Risk Analysis: Efficient Computation of Failure Probability

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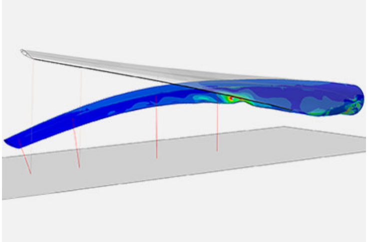
Motivation: Reliability

- v: Wind Speed [Random Variable]
- F_c: Critical Load [Scalar]
- Load on the Blade x: $\mathbf{F}(v,x)$ [Expensive Function]

Goal: Assess Blade Robustness

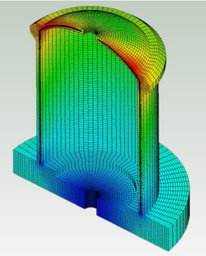
$$p_f = \mathbb{P}_v[\mathbf{F}(v,x) > F_c] < 10^{-6} \ ?$$

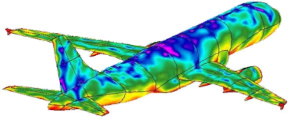


Blade x: Analysis

Damaged Wind Turbine

Motivation: Very Low Failure Probability





Industry Design Assessment

- Aeronautics
- Nuclear Power Plants

$\Longrightarrow \; p_f < 10^{-8}$

How to Compute p_f?

- Accurately
- Low Number of Evaluations [CFD/Structural Analysis Models]

Contents

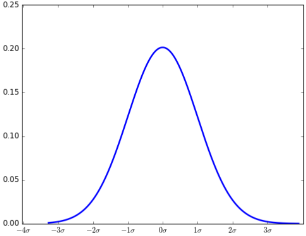
Section 1

Introduction

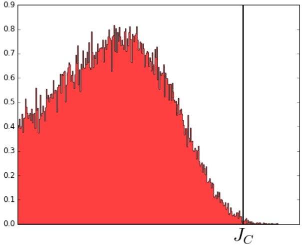
Motivation : Tail Probability

Failure Risk/Tail Probability : $\mathbb{P}(\mathbf{J} > J_C)$

- Quantity of Interest : $\mathbf{J} \ : \ \mathbb{R}^d \to \mathbb{R}$
- $J_C \in \mathbb{R}$ Critical value
- $\xi \in \mathbb{R}^d$ Random variable
- Goal : $\mathbb{P}(\mathbf{J}(\xi) > J_C)$



ξ pdf - Uncertain Input



$\mathbf{J}(\xi)$ pdf - Quantity of Interest

Basic Strategy: Monte-Carlo

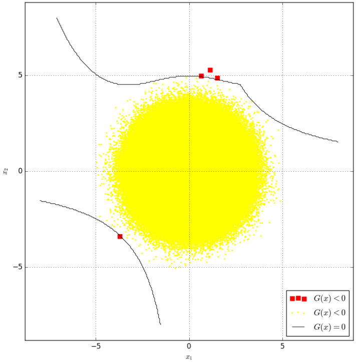
In the Standard Space

- $p_f = \mathbb{P}(G(\mathbf{X}) < 0) = \mathbb{E}[\mathbb{1}_{G<0}(\mathbf{X})]$
- Standard Space: $\mathbf{X} \sim \mathcal{N}(0, I_d)$
- $p_f \ll 1$
- Disconnected Failure Regions

Crude Monte-Carlo

$$\delta_{target} = \frac{\hat{\sigma}_f}{\hat{p}_f} = 1\%$$

$$\Longrightarrow \mathbf{N_{MC} \approx 10^{10}}!!$$



5×10^6 MC points [Expensive Computations]

$p_f = 9 \times 10^{-7} \Longrightarrow$ **4 Failure points...**

Section 2

Different Approaches

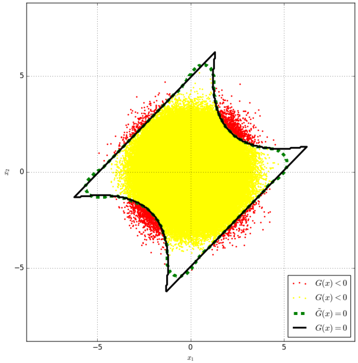
Metamodel Substitution

- **No Error Control**

Requirements

- Few Expensive Calls
- Different Failure Branches
- No points Clustering

$\mathbf{p_{\tilde{f}} \ll 1 \implies \tilde{N}_{MC}}$ **very high**



10^6 MC points [Cheap Computations]

