Efficient simulation schemes for some multidimensional stochastic volatility models

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Structure of the talk

1 Wishart and Affine processes on nonnegative symmetric matrices
2 Exact simulation of Wishart processes
3 Discretization schemes obtained by composition
4 High order discretization schemes for Wishart and Affine processes
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What is a Wishart process?

Wishart processes have initially been introduced and studied by Bru in her PhD thesis on Escherichia Coli (1987), and have recently been extended by Cuchiero, Filipovic, Mayerhofer and Teichmann (2009). A Wishart process \((X_t)_{t \geq 0}\) of dimension \(d\) is defined on nonnegative symmetric matrices \(S^+_d(\mathbb{R})\) and solves the following SDE:

\[
\begin{align*}
    dX_t &= (\alpha a^T a + bX_t + X_t b^T)dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}, \\ X_0 &= x \in S^+_d(\mathbb{R}).
\end{align*}
\]

Here, \(\alpha \in \mathbb{R}, a, b \in \mathcal{M}_d(\mathbb{R})\) and \(\sqrt{X_t}\) is the square root of the nonnegative matrix \(X_t\): if \(X_t = O_t \text{diag}(\Lambda^1_t, \ldots, \Lambda^d_t)O_t^{-1}\),

\[
\sqrt{X_t} := O_t \text{diag}(\sqrt{\Lambda^1_t}, \ldots, \sqrt{\Lambda^d_t})O_t^{-1}.
\]

\(W_t\) denotes a \(d \times d\) matrix whose components are independent standard Brownian motions.

\[d = 1\textbf{ CIR diffusion}: \] \(dX_t = (\alpha a^2 + 2bX_t)dt + 2a\sqrt{X_t}dW_t, t \geq 0.\)
When is it well-defined?

We have the following results (Bru, Cuchiero et al., Mayerhofer et al.):

- When $\alpha \geq d + 1$, the SDE has a unique strong solution on the positive symmetric matrices $S_d^{+,*}(\mathbb{R})$.
- When $d - 1 \leq \alpha < d + 1$, the SDE has a unique weak solution on $S_d^+(\mathbb{R})$.

When $d = 1$, $\alpha \geq 2$ ensures that the CIR never reaches 0. However, we know in that case that there is a strong solution for any $\alpha \geq 0$. 
An explicit characteristic function

Let \( X_t^x \sim WIS_d(x, \alpha, b, a; t) \) follow a Wishart distribution. Its Fourier transform is known explicitly:

\[
\forall \nu \in S_d(\mathbb{R}), \quad \mathbb{E}[\exp(i \text{Tr}(\nu X_t^x))] = \frac{\exp(\text{Tr}[i \nu (I_d - 2iq_t \nu)^{-1} m_t m_T^T])}{\det(I_d - 2iq_t \nu)^{\alpha/2}},
\]

where \( q_t = \int_0^t \exp(sb)a^T a \exp(sb^T) ds, \) \( m_t = \exp(tb). \)

In particular, if \( \tilde{X}_t^x \sim WIS_d(x, \alpha, 0, I_n^d; t), \) where \((I_n^d)_{i,j} = 1_{i=j \leq n},\)

\[
\forall \nu \in S_d(\mathbb{R}), \quad \mathbb{E}[\exp(i \text{Tr}(\nu \tilde{X}_t^x))] = \frac{\exp(\text{Tr}[i \nu (I_d - 2itI_n^d \nu)^{-1} x])}{\det(I_d - 2itI_n^d \nu)^{\alpha/2}}.
\]
Affine diffusions on nonnegative symmetric matrices

Cuchiero et al. consider the following dynamics for General Affine processes (we exclude here jumps):

\[ dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_t a + a^T dW_t^T \sqrt{X_t}, X_0 = x \in S_d(\mathbb{R}). \quad (3) \]

\( \bar{\alpha} \in S_d(\mathbb{R}), a \in M_d(\mathbb{R}) \) and \( B : S_d(\mathbb{R}) \to S_d(\mathbb{R}) \) is a linear mapping such that \( \text{Tr}(B(x)z) \geq 0 \) if \( \text{Tr}(xz) = 0 \) for \( x, z \in S_d^+(\mathbb{R}) \).

The Wishart SDE (1) is the particular case where

\[ \bar{\alpha} = \alpha a^T a, \quad B(x) = bx + xb^T. \]

- If \( \bar{\alpha} - (d + 1)a^T a \in S_d^+(\mathbb{R}) \), (3) has a unique strong solution.
- If \( \bar{\alpha} - (d - 1)a^T a \in S_d^+(\mathbb{R}) \), (3) has a unique weak solution.

The characteristic function of \( X_t \) can be obtained by solving ODEs.
An application of Wishart processes in finance

Gourieroux and Sufana (2004) consider the following dynamics for \( d \) assets:

\[
\begin{pmatrix}
\log(S^1_t) \\
\vdots \\
\log(S^d_t)
\end{pmatrix}
\begin{pmatrix}
\log(S^1_t) \\
\vdots \\
\log(S^d_t)
\end{pmatrix} = \left( r - \begin{pmatrix}
(X_t)_{1,1}/2 \\
\vdots \\
(X_t)_{d,d}/2
\end{pmatrix} \right) dt + \sqrt{X_t} dZ_t,
\]

(4)

where \( X_t \) solves (1) and \((Z_t, t \geq 0)\) is a \( d \)-dimensional Brownian motion independent of \((W_t, t \geq 0)\), and \( D_i \in M_d(\mathbb{R}) \).

- \( d = 1 \): Heston model without correlation.
- The characteristic function of \((\log(S^1_t), \ldots, \log(S^d_t))^T\) can be calculated by solving ODEs.
- Dependence between \( W \) and \( Z \) has been considered in Da Fonseca and al. (2008) to keep the Affine structure.
- \( X_t \) is the instantaneous covariance matrix, i.e.
  \[
  \langle d \log(S^i_t), d \log(S^j_t) \rangle = (X_t)_{i,j} dt.
  \]
Existing results on the simulation of Wishart processes

**Exact simulation.** To the best of our knowledge, exact simulation algorithms for Wishart distribution only exist in the literature for integer degrees $\alpha \in \mathbb{N}, \alpha \geq d - 1$ (Odell and Feiveson (1966) and Gleser (1975)).

