Inverse Problems in Plasma Diffusion

A Probabilistic Approach for Physical Systems

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Outline

1. Problem
   - Magnetosphere
   - Plasma Dynamics: Adiabatic Theory
   - Plasma Dynamics: Diffusion Theory
   - Parameter Estimation
2. Methodology
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   - Parameterization
   - Model Surrogate
   - Bayesian Inference
3. Experiments
   - Synthetic Data
   - Preliminary Results

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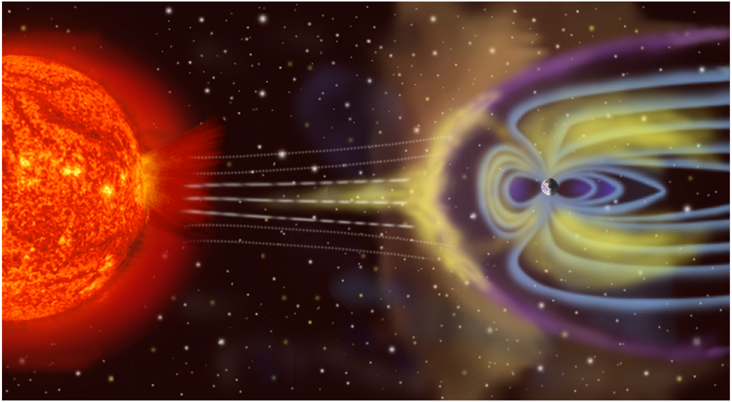

Figure: Sun-Magnetosphere System

Radiation Belts

Van Allen radiation belts are a layer of trapped charged particles around the Earth.

- **Outer radiation belt**, mainly composed of high energy electrons (energetic range around 0.1 to 10 MeV)
- **Inner radiation belt**, composed of high concentrations of electrons (energetic range of hundreds of keV) and energetic protons.

Plasma Motions

Motion of charged particles is decomposed into three components.

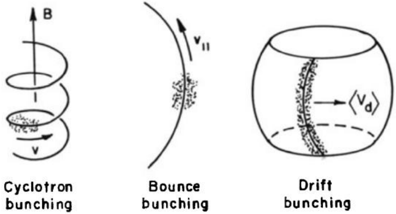

Figure: Adiabatic Motions

Adiabatic Invariants

For each component of plasma motion, there is one *adiabatic invariant* which is assumed to be conserved in plasma motion.

- Cyclotron Motion: $\mathcal{M} = m_0 v_{\perp}^2 / 2B = \frac{m_0 v^2 sin^2 \alpha}{2B}$

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- Drift Motion: $\Phi = \int \int \mathbf{B} d\mathbf{S}$

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Phase Space Density

Quantity of interest in plasma diffusion is the so called *phase space density* $f(\mathcal{M}, J, \Phi, t)$.

Fokker-Plank Equation

Canonical form of plasma diffusion [Schulz and Lanzerotti, 1974]

$$\frac{\partial f}{\partial t} = \sum_{p,q=1}^{3} \frac{\partial}{\partial J_p} \left(\kappa_{pq} \frac{\partial f}{\partial J_q} \right)$$

$$J_1 = \mathcal{M}$$

$$J_2 = J$$

$$J_3 = \Phi$$

Radial Diffusion

The simplified system [Walt, 1970], now looks like

$$\frac{\partial f}{\partial t} = \ell^2 \frac{\partial}{\partial \ell}\left(\frac{\kappa(\ell,t)}{\ell^2}\frac{\partial f}{\partial \ell}\right)_{\mathcal{M},J} - \lambda(\ell,t) f$$

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Key Quantities

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- $\lambda(\ell, t)$: Particle loss rate.

Inverse Problem

Data

Given a set of (noisy) observations

$$\mathcal{D} = \{((\ell_1, t_1), y_1), ((\ell_2, t_2), y_2), \cdots, ((\ell_n, t_n), y_n)\}$$
$$y(\ell, t) = f(\ell, t) + \epsilon$$
$$\epsilon \sim \mathcal{N}(0, \sigma^2)$$

Inference

The electron loss rate $\lambda(\ell, t)$

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Challenges

- Data sparsity & irregularity.