Strategy 2: Metamodel Substitution + IS

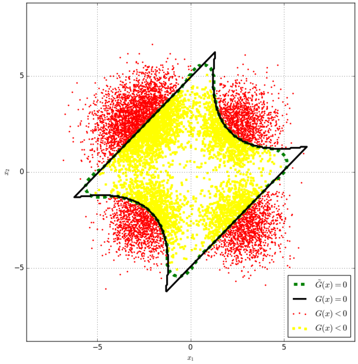
If $p_{\tilde{f}} \ll 1$

Metamodel Substitution + IS

- **No Error Control**
- Suitable for $p_{\tilde{f}} \ll 1$.

Requirements

- Few Expensive Calls
- Different Failure Branches
- No points Clustering
- Suitable for very Low probability



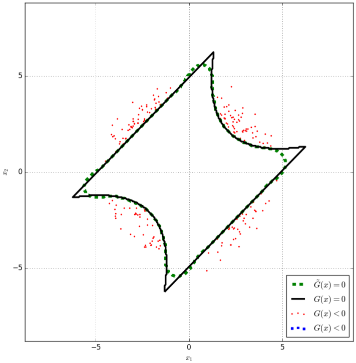
10000 IS points [Cheap Computations]

Strategy 3: Metamodel + (quasi-Optimal) IS

Metamodel + IS

- Construct a Metamodel $\tilde{G}$
- Define a quasi-optimal Importance Sampling Density (ISD) based on $\tilde{G}$
- Run performance function G with IS

Unbiased Estimation $\hat{p}_f$.



200 IS points [Expensive Computations]

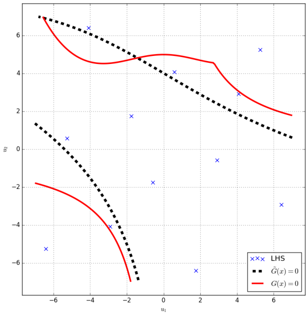
Section 3

Some Examples

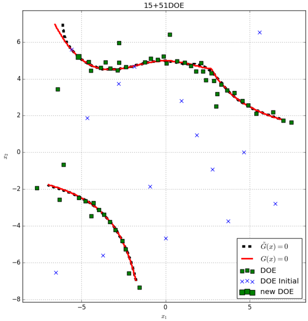
Metamodel Construction

2D Example

- Two Disconnected Failure Regions
- Low Failure probability: $p_f = 9 \times 10^{-7}$