**Approximation schemes.**

- the Euler-Maruyama scheme is not well-defined exactly as for the CIR process.
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2. Exact simulation of Wishart processes

3. Discretization schemes obtained by composition

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6. A mean-reverting SDE on correlation matrices
A first simple remark

We consider two SDEs \( X_{t,x}^1 \) and \( X_{t,x}^2 \) associated respectively to the operators \( L_1 \) and \( L_2 \) and defined on the same domain \( \mathbb{D} \). We assume that:

\[
L_1 L_2 = L_2 L_1.
\]

Then,

\[
\mathbb{E}[f(X_{t,x}^1, X_{t,x}^2, x)] = \mathbb{E}[\mathbb{E}[f(X_{t,x}^1, X_{t,x}^2, x)|X_{t,x}^2, x]] = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathbb{E}[L_1^k f(X_{t,x}^2, x)] = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{t^{k+l}}{k!l!} L_2^l L_1^k f(x) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (L_1 + L_2)^k f(x).
\]

Thus, if we have exact schemes for \( L_1 \) and \( L_2 \), then we have an exact scheme for \( L_1 + L_2 \) simply by taking the composition of both schemes.
The extended Cholesky decomposition

Let \( q \in S^+_d(\mathbb{R}) \) be a matrix with rank \( r \). Then there is a permutation matrix \( p \), an invertible lower triangular matrix \( c_r \in G_r(\mathbb{R}) \) and \( k_r \in M_{d-r \times r}(\mathbb{R}) \) such that:

\[
pqp^T = cc^T, \quad c = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix}.
\]

The triplet \((c_r, k_r, p)\) is called an extended Cholesky decomposition of \( q \). Besides, \( \tilde{c} = \begin{pmatrix} c_r & 0 \\ k_r & I_{d-r} \end{pmatrix} \in G_d(\mathbb{R}) \), and we have:

\[
q = (\tilde{c}^T p)^T I_d^r \tilde{c}^T p,
\]

where \((I_d^r)_{i,j} = 1_{i=j \leq r} \).
Reduction to the case $b = 0, a = I^n_\alpha$

We use the characteristic function (2). Let $n = \text{Rk}(q_t)$. There is $\theta_t \in \mathcal{G}_d(\mathbb{R})$ such that $q_t/t = \theta_t I^n_d \theta_t^T$. 

\[
\begin{align*}
\det(I_d - 2iq_t \nu) &= \det(\theta_t (\theta_t^{-1} - 2iI^n_d \theta_t^T \nu)) = \det(I_d - 2iI^n_d \theta_t^T \nu \theta_t), \\
\text{Tr}[i\nu(I_d - 2iq_t \nu)^{-1} m_t x m_t^T] &= \text{Tr}[i(\theta_t^{-1})^T \theta_t^T \nu (\theta_t \theta_t^{-1} - 2i\theta_t I^n_d \theta_t^T \nu \theta_t \theta_t^{-1})^{-1} m_t x m_t^T] \\
&= \text{Tr}[i\theta_t^T \nu \theta_t (I_d - 2iI^n_d \theta_t^T \nu \theta_t)^{-1} \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T].
\end{align*}
\]

\[
\implies \mathbb{E}[\exp(i\text{Tr}(\nu X_t))] = \mathbb{E}[\exp(i\text{Tr}(\theta_t^T \nu \theta_t^T \tilde{X}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T}))] = \\
\mathbb{E}[\exp(i\text{Tr}(\nu \theta_t \tilde{X}_t^{\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T} \theta_t^T))], \text{ i.e.}
\]

\[
\text{WIS}_d(x, \alpha, b, a; t) = \theta_t \text{WIS}_d(\theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T, \alpha, 0, I^n_d; t) \theta_t^T.
\] (5)

It is therefore sufficient to simulate exactly a Wishart process with $a = I^n_\alpha$ and $b = 0.$
A remarkable splitting when $a = I^n_d$ and $b = 0$

The infinitesimal generator of a Wishart process with $b = 0$ and $a = I^n_d$ is:

$$L = \text{Tr}(\alpha I^n_d D) + 2\text{Tr}(xDI^n_d D), \text{ with } D_{i,j} = \partial_{i,j}$$

for $f : \mathcal{M}_d(\mathbb{R}) \to \mathbb{R}$ s.t. $f(x) = f(x^T)$ for $1 \leq i, j \leq d$.

$I^n_d = \sum_{i=1}^n e^i_d$ with $(e^i_d)_{k,l} = 1_{k=l=i}$. We set $L_i = \text{Tr}(\alpha e^i_d D) + 2\text{Tr}(xDe^i_d D)$.

**Proposition 1**

$$L = L_1 + \cdots + L_n, \text{ with } L_i L_j = L_j L_i, \text{ and where}$$

- $L_i$ is the same operator as $L_1$ by permuting $i$th and first coordinates.
- $L_1$ is the operator of a Wishart process with $b = 0$ and $a = I^1_d$, which is thus well defined on $S_d^+ (\mathbb{R})$.

