\in H^2_D(\Omega;\mathcal{T}^+)$ then $v|_K\in H^2(K)$ for all $K\in\mathcal{T}^+$.
- Also note $V^s_k\not\subset V^s_\infty$ and $V^s_\infty\not\subset V^s_k$ in general.

Limit spaces: symmetry of Hessians

- Recall definition $\nabla^2 v = \nabla(\nabla v)$ — symmetry of $\nabla^2 v$ does not follow directly from definition
- However symmetry is necessary for convergence of FEM approximations!
- **Alberti 91; Fonseca, Leoni & Paroni 05**: there exists functions $v \in SBV^2(\Omega)$ such that $\nabla^2 v = \nabla(\nabla v)$ is **not symmetric!**
  (note $SBV^2(\Omega)$ includes $H^2_D(\Omega; \mathcal{T}^+)$ as a subspace)

Thm: Symmetry of the Hessian — Kawecki & S. 21 (FoCM)

For every $v \in H^2_D(\Omega; \mathcal{T}^+)$ there exists a $w \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$v = w, \quad \nabla v = \nabla w, \quad \nabla^2 v = \nabla^2 w \qquad \text{a.e. on } \Omega^-.$$

Furthermore, $\nabla^2 v$ is symmetric a.e. on Ω.

Limit spaces: symmetry of Hessians

- Recall definition $\nabla^2 v = \nabla(\nabla v)$ — symmetry of $\nabla^2 v$ does not follow directly from definition
- However symmetry is necessary for convergence of FEM approximations!
- **Alberti 91; Fonseca, Leoni & Paroni 05**: there exists functions $v \in SBV^2(\Omega)$ such that $\nabla^2 v = \nabla(\nabla v)$ is **not symmetric!**
  (note $SBV^2(\Omega)$ includes $H^2_D(\Omega; \mathcal{T}^+)$ as a subspace)

Thm: Symmetry of the Hessian — Kawecki & S. 21 (FoCM)

For every $v \in H^2_D(\Omega; \mathcal{T}^+)$ there exists a $w \in H^2(\Omega) \cap H^1_0(\Omega)$ such that

$$v = w, \quad \nabla v = \nabla w, \quad \nabla^2 v = \nabla^2 w \qquad \text{a.e. on } \Omega^-.$$

Furthermore, $\nabla^2 v$ is symmetric a.e. on Ω.

Limit space: strong approximation

Thm: Strong approximation by FE functions — Kawecki & S. 21 (FoCM)

For any $v \in V_\infty^s$, there exists a sequence of finite element functions $v_k \in V_k^s$ for each $k \in \mathbb{N}$, such that

$$\lim_{k\to\infty} \|v - v_k\|_k = 0, \qquad \sup_{k\in\mathbb{N}} \|v_k\|_k < \infty.$$

Moreover, the sequence $\{v_k\}_{k\in\mathbb{N}}$ above can be chosen such that

$$\lim_{k\to\infty} \int_\Omega \left[h_k^{-2} |\nabla(v - v_k)|^2 + h_k^{-4} |v - v_k|^2 \right] = 0.$$

Proof uses

- Symmetry of Hessians of functions in $H_D^2(\Omega; \mathcal{T}^+)$
- Poincaré and trace inequalities of functions in $H^1(\Omega; \mathcal{T}^+)$
- Quasi-interpolation, enrichment operators…

Weak convergence

Thm: weak convergence — Kawecki & S. 21 (FoCM)

Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of functions such that $v_k \in V_k^s$ for each $k \in \mathbb{N}$, and such that $\sup_{k\in\mathbb{N}} \|v_k\|_k < \infty$. Then, there exist a $v \in V_\infty^s$ and a $\boldsymbol{r} \in L^2(\Omega;\mathbb{R}^{d\times d})$ s.t.

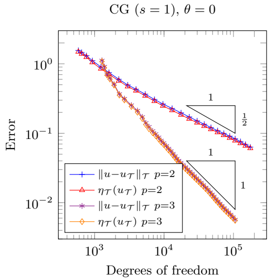
- there exists a subsequence $\{v_{k_j}\}_{j\in\mathbb{N}}$ such that, as $j \to \infty$,
  - $v_{k_j} \to v$ in $L^2(\Omega)$,
  - $\nabla v_{k_j} \to \nabla v$ in $L^2(\Omega;\mathbb{R}^d)$,
  - $\boldsymbol{H}_{k_j} v_{k_j} \rightharpoonup \boldsymbol{H}_\infty v$ in $L^2(\Omega;\mathbb{R}^{d\times d})$
  - $\boldsymbol{r}_{k_j}([\![\nabla v_{k_j}]\!]) \rightharpoonup \boldsymbol{r}$ in $L^2(\Omega;\mathbb{R}^{d\times d})$.
- $\boldsymbol{r}\chi_{\Omega^+} = \boldsymbol{r}_\infty([\![\nabla v]\!])$ a.e. in Ω

NB: In general, liftings are necessary because $\nabla^2 v_{k_j} \not\rightharpoonup \nabla^2 v$ is possible.

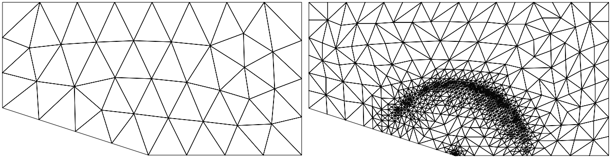
$$\boldsymbol{H}_k v_k := \nabla^2 v_k - \boldsymbol{r}_k([\![\nabla v_k]\!]), \qquad \boldsymbol{r}_k([\![\nabla v_k]\!]) := \sum_{F\in\mathcal{F}_k} \boldsymbol{r}_k^F([\![\nabla v_k]\!]).$$

$$\boldsymbol{H}_\infty v := \nabla^2 v - \boldsymbol{r}_\infty([\![\nabla v]\!]), \qquad \boldsymbol{r}_\infty([\![\nabla v]\!]) := \sum_{F\in\mathcal{F}^+} \boldsymbol{r}_\infty^F([\![\nabla v]\!]).$$

Experiment with nonsmooth solution $u \in H^{2+\frac{1}{9}-\epsilon}$ from **Kawecki & S. 21 (M2AN)**



Initial and adaptively refined meshes (right: $k = 14$)



NGSolve

Conclusion

- Analysis of limit spaces and new function spaces to describe limiting behaviour of adaptive nonconforming methods
- Proof of essential properties of limit spaces: symmetry of Hessians, strong approximation by FE spaces and weak convergence properties
- Proof of convergence of adaptive DG and C^0-IP for HJB and Isaacs equations with Cordes coefficients

Reference

- **Kawecki & S. 21 (FoCM)**: *Convergence of adaptive discontinuous Galerkin and C^0-interior penalty finite element methods for Hamilton–Jacobi–Bellman and Isaacs equations*, **Foundations of Computational Mathematics** (2021), `https://doi.org/10.1007/s10208-021-09493-0`.

Thank you for your attention!

Acknowledgement: EPSRC Doctoral Prize Fellowship for E. L. Kawecki (under ESPRC Grant EP/R513143/1/).