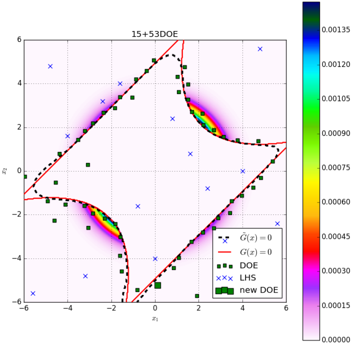


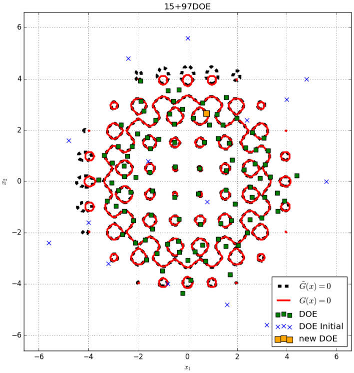
LHS Sampling in Standard Space



Refined Metamodel: $N_{calls} = 66$

Refined Metamodel: 2D examples

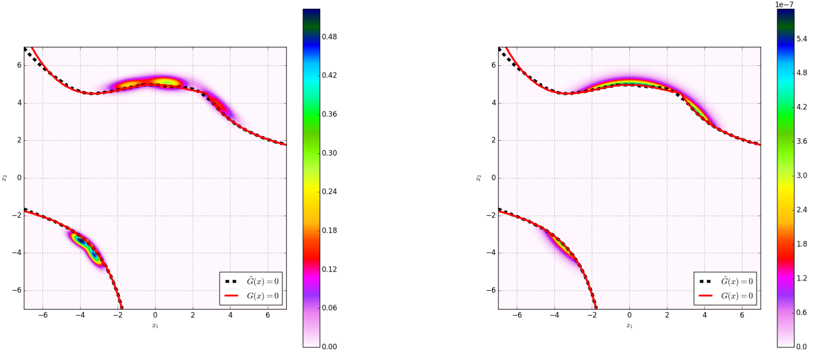




Refined Metamodel ; $N_{calls} = 68$; $p_f = 2.21 \times 10^{-3}$

Refined Metamodel ; $N_{calls} = 112$; $p_f = 7.43 \times 10^{-2}$

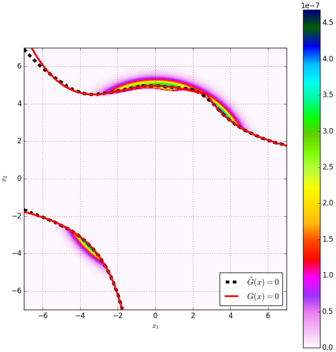
IS Metamodel: $\tilde{p}_f = \mathbb{E}[\mathbb{1}_{\tilde{G}<0}(\mathbf{X})]$



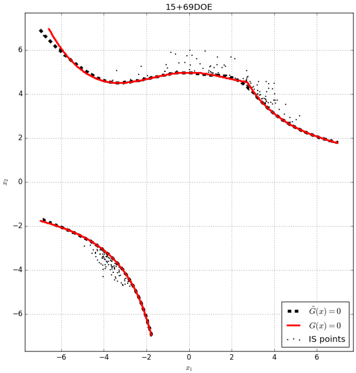
ISD used on Metamodel Contour ; $\tilde{N}_{calls} \sim 2 \times 10^6$; $\tilde{\delta} < 0.1\%$; $\tilde{p}_f = 9 \times 10^{-7}$

Optimal ISD Contour

Quasi-Optimal IS: $p_f = \mathbb{E}[\mathbb{1}_{G<0}(\mathbf{X})]$, using MCMC



Quasi-opt IS Contour



Sampled Points using MCMC ; $N_{calls} = 84 + 496$; $\delta < 1\%$

Section 4

Conclusion

Conclusion

Tail Probability

- Accurate Metamodel Refinement
- Suitable for very low p_f, multiple failure regions
- Sharp IS accuracy on metamodel
- Statistically consistent error
- Low number of performance function evaluations

Perspectives

- Parallel Metamodel Refinement
- Extreme Quantile Evaluation: $\mathbb{P}(\mathbf{J}(\xi) > q) = 10^{-6}$
- Optimization under Failure Probability Constraint

References

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- Echard B, Gayton N, Lemare M. AK-MCS: An active learning reliability method combining Kriging and Monte Carlo Simulation. *Reliability Engineering and System Safety*, **33 (2011) 145-154.**
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- Cadini F, Santos F, Zio E. An improved adaptive kriging-based importance technique for sampling multiple failure regions of low probability. *Reliability Engineering and System Safety*, **131 (2014) 109-117.**
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Isoprobabilistic Transformation T

From Physical Space to Standard Space

$p_f = \mathbb{P}(F(\mathbf{X}) > F_c)$, with $\mathbf{X} = (X_1, ..., X_d)$ independant.

- $J(\mathbf{X}) = F_c - F(\mathbf{X}) \implies p_f = \mathbb{P}(J(\mathbf{X}) < 0)$
- $\mathbf{U} = T(\mathbf{X}), \mathbf{U} \sim \mathcal{N}(0, I_d)$
- $T(\mathbf{x}) = \left[\phi^{-1} \circ F_{X_k}(x_k)\right]_{k \in [\![1,d]\!]}$
- $G = J \circ T^{-1} \implies G(\mathbf{U}) = J \circ T^{-1}(\mathbf{X}) = J(\mathbf{X})$

$\implies p_f = \mathbb{P}(G(\mathbf{U}) < 0)$

If **X** not independent, use Rosenblatt Transformation.