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Challenges

- Data sparsity & irregularity.
- Non-identifiable and Ill-posed
- Combine sparse data with reduced physics

- Parameterize $\kappa(\ell, t), \lambda(\ell, t)$

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- Fit a surrogate for the physical model
- Quantify likelihood of observed data conditioned on parameters of λ
- Perform MCMC based sampling

κ & λ

Space weather literature outlines a way to parameterize radial diffusion unknowns [Brautigam and Albert, 2000].

$$\kappa(\ell, t), \lambda(\ell, t) \sim \alpha \ell^{\beta} 10^{bKp(t)}$$

The parameters of κ are set to values already used in literature, we focus on inference of λ parameters only.

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Radial Basis Functions

Basis Expansion

The approximation to $f(\ell, t)$ is constructed as a radial basis expansion.

$$x = (\ell, t)$$
$$\hat{f}(x) = \sum_i w_i \phi(||x - x_i||/\rho_i)$$

Goodness of fit

The coefficients w_i are calculated by optimizing squared error with respect to observed data and *ghost points* where deviations from radial diffusion dynamics are penalized.

$$\mathcal{J}(w|\theta, \nu, x_i, f_i, g_j) = \frac{1}{2}\sum_i |\hat{f}(x_i) - y_i|^2 + \frac{1}{2}\nu \sum_j |\mathcal{L}_\theta[\hat{f}(x_j)]|^2$$

Likelihood

Multivariate normal distribution is used to specify the likelihood.

$$y_1, \cdots, y_n | \theta\{\lambda\} \sim \mathcal{N}\left(\begin{pmatrix} \hat{f}_\theta(x_1) \\ \vdots \\ \hat{f}_\theta(x_n) \end{pmatrix}, \begin{bmatrix} C(x_1, x_1) & \cdots & C(x_1, x_n) \\ \vdots & \ddots & \vdots \\ C(x_n, x_1) & \cdots & C(x_n, x_n) \end{bmatrix} \right)$$

$$C(x, y) = s^2 exp\left(-\frac{||x - y||^2}{2u^2} \right)$$

Synthetic phase space density data is generated using a discretized solver and assumed values of unknown parameters.