**Consequence:** It is sufficient to sample an exact scheme for $L_1$ to get an exact scheme for $L$. This can be done!
Exact scheme for $L_1$ when $d = 2$ ($\alpha \geq d - 1 = 1$)

For $f : S_d(\mathbb{R}) \to \mathbb{R}$, $L_1 f(x) = \alpha \partial_{\{1,1\}} f(x) + 2x_{1,1} \partial^2_{\{1,1\}} f(x) + 2x_{1,2} \partial_{\{1,1\}} \partial_{\{1,2\}} f(x) + \frac{x_{2,2}}{2} \partial^2_{\{1,2\}} f(x)$. It is associated to the following SDE when $(X_0)_{2,2} > 0$

$$d(X_t)_{1,1} = \alpha dt + 2 \sqrt{(X_t)_{1,1}} dB^1_t + 2 \frac{(X_t)_{1,2}}{\sqrt{(X_t)_{2,2}}} dB^2_t$$
$$d(X_t)_{1,2} = \sqrt{(X_t)_{2,2}} dB^2_t, (X_t)_{2,1} = (X_t)_{1,2},$$
$$d(X_t)_{2,2} = 0$$

and if $(X_0)_{2,2} = 0$:

$$d(X_t)_{1,1} = \alpha dt + 2 \sqrt{(X_t)_{1,1}} dB^1_t, d(X_t)_{1,2} = d(X_t)_{2,2} = 0.$$ 

In the second case: CIR that can be simulated exactly (e.g. Glasserman) In the first case, we set $U_t = (X_t)_{1,1} - ((X_t)_{1,2})^2 / (X_t)_{2,2}$:

$$dU_t = (\alpha - 1) dt + \sqrt{U_t} dB^1_t : \text{CIR indep. of } (X_t)_{1,2} \sim \mathcal{N}((X_0)_{1,2}, (X_0)_{2,2} t).$$
Exact scheme for $L_1$ when $d \geq 3$ ($\alpha \geq d - 1$)

Up to a permutation, $(x)_{2 \leq i, j \leq d} = \begin{pmatrix} c_r & 0 \\ k_r & 0 \end{pmatrix} \begin{pmatrix} c_r^T \\ k_r^T \end{pmatrix} =: cc^T$.

We can show that $L_1$ is the generator of the SDE:

\[
\begin{align*}
    d(X^x_t)_{1,1} &= \alpha dt + 2\sqrt{(X^x_t)_{1,1} - \sum_{k=1}^r \left( \sum_{l=1}^r (c^{-1}_r)_{k,l} (X^x_t)_{1,l+1} \right)^2} dZ^1_t \\
    &\quad + 2 \sum_{k=1}^r \sum_{l=1}^r (c^{-1}_r)_{k,l} (X^x_t)_{1,l+1} dZ^{k+1}_t \\
    d(X^x_t)_{1,i} &= \sum_{k=1}^r c_{i-1,k} dZ^{k+1}_t = d(X^x_t)_{i,1}, \; i = 2, \ldots, d \\
    d(X^x_t)_{l,k} &= 0, \text{ for } 2 \leq k, l \leq d.
\end{align*}
\]

The SDE associated to $L_1$ can be solved explicitly as for $d = 2$, and requires the sampling of 1 CIR distribution and $r - 1$ standard Gaussian variables that are independent:
Exact scheme for $L_1$ when $d \geq 3$ ($\alpha \geq d - 1$) II

$$X_t^x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r & 0 \\ 0 & k_r & I_{d-r-1} \end{pmatrix} \times \begin{pmatrix} (U_t^u)_{1,1} + \sum_{k=1}^r ((U_t^u)_{1,k+1})^2 & ((U_t^u)_{1,l+1})^T_{1 \leq l \leq r} & 0 \\ ((U_t^u)_{1,l+1})_{1 \leq l \leq r} & I_r & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_r^T & k_r^T \\ 0 & 0 & I_{d-r-1} \end{pmatrix},$$

where

$$d(U_t^u)_{1,1} = (\alpha - r) dt + 2\sqrt{(U_t^u)_{1,1}} dZ_t^1,$$

$$u_{1,1} = x_{1,1} - \sum_{k=1}^r (u_{1,k+1})^2 \geq 0,$$

$$d((U_t^u)_{1,l+1})_{1 \leq l \leq r} = (dZ_t^{l+1})_{1 \leq l \leq r},$$

$$(u_{1,l+1})_{1 \leq l \leq r} = c_r^{-1}(x_{1,l+1})_{1 \leq l \leq r}. \quad (7)$$
Remarks on the exact scheme

- when the initial value $x \in S_d^+;^*(\mathbb{R})$, we only make usual Cholesky decompositions.

- For $x = 0$, $b = 0$ and $t = 1$, our exact scheme gives back the Bartlett’s decomposition (1933):

  $$
  \begin{pmatrix}
    (L_{i,j})_{1 \leq i,j \leq n} & 0 \\
    0 & 0
  \end{pmatrix}
  \begin{pmatrix}
    (L_{i,j}^T)_{1 \leq i,j \leq n} & 0 \\
    0 & 0
  \end{pmatrix}
  \sim WIS_d(0, \alpha, 0, I_d^n; 1),
  $$

  where $L_{i,j} \sim \mathcal{N}(0, 1)$, $i > j$ and $(L_{i,i})^2 \sim \chi^2(\alpha - i + 1)$ are independent and $L_{i,j} = 0$ for $i < j$.

- The exact scheme has a complexity of $O(d^4)$ operations ($n \leq d$ Cholesky), $O(d^2)$ Gaussian samples, $O(d)$ CIR samples. [A totally different exact scheme in $O(d^3)$ is possible for $\alpha \geq 2d - 1$]
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Assumptions

\[ t \geq 0, \quad X_t^x = x + \int_0^t b(X_s^x)ds + \int_0^t \sigma(X_s^x)dB_s. \]

Assumption : domain \( \mathbb{D} \subset \mathbb{R}^{\zeta}, \forall x \in \mathbb{D}, \mathbb{P}(\forall t \geq 0, X_t^x \in \mathbb{D}) = 1 ; b_i(x), (\sigma(x)\sigma^T(x))_{i,j} \in C^\infty_{\text{pol}}(\mathbb{D}). \)

\[
C^\infty_{\text{pol}}(\mathbb{D}) = \{ f \in C^\infty(\mathbb{D}, \mathbb{R}), \forall \gamma \in \mathbb{N}^{\zeta}, \exists C_\gamma > 0, e_\gamma \in \mathbb{N}^*, \forall x \in \mathbb{D},
|\partial_\gamma f(x)| \leq C_\gamma (1 + \|x\|^{e_\gamma}) \}
\]

Associated operator :

\[
Lf(x) = \sum_{i=1}^\zeta b_i(x) \partial_i f(x) + \frac{1}{2} \sum_{i,j,k=1}^\zeta \sigma_{i,k}(x)\sigma_{j,k}(x) \partial_i \partial_j f(x).
\]

Rem : \( f \in C^\infty_{\text{pol}}(\mathbb{D}) \implies Lf \in C^\infty_{\text{pol}}(\mathbb{D}). \)
Notations for discretization schemes

**Definition 2**

A family of transition probabilities $(\hat{p}_x(t)(dz), t > 0, x \in \mathbb{D})$ on $\mathbb{D}$ is s.t. $\hat{p}_x(t)$ is a probability law on $\mathbb{D}$ for $t > 0$ and $x \in \mathbb{D}$. We note $\hat{X}_t^x$ a r.v. with law $\hat{p}_x(t)(dz)$.