Two Failure Regions Example in 2D

Equation

$c = 5$, $\mathbf{X} = (X_1, X_2) \sim \mathcal{N}(0, I_2)$

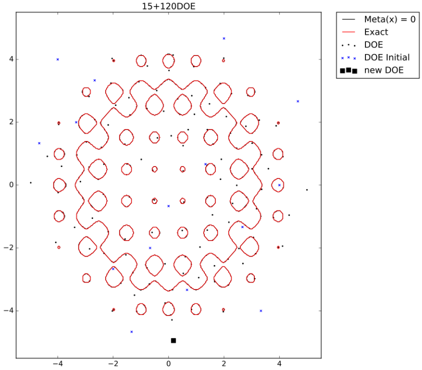
$$G(x_1, x_2) = \min \left\{ \begin{array}{c} c - 1 - x_2 + e^{\frac{-x_1^2}{10}} + \left(\frac{x_1}{5}\right)^4 \\ \frac{c^2}{2} - x_1 \cdot x_2 \end{array} \right\}$$

Method	N_{calls}	$\hat{p}_f$	$\hat{\delta}_f$	$3-\hat{\sigma}_f$ Interval	$\tilde{N}_{calls}$	$\hat{p}_{\tilde{G}}$	$\hat{\delta}_{p_{\tilde{G}}}$	$3-\hat{\sigma}_{p_{\tilde{G}}}$ Interval
Crude MC	$422,110,000$	9.48×10^{-7}	$< 5\%$	$\subseteq [8.06, 11.9] \times 10^{-7}$				
FORM	7	2.87×10^{-7}						
Subset	$700,000$	6.55×10^{-7}	$< 5\%$	$\subseteq [5.57, 7.53] \times 10^{-7}$				
Meta-IS	$40 + 2900$	9.17×10^{-7}	$< 5\%$	$\subseteq [7.80, 10.5] \times 10^{-7}$				
MetaAK-IS^2	$117 + 119$					8.16×10^{-7}	$< 5\%$	$\subseteq [6.94, 9.38] \times 10^{-7}$
MetaAL-OIS	$\mathbf{56 + 1000}$	$\mathbf{8.90 \times 10^{-7}}$	**1.23%**	$\mathbf{[8.58, 9.24] \times 10^{-7}}$	$\mathbf{1.62 \times 10^{6}}$	$\mathbf{9.18 \times 10^{-7}}$	**0.09%**	$\mathbf{[9.16, 9.21] \times 10^{-7}}$
Perf + IS	2.62×10^{6}	8.97×10^{-7}	*0.09%*	$[8.942, 8.996] \times 10^{-7}$				

Results

Metamodel $\tilde{G}$: Tricky Case

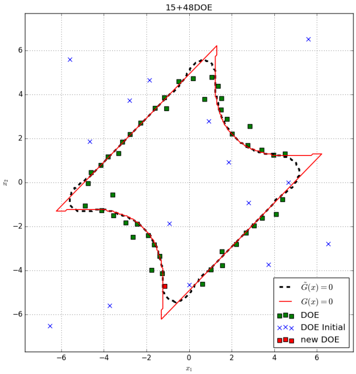
$G(x_1, x_2) =$
$10 - \sum_{i=1}^{2}(x_i^2 - 5\cos(2\pi x_i))$



Refined Metamodel: $N_{calls} = 135$

Metamodel $\tilde{G}$: Four Failure Case

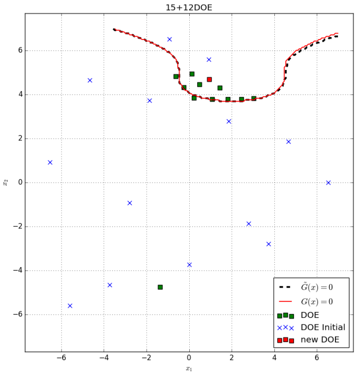
$$G(x_1, x_2) = \min \begin{cases} 3 + \frac{(x_1 - x_2)^2}{10} - \frac{x_1 + x_2}{\sqrt{2}} \\ 3 + \frac{(x_1 - x_2)^2}{10} + \frac{x_1 + x_2}{\sqrt{2}} \\ x_1 - x_2 + \frac{7}{\sqrt{2}} \\ -(x_1 - x_2) + \frac{7}{\sqrt{2}} \end{cases}$$



Refined Metamodel: $N_{calls} = 73$

Metamodel $\tilde{G}$: One Failure Case

$$G(x_1, x_2) = \frac{1}{2}(x_1 - 2)^2 - \frac{3}{2}(x_2 - 5)^3 - 3$$



Refined Metamodel: $N_{calls} = 27$

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