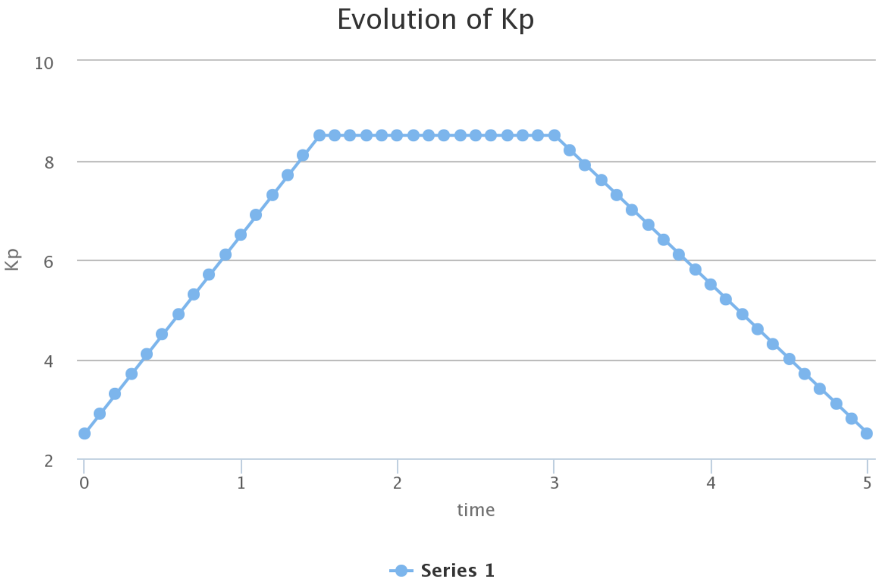


Figure: Kp Index

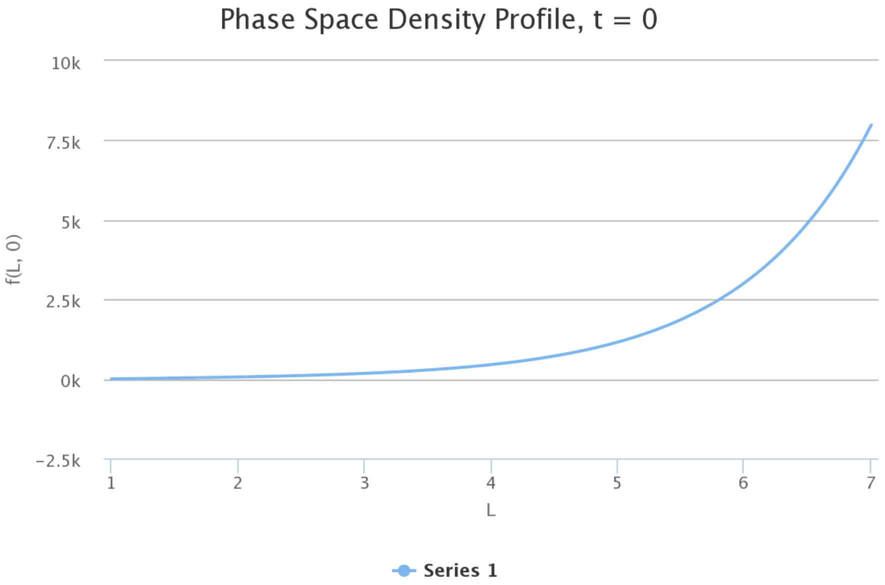
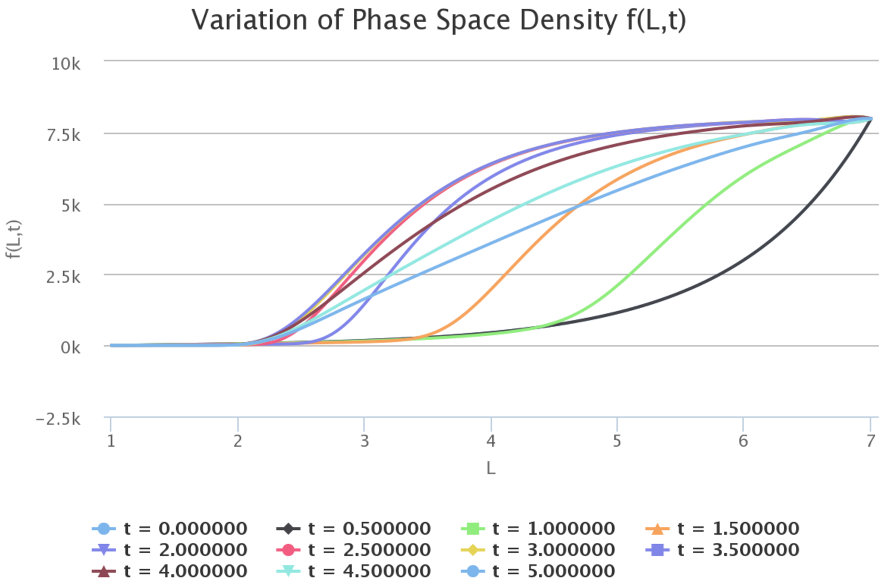


Figure: Initial Condition $f(\ell, 0)$

Generated profiles $f(\ell, t)$



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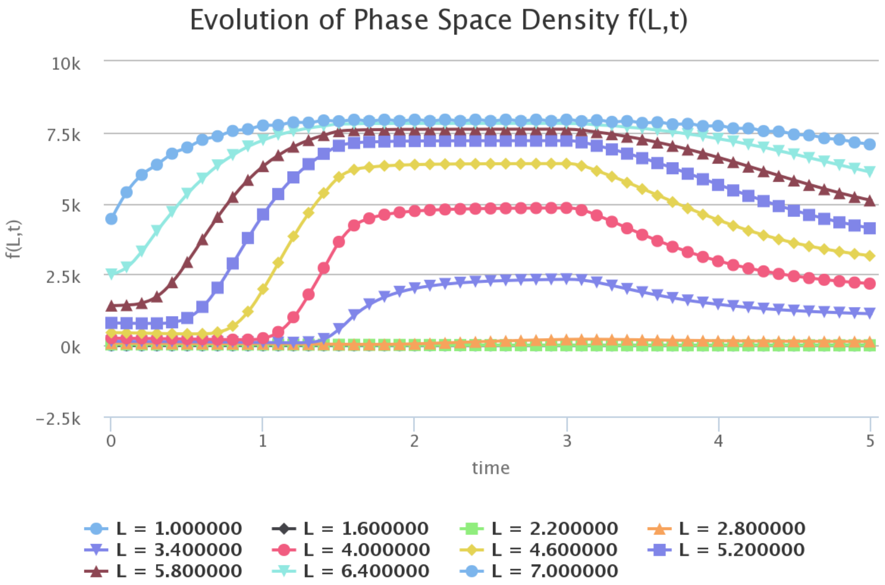