Associated discretization scheme: $(\hat{X}_{t_i}^n, 0 \leq i \leq n)$ sequence of $\mathbb{D}$-valued r.v. s.t. $\hat{X}_{t_{i+1}}^n$ is sampled according to $\hat{p}_{\hat{X}_{t_i}^n}(T/n)(dz)$.

Example (Euler): $\hat{X}_t^x = x + b(x)t + \sigma(x)W_t$, $\hat{p}_x(t)$: law density of $\hat{X}_t^x$. 
Talay-Tubaro Theorem (1990)

If

- \( f : \mathbb{D} \to \mathbb{R} \) s. t. \( u(t, x) = \mathbb{E}[f(X^x_{T-t})] \) is defined on \([0, T] \times \mathbb{D}\), solves for \( t \in [0, T], x \in \mathbb{D}, \partial_t u(t, x) = -Lu(t, x) \), and has “good bounds” on all its derivatives \( \partial^l \partial^\gamma u \), i.e.

\[
\forall l \in \mathbb{N}, \gamma \in \mathbb{N}^c, \exists C_{l, \gamma}, e_{l, \gamma} > 0, \forall x \in \mathbb{D}, t \in [0, T], |\partial^l \partial^\gamma u(t, x)| \leq C_{l, \gamma}(1 + ||x||^{e_{l, \gamma}}).
\]

- the scheme is a potential weak \( \nu \)th-order discr. scheme for \( L \):

\[
\mathbb{E}[f(\hat{X}_t^x)] \quad \text{as} \quad t \to 0^+ \quad f(x) + \sum_{k=1}^{\nu} \frac{1}{k!} t^k L^k f(x) + \text{Remainder in } t^{\nu+1}
\]

and \( (\hat{X}_t^x, i = 0, \ldots, n) \) has uniformly bounded moments.

then, \( |\mathbb{E}[f(\hat{X}_{t_i}^n)] - \mathbb{E}[f(X^x_T)]| \leq K/n^\nu \).
Composition of discretization schemes I

\[ \hat{p}_x^1(t)(dz), \hat{p}_x^2(t)(dz) : \text{potential } \nu\text{-th-order schemes for } L_1, L_2. \]

\[ \hat{p}^2(\lambda_2 t) \circ \hat{p}_x^1(\lambda_1 t)(dz) = \int_{\mathbb{D}} \hat{p}_y^1(\lambda_2 t)(dz) \hat{p}_x^1(\lambda_1 t)(dy) : \text{scheme that amounts to first use the scheme 1 with a time step } \lambda_1 t \text{ and then the scheme 2 with a time step } \lambda_2 t. \]

\[ \hat{X}_{\lambda_2 t, \lambda_1 t}^{2 \circ 1, x} \text{ a r.v. with this law.} \]

**Proposition 3**

\[
E[f(\hat{X}_{\lambda_2 t, \lambda_1 t}^{2 \circ 1, x})] = \lim_{t \to 0^+} \sum_{l_1 + l_2 \leq \nu} \frac{\lambda_1^{l_1} \lambda_2^{l_2} t^{l_1+l_2} L_1^{l_1} L_2^{l_2} f(x)}{l_1! l_2!} + \text{Remainder}
\]

\[
(= [I + \lambda_1 t L_1 f + \ldots + \frac{(\lambda_1 t)^\nu}{\nu!} L_1^\nu f][I + \lambda_2 t L_2 f + \ldots + \frac{(\lambda_2 t)^\nu}{\nu!} L_2^\nu f] + \text{Rem})
\]

Csq : a scheme acts on \( f \) "as" the operator \( I + tLf + \ldots + \frac{t^\nu}{\nu!} L_2^\nu f + \text{Rem} \).

**Composition of schemes = Composition of operators.**
Composition of discretization schemes II

**Corollary 4**

If \( \hat{p}_x^1 \) and \( \hat{p}_x^2 \) are potential \( \nu \)th-order schemes for \( L_1, L_2 \) and \( L_1L_2 = L_2L_1 \),
\[ \hat{p}^1(t) \circ \hat{p}_x^2(t) \] is a potential \( \nu \)th-order schemes for \( L_1 + L_2 \).

**Corollary 5**

\( \hat{p}_x^1, \hat{p}_x^2 \) : potential 2nd order schemes for \( L_1, L_2 \). Then,
\[ \hat{p}^2(t/2) \circ \hat{p}^1(t) \circ \hat{p}_x^2(t/2) \] (Strang 1968)
\[ \frac{1}{2} (\hat{p}^2(t) \circ \hat{p}_x^1(t) + \hat{p}^1(t) \circ \hat{p}_x^2(t)) \] (8)

are potential second order schemes for \( L_1 + L_2 \).

Proof for (9) : \( (I + tL_1 + t^2/2L_1^2 + ...) (I + tL_2 + t^2/2L_2^2 + ...) = I + t(L_1 + L_2) + t^2/2(L_1^2 + L_2^2 + 2L_1L_2) + ... \)
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High order schemes for the operator $L_1$

**Result:** If we replace in the exact scheme for $L_1$ (7),

- $(U_{t}^u)_{1,1}$ by $(\hat{U}_{t}^u)_{1,1}$ sampled with a potential $\nu$th-order scheme for the CIR: $dU_{t} = (\alpha - r)dt + 2\sqrt{U_{t}}dZ_{t}$,

- $(U_{t}^u)_{1,l+1}$ by $\sqrt{l}\hat{G}^{l+1}$ where $\hat{G}^{l+1}$ is a bounded variable s.t. $\forall k \leq 2\nu + 1, \mathbb{E}[(\hat{G}^{l+1})^k] = \mathbb{E}[G^k]$, where $G \sim \mathcal{N}(0, 1)$,

we can show that we get a potential $\nu$th order scheme for $L_1$.

Second and third order schemes for the CIR can be found in A. 2008. Here are some matching-moment variables for $\mathcal{N}(0, 1)$ for $\nu = 2, 3$:

\[
\mathbb{P}(\hat{G}^i = \sqrt{3}) = \mathbb{P}(\hat{G}^i = -\sqrt{3}) = \frac{1}{6} \text{ and } \mathbb{P}(\hat{G}^i = 0) = \frac{2}{3}
\]

(resp. $\mathbb{P} \left( \hat{G}^i = \varepsilon \sqrt{3 + \sqrt{6}} \right) = \frac{\sqrt{6} - 2}{4\sqrt{6}}$, $\mathbb{P} \left( \hat{G}^i = \varepsilon \sqrt{3 - \sqrt{6}} \right) = \frac{1}{2} - \frac{\sqrt{6} - 2}{4\sqrt{6}}$, $\varepsilon \in \{-1, 1\}$).
A third order scheme for Wishart processes

We use once again the splitting given by Proposition 1.

- We have a third order scheme for $L_1$.
- By a permutation of the first and $i^{th}$ coordinate, we get also a third order scheme for $L_i$.
- By Corollary 4, we get a third order scheme $\hat{X}_t^x$ for a Wishart process with $a = I^n_d$ and $b = 0$.
- Last, we can show from (5) (under some assumptions) that $\theta_t \hat{X}_t \theta_t^{-1} m_t x m_t^T (\theta_t^{-1})^T \theta_t^T$ is a third order scheme.
Second order scheme for general Affine processes I

A First remark:

Let \( dX_t = (\bar{\alpha} + B(X_t))dt + \sqrt{X_t}dW_t a + a^TdW_t^T \sqrt{X_t} \), \( X_0 = x \in S_d(\mathbb{R}) \).

There is \( u \in \mathcal{G}_d(\mathbb{R}) \) such that

\[
(X_t)_{t \geq 0} = (u^T \tilde{X}_t u)_{t \geq 0},
\]

where

- \( d\tilde{X}_t = (\tilde{\delta} + B_u(\tilde{X}_t))dt + \sqrt{\tilde{X}_t}dW_t I_d^n + I_d^n dW_t^T \sqrt{\tilde{X}_t} \), \( \tilde{X}_0 = (u^{-1})^T x u^{-1} \),

- \( B_u(x) = (u^{-1})^T B(u^T x u) u^{-1} \),

- \( \tilde{\delta} \) is a diagonal matrix such that \( \tilde{\delta} - (d - 1)I_d^n \in S_d^+(\mathbb{R}) \)

\[\implies\] It is sufficient to get a scheme for \( \tilde{X}_t \) (i.e. when \( a = I_d^n \) and \( \bar{\alpha} \) is a diagonal matrix).
Second order scheme for general Affine processes II

Let $\delta_{\text{min}} = \min_{i=1,...,n} \delta_{i,i} \geq d - 1$. We split the generator of $\tilde{X}_t$:

$$L = \text{Tr}(\lbrack \delta + B(x) \rbrack D^S) + 2\text{Tr}(xD^S I_d^m D^S)$$

$$= \underbrace{\text{Tr}(\lbrack \delta - \delta_{\text{min}} I_d^m + B_u(x) \rbrack D^S)}_{L_{\text{ODE}}} + \delta_{\text{min}} \text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^m D^S),$$

where $L_{\text{ODE}}$ is associated to the affine ODE $x'(t) = \delta - \delta_{\text{min}} I_d^m + B_u(x(t))$ that can be solved explicitly and is such that $x(t) \in S_d^+ (\mathbb{R})$ for $t \geq 0$. By Corollary 5, we get a second order scheme for $\tilde{X}_t$ and thus for $X_t$. 
A faster second order scheme when $\bar{\alpha} - dI_d^n \in S_d^+(\mathbb{R})$

All the previous schemes rely on the splitting given by Proposition 1 and require thus $O(d^4)$ operations.

**Remark**: We can check that if $c^Tc = x$, $(c + W_tI_d^n)^T(c + W_tI_d^n)$ is a Wishart process with $\alpha = d$, $a = I_d^n$, $b = 0$ starting from $x$. Also, $(c + \sqrt{t}\hat{G}I_d^n)^T(c + \sqrt{t}\hat{G}I_d^n)$ is a potential second order scheme for $WIS_d(x, d, 0, I_d^n)$ where $\hat{G}$ is a matrix with independent elements sampled according to (10).

**Consequence**: By using the splitting:

$$L = \underbrace{\text{Tr}([\bar{\delta} - dI_d^n + B_u(x)]D^S)}_{\tilde{L}_{ODE}} + \underbrace{d\text{Tr}(D^S) + 2\text{Tr}(xD^S I_d^n D^S)}_{L_{WIS_d}(x, d, 0, I_d^n)},$$

we get a by Corollary 5 a second order scheme for $\tilde{X}_t$ in $O(d^3)$ operations.
1. Wishart and Affine processes on nonnegative symmetric matrices

2. Exact simulation of Wishart processes

3. Discretization schemes obtained by composition

4. High order discretization schemes for Wishart and Affine processes

5. **Numerical results**

6. A mean-reverting SDE on correlation matrices
A modified Euler scheme

The Euler scheme for the Affine diffusion (3) is:

\[ \hat{X}_{t_{i+1}} = \hat{X}_t + (\bar{\alpha} + B(\hat{X}_t))(t_{i+1} - t_i) + \sqrt{\hat{X}_t} (W_{t_{i+1}} - W_t) a + a^T (W_{t_{i+1}} - W_t)^T \sqrt{\hat{X}_t}, \]

It is not well-defined since \( \hat{X}_{t_{i+1}} \) may not be nonnegative.