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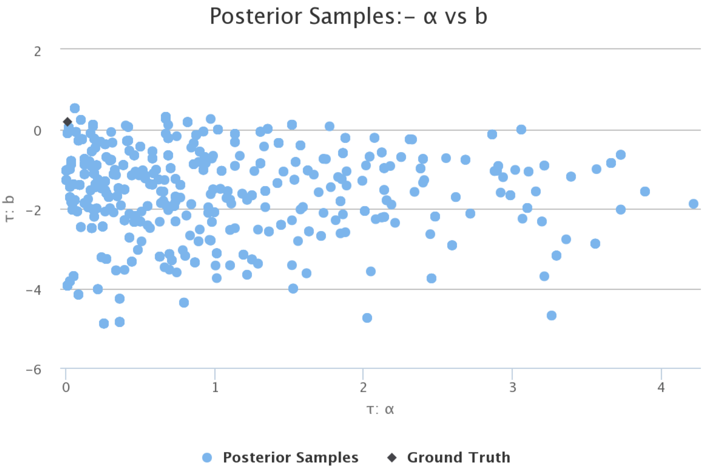

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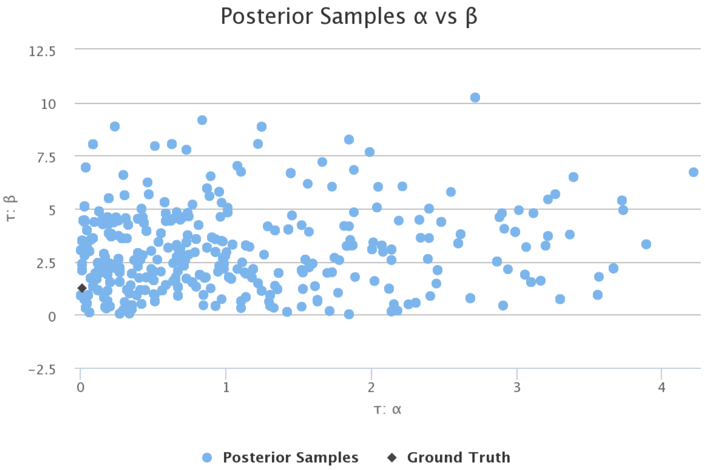

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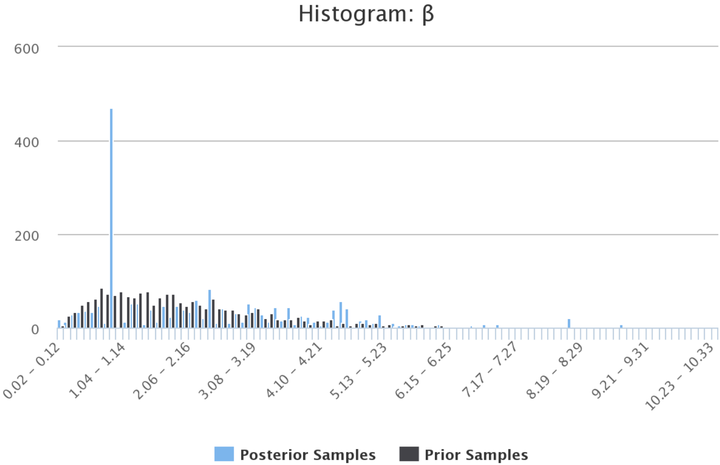

Summary

- Uncertainty creeps into deterministic physical systems.
- Need to intelligently combine Bayesian statistics with existing phyiscal models.
- Outlook
  - Decrease vagueness of posterior distribution by incorporating boundary/initial conditions.

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