Corrected Euler scheme:

\[
\hat{X}_{t_{i+1}} = \hat{X}_t + (\bar{\alpha} + B(\hat{X}_t))(t_{i+1} - t_i) + \sqrt{(\hat{X}_t)^+} (W_{t_{i+1}} - W_t) a + a^T (W_{t_{i+1}} - W_t)^T \sqrt{(\hat{X}_t)^+},
\]

where \( \sqrt{x^+} := \text{odiag}(\sqrt{\lambda_1^+}, \ldots, \sqrt{\lambda_d^+})o^{-1} \) for \( x \in S_d(\mathbb{R}) \) and \( x = \text{odiag}(\lambda_1, \ldots, \lambda_d^+)o^{-1} \).
# Simulation of processes on matrices

## Numerical results

### A time comparison (10⁶ samples, N time-steps)

<table>
<thead>
<tr>
<th>Schemes</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact (1 step)</td>
<td>−0.526852</td>
<td>−0.227962</td>
<td>12</td>
<td>−0.526486</td>
<td>−0.229078</td>
<td>125</td>
</tr>
<tr>
<td>2\text{nd} order bis</td>
<td>−0.526229</td>
<td>−0.228663</td>
<td>41</td>
<td>−0.526574</td>
<td>−0.228133</td>
<td>229</td>
</tr>
<tr>
<td>2\text{nd} order</td>
<td>−0.526577</td>
<td>−0.228923</td>
<td>76</td>
<td>−0.526574</td>
<td>−0.228133</td>
<td>229</td>
</tr>
<tr>
<td>3\text{rd} order</td>
<td>−0.527021</td>
<td>−0.227286</td>
<td>82</td>
<td>−0.527613</td>
<td>−0.228376</td>
<td>244</td>
</tr>
<tr>
<td>Exact (N steps)</td>
<td>−0.526963</td>
<td>−0.228303</td>
<td>123</td>
<td>−0.526891</td>
<td>−0.227729</td>
<td>369</td>
</tr>
<tr>
<td>Corrected Euler</td>
<td>−0.525627*</td>
<td>−0.233863*</td>
<td>225</td>
<td>−0.525638*</td>
<td>−0.231449*</td>
<td>687</td>
</tr>
</tbody>
</table>

\begin{align*}
\alpha &= 3.5, d = 3, \Delta_R = 10^{-3}, \Delta_{Im} = 10^{-3}, \text{exact value R.} = -0.527090, \text{Im.} = -0.228251
\end{align*}

<table>
<thead>
<tr>
<th>Schemes</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact (1 step)</td>
<td>−0.591579</td>
<td>−0.037651</td>
<td>12</td>
<td>−0.590808</td>
<td>−0.036487</td>
<td>229</td>
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<tr>
<td>2\text{nd} order</td>
<td>−0.590444</td>
<td>−0.037024</td>
<td>77</td>
<td>−0.590818</td>
<td>−0.036210</td>
<td>246</td>
</tr>
<tr>
<td>3\text{rd} order</td>
<td>−0.591234</td>
<td>−0.034847</td>
<td>82</td>
<td>−0.592145</td>
<td>−0.037411</td>
<td>920</td>
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<tr>
<td>Exact (N steps)</td>
<td>−0.591169</td>
<td>−0.036618</td>
<td>174</td>
<td>−0.59079*</td>
<td>−0.039937*</td>
<td>680</td>
</tr>
<tr>
<td>Corrected Euler</td>
<td>−0.589735*</td>
<td>−0.042002*</td>
<td>223</td>
<td>−0.590079*</td>
<td>−0.039937*</td>
<td>680</td>
</tr>
</tbody>
</table>

\begin{align*}
\alpha &= 2.2, d = 3, \Delta_R = 0.9 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}, \text{exact value R.} = -0.591411, \text{Im.} = -0.036346
\end{align*}

<table>
<thead>
<tr>
<th>Schemes</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact (1 step)</td>
<td>0.062712</td>
<td>−0.063757</td>
<td>181</td>
<td>0.064573</td>
<td>−0.062747</td>
<td>2762</td>
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<tr>
<td>2\text{nd} order</td>
<td>0.064237</td>
<td>−0.063825</td>
<td>921</td>
<td>0.064534</td>
<td>−0.063280</td>
<td>4283</td>
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<tr>
<td>3\text{rd} order</td>
<td>0.064922</td>
<td>−0.064103</td>
<td>1431</td>
<td>0.064120</td>
<td>−0.063122</td>
<td>4343</td>
</tr>
<tr>
<td>Exact (N steps)</td>
<td>0.063418</td>
<td>−0.064636</td>
<td>1806</td>
<td>0.063469</td>
<td>−0.064380</td>
<td>5408</td>
</tr>
<tr>
<td>Corrected Euler</td>
<td>0.068298*</td>
<td>−0.058491*</td>
<td>2312</td>
<td>0.061732*</td>
<td>−0.056882*</td>
<td>7113</td>
</tr>
</tbody>
</table>

\begin{align*}
\alpha &= 10.5, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.3 \times 10^{-3}, \text{exact value R.} = 0.063960, \text{Im.} = -0.063544
\end{align*}

<table>
<thead>
<tr>
<th>Schemes</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
<th>R. value</th>
<th>Im. value</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact (1 step)</td>
<td>−0.036869</td>
<td>−0.094156</td>
<td>177</td>
<td>−0.035944</td>
<td>−0.092770</td>
<td>4285</td>
</tr>
<tr>
<td>2\text{nd} order</td>
<td>−0.036246</td>
<td>−0.094196</td>
<td>1430</td>
<td>−0.036277</td>
<td>−0.093178</td>
<td>4327</td>
</tr>
<tr>
<td>3\text{rd} order</td>
<td>−0.035408</td>
<td>−0.093479</td>
<td>1441</td>
<td>−0.036145</td>
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</tr>
<tr>
<td>Exact (N steps)</td>
<td>−0.036478</td>
<td>−0.092860</td>
<td>1866</td>
<td>−0.030110*</td>
<td>−0.088988*</td>
<td>7144</td>
</tr>
<tr>
<td>Corrected Euler</td>
<td>−0.028685*</td>
<td>−0.094281*</td>
<td>2321</td>
<td>−0.030118*</td>
<td>−0.088988*</td>
<td>7144</td>
</tr>
</tbody>
</table>

\begin{align*}
\alpha &= 9.2, d = 10, \Delta_R = 1.4 \times 10^{-3}, \Delta_{Im} = 1.4 \times 10^{-3}, \text{exact value R.} = -0.036064, \text{Im.} = -0.093275
\end{align*}
**Weak convergence**

**Figure:** \(d = 10, 10^7\) MC samples, \(T = 10\). Left: \(\mathcal{I} (\mathbb{E}[\exp(-\text{Tr}(i\nu \hat{X}^{N}_{tN}))])\) with \(\nu = 0.009I_d\) in fct of \(T/N\). \(x = 0.4I_d, \alpha = 12.5, b = 0\) and \(a = I_d\). Ex. value: \(-0.361586\). Right: \(\mathcal{M}(\mathbb{E}[\exp(-\text{Tr}(i\nu \hat{X}^{N}_{tN}))])\) with \(\nu = 0.009I_d\) in fct of \(T/N\). \(x = 0.4I_d, \alpha = 9.2, b = -0.5I_d\) and \(a = I_d\). Ex. value 0.572241.
A scheme for the Gourieroux-Sufana model

In (4), the joint operator of \( (S_t, X_t) \) is

\[
L = L^S + L^X, \quad \text{where} \quad L^S = \sum_{i=1}^{d} r s_i \partial_{s_i} + \frac{1}{2} \sum_{i,j=1}^{d} s_i s_j x_{i,j} \partial_{s_i} \partial_{s_j},
\]

and \( L^X \) is the generator of a Wishart process. We can solve explicitly the SDE associated to \( L^S : S^l_t = S^l_0 \exp[(r - x_{l,l}/2)t + (\sqrt{x}Z_t)_l] \).

By using a second order scheme for \( L^X \), we get a second order scheme for \( L \) by Corollary 5.
Put option in the Gourieroux-Sufana model

**Figure:** $\mathbb{E}[e^{-rT}(K - \max(\hat{S}_{1,N}^{1}, \hat{S}_{2,N}^{2}))^+]$ in fct of $T/N$. $d = 2$, $T = 1$, $K = 120$, $S_{0}^{1} = S_{0}^{2} = 100$, and $r = 0.02$. $x = 0.04I_{d} + 0.02q$ with $q_{i,j} = 1_{i \neq j}$, $a = 0.2I_{d}$, $b = 0.5I_{d}$ and $\alpha = 4.5$ (left), $\alpha = 1.05$ (right). $10^6$ Monte-Carlo samples.
Summary of the different schemes

We have obtained using splitting methods:

- an exact simulation algorithm for Wishart processes,
- second and third order schemes for Wishart processes,
- second order scheme for affine processes on nonnegative matrices.

The discretization schemes are much more accurate and less time-consuming than the modified Euler scheme.

Which scheme to use? We recommend the exact scheme to calculate expectations that depends on one or few dates. For pathwise expectations, we recommend instead to use discretization schemes: the second order scheme “bis” if it is defined and the second/third order scheme otherwise.
A mean-reverting SDE on correlation matrices

1. Wishart and Affine processes on nonnegative symmetric matrices
2. Exact simulation of Wishart processes
3. Discretization schemes obtained by composition
4. High order discretization schemes for Wishart and Affine processes
5. Numerical results

6. A mean-reverting SDE on correlation matrices
Wright-Fisher (or Jacobi) processes

In dimension 2, $X_t$ is a correlation matrix iff $X_t = \begin{bmatrix} 1 & \rho_t \\ \rho_t & 1 \end{bmatrix}$, with $\rho_t \in [-1, 1]$. The processes that we will present extends in dimension $d$ the following one

$$d\rho_t = \kappa(\bar{\rho} - \rho_t)dt + \sigma \sqrt{1 - \rho_t^2}dB_t,$$

known as a Wright-Fisher (or Jacobi) process.

Properties: mean-reversion, explicit calculation of moments, ergodic law.

Up to our knowledge, there is no literature on particular diffusions defined on correlation matrices in larger dimension.
The full MRC process

\[ X_t = x + \int_0^t (\kappa(c - X_s) + (c - X_s)\kappa) \, ds \]
\[ + \sum_{n=1}^d a_n \int_0^t \left( \sqrt{X_s - X_se_d^T X_s} dW_s e_d^n + e_d^n dW_s^T \sqrt{X_s - X_se_d^T X_s} \right), \]

where \( x, c \in \mathfrak{C}_d(\mathbb{R}) \) and \( \kappa = \text{diag}(\kappa_1, \ldots, \kappa_d) \) and \( a = \text{diag}(a_1, \ldots, a_d) \) are nonnegative diagonal matrices.

Weak existence & uniqueness if \( \kappa c + c\kappa - (d - 2)a^2 \in S_d^+(\mathbb{R}) \) or \( d = 2 \).

Strong uniqueness if \( \kappa c + c\kappa - da^2 \in S_d^+(\mathbb{R}) \) and \( X_0 \in \mathfrak{C}_d^*(\mathbb{R}) \).

Intuitive parameters: mean-reversion towards \( c \) with a speed and a noise respectively tuned by \( \kappa, a \).

Notations: \( \text{MRC}_d(x, \kappa, c, a) \) law of \( (X_t)_{t \geq 0} \), \( \text{MRC}_d(x, \kappa, c, a; t) \) law of \( X_t \).
Some properties

• Each cross correlation follows a 1D WF process:

\[ d(X_t)_{i,j} = (\kappa_i + \kappa_j)(c_{i,j} - (X_t)_{i,j})dt + \sqrt{a_i^2 + a_j^2} \sqrt{1 - (X_t)_{i,j}^2} d\beta_t^{i,j}. \]

• Any principal sub-matrix of \( X_t \) follows a MRC process: Let \( I = \{k_1 < \cdots < k_{d'}\} \subset \{1, \ldots, d\} \) and denote for \( x \in \mathcal{M}_d(\mathbb{R}) \),

\[ (x^I)_{i,j} = x_{k_i,k_j} \text{ for } 1 \leq i, j \leq d'. \]

We have:

\[ (X^I_t)_{t \geq 0} \overset{\text{law}}{=} \text{MRC}_{d'}(x^I, \kappa^I, c^I, a^I). \]

• Explicit calculation of moments (\( \Rightarrow \) weak uniqueness) and ergodic law.
How to get such a process? I

For $x \in S_d^+(\mathbb{R})$ such that $x_{i,i} > 0$ for all $1 \leq i \leq d$, we define $p(x) \in \mathcal{C}_d(\mathbb{R})$ by

$$(p(x))_{i,j} = \frac{x_{i,j}}{\sqrt{x_{i,i}x_{j,j}}}, \; 1 \leq i, j \leq d. \quad (11)$$

A natural idea to construct a process on $\mathcal{C}_d(\mathbb{R})$ from a Wishart process $Y_t$ is to consider $X_t = p(Y_t)$. **Problem**: this does not lead in general to an autonomous SDE... unless in special cases!
How to get such a process? II

**Result:** Let $\alpha \geq \max(1, d - 2)$ and $y \in S_d^+(\mathbb{R})$ such that $y_{i,i} > 0$ for $1 \leq i \leq d$. Let $(Y^y_t)_{t \geq 0} \sim WIS_d(y, \alpha + 1, 0, e_1^d)$. Then, $(Y^y_t)_{i,i} = y_{i,i}$ for $2 \leq i \leq d$ and $(Y^y_t)_{1,1}$ follows a squared Bessel process of dimension $\alpha + 1$ and a.s. never vanishes. We set

$$X_t = p(Y^y_t), \quad \phi(t) = \int_0^t \frac{1}{(Y^y_s)_{1,1}} ds.$$ 

The function $\phi$ is a.s. one-to-one on $\mathbb{R}_+$ and defines a time-change such that:

$$(X_{\phi^{-1}(t)}, t \geq 0) \overset{\text{law}}{=} MRC_d(p(y), \frac{\alpha}{2} e_1^d, I_1, e_1^d).$$

In particular, there is a weak solution to $MRC_d(p(y), \frac{\alpha}{2} e_1^d, I_1, e_1^d)$. Besides, the processes $(X_{\phi^{-1}(t)}, t \geq 0)$ and $((Y^y_t)_{1,1}, t \geq 0)$ are independent.
How to get such a process? III

Let $L^i,C$ the infinitesimal generator of $MRC_d(p(y), \frac{d-2}{2}e_d, I_d, e_i)$. The SDE (10) is nothing but the one associated to generator:

$$\sum_{i=1}^{d} a_i^2 L^i,C + L^{ODE},$$

where $L^{ODE}$ is the operator associated to

$$\xi'(t, x) = \kappa(c-x)+(c-x)\kappa-\frac{d-2}{2}[a^2(I_d-x)+(I_d-x)a^2], \ x \in \mathcal{C}_d(\mathbb{R}).$$

This (linear) ODE can be solved explicitly and such that $\forall t \geq 0, x \in \mathcal{C}_d(\mathbb{R}), \xi(t, x) \in \mathcal{C}_d(\mathbb{R})$ if $\kappa c + c\kappa - (d - 2)a^2 \in S_d^+(\mathbb{R})$.

Remark: the operators $L^i,C$ and $L^j,C$ commute.
A second order scheme for MRC processes

Let $\hat{Y}_{t}^{1,x}$ be the second-order scheme for $WIS_{d}(x, d - 1, 0, e_{d}^{1})$. We can prove that:

$$p(\hat{Y}_{\phi(t)}^{1,x})$$

with:

$$\phi(t) = \begin{cases} 
    t - (5 - d) \frac{t^2}{2} & \text{if } d \geq 5 \\
    \frac{-1 + \sqrt{1 + 2(5-d)t}}{5-d} & \text{otherwise,} 
\end{cases}$$

is a potential second order scheme for $MRC_{d}(p(y), \frac{d-2}{2}e_{d}^{1}, I_{d}, e_{d}^{1})$. Then, by composition (8), we get a potential second order scheme for any MRC process.
Some Remarks

- The complexity of the simulation scheme is in $O(d^4)$. However, when $a$ is proportional to the identity and 
  $\kappa c + c\kappa - (d - 1)a^2 \in S_d^+(\mathbb{R})$, we can get a second-order scheme in $O(d^3)$.

- Let $\hat{X}_t^x$ be potential weak $\nu$th-order scheme for $MRC_d(x, \kappa, c, a)$. Let $f$ be a continuous function on $\mathcal{C}_d(\mathbb{R})$. Then,

  $$\forall \varepsilon > 0, \exists K > 0, |E[f(\hat{X}_{tN}^x)] - E[f(X_T^x)]| \leq \varepsilon + K/N^\nu.$$
Weak convergence

**Figure:** $d = 3$, same parameters. Left:

$$
\mathbb{E} \left[ \sum_{\substack{1 \leq i \neq j \leq 3 \\ 1 \leq k \neq l \leq 3}} \left( \hat{X}_T^N \right)_{i,j} \left( \hat{X}_T^N \right)_{k,l}^2 + (\hat{X}_T^N)_{1,2}(\hat{X}_T^N)_{2,3}(\hat{X}_T^N)_{1,3} \right],
$$

Right:

$$
\mathbb{E} \left[ \sum_{1 \leq i \neq j \leq d} (\hat{X}_T^N)_{i,j} \right],
$$

in function of the time step $1/N$. The width of each point represents the 95% confidence interval.
To sum up

- We have proposed a new diffusion defined on Correlation matrices, with intuitive parameters.
- We have given efficient simulation schemes for affine diffusions on positive semidefinite matrices and for these Correlation processes: this was a sine qua non condition to implement such processes in finance.
- Correlation processes seem to be more suited than covariance processes to model the dependence in finance (more intuitive for practitioners, allow a bottom-up approach to use well-calibrated single asset models).
- However, it still remains to do numerical study on the practical use of MRC processes in financial modelling.