Counterparty Wrong Way and Gap Risks Modeling: A Marked Default Time Approach

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Sequential Monte Carlo methods and Efficient simulation in Finance
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Counterparty Risk

- Risk that some value is lost in OTC derivatives contracts due to the default of a counterparty
  - Early termination of a contract with positive value at time of default of the other party
  - Volatility of the pricing (CVA for Credit Valuation Adjustment) of this risk

  “During the financial crisis of 2007-09, roughly two-thirds of losses attributed to counterparty credit risk were due to CVA losses and only about one-third were due to actual defaults” (Bank of International Settlements press release, June 2011)

- The primary form of financial (credit) risk
  - Vulnerability
  - As opposed to reference credit risk which is present in credit derivatives cashflows
Very significant and a major driver of banks’ P&Ls since the crises

- Also **DVA** for Debt Valuation Adjustment in a bilateral counterparty risk perspective..
- .. and the related nonlinear funding issue → Liquidity Adjustment **LVA**
- .. and Replacement Cost **RC**
- **Total Valuation Adjustment** TVA = CVA + DVA + LVA + RC

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An Important Dynamic Modeling Issue / Challenge

- Pricing at any future time..
  ..(hundreds of) thousands of contracts
  ..consistently in time and across all classes of assets
- Related nonlinear funding issue tricky under bilateral counterparty risk


- Dependence modeling
  - Right/Wrong-Way risk between the parties and the reference portfolio
    - Multiname defaults dependence in case of counterparty risk on credit derivatives

- Collateral modeling
  - AIG bailout on 16 September 2008 triggered by AIG’s inability to face increasing margin calls on sell-protection CDS positions on the distressed Lehman
  - Collateralization “leverages” rather than eliminates counterparty risk
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Crépey, S.: Bilateral Counterparty Risk under Funding Constraints - Part II: CVA. Forthcoming in *Mathematical Finance*.

- Computationally efficient
- Allows one to deal in an integrated setting with funding costs (also in a bilateral counterparty risk setup)
- But subject to an immersion hypothesis between a reference filtration and the filtration of the first-to-default time $\tau$ of the two parties..
  
  ..and to the assumption that “data do not jump at $\tau$”
- Assumptions fine in standard cases (e.g. interest rate derivatives, cf. the numerics later in this talk)..
  
  ..but too strong for cases of strong dependence between the underlying portfolio and the default of the two parties
  
  → Most stringent: Counterparty risk on credit derivatives
  
  → Also: Collateral posted in other currencies
Dynamized copulas used for counterparty risk on credit derivatives


Unified Perspective
- Modeling a default time \( \tau \) with a mark
- Playing with the probability measure and/or the filtration

Reduced-form approaches possible in this perspective in the above dynamized copula setups
- Addressing the TVA high-dimensional nonlinear challenge of bilateral counterparty risk on credit derivatives under funding constraints
  - “Purely forward” particle simulation schemes might be applied to the pre-default Markovian TVA equations over \([0, \bar{T}]\).
  - As opposed to the full model TVA equations over the random interval \([0, \bar{\tau}]\)
- Markovianity of a pre-default setup less rigid than Markovianity of the full model
Modeling Default Times with a Mark

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Outline

1. Generic TVA Setup
2. Pre-default Intensity Setup
3. Wrong Way Risk Setup
4. Dynamized Gaussian Copula TVA Model
5. Dynamized Marshall-Olkin TVA Model
Contract (netted portfolio) with maturity $T$ between the bank and her counterparty

- First default time $\tau$ of the bank and her counterparty
  Survival indicator process $J_t = 1_{t < \tau}$

- Promised (algebraic) dividends $dD_t$ from the bank to her counterparty
  Effective dividends $dC_t = J_t dD_t$

- Effective time horizon $\bar{\tau} = \tau \wedge T$
  Full model filtration $\mathcal{G}$ over $[0, \bar{\tau}]$
  Conditional expectation given $\mathcal{G}_t$ denoted by $\mathbb{E}_t$
Two-step price-and-hedge methodology, “clean minus TVA”

- Introduced in banks for organizational reasons
  - Industry trading desks lack the global view, and specifically the aggregated data, needed to properly value the default (“CSA” for Credit Support Annex) cash-flows
  → Industry trading desks in charge of clean price-and-hedge
    - Clean of counterparty risk and funding costs
  → Central TVA desk in charge of TVA price-and-hedge
- Also useful mathematically

Clean valuation

- Risk-free (or overnight indexed swap OIS) rate $r_t$ and discount factor $\beta_t = \exp(-\int_0^t r_s ds)$
- Clean dividend process $D$ and clean value process $P$ of the contract such that, for $t \in [0, \bar{\tau}]$, 
  $$\beta_t P_t = \mathbb{E}_t \left( \int_t^\tau \beta_s dD_s \right) = \mathbb{E}_t \left[ \int_t^{\bar{\tau}} \beta_s dD_s + \beta_{\bar{\tau}} P_{\bar{\tau}} \right]$$
Two-step price-and-hedge methodology, “clean minus TVA”

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TVA

- **Close-out cash-flow** $R(\pi)$ paid by the bank at time $\tau$ if $\tau < T$, depending in particular on the wealth $\pi$ of the bank right before time $\tau$.

- **$G_\tau$-measurable exposure at default** given as
  \[ \xi(\pi) = P_\tau + \Delta_\tau - R(\pi) \]
  where $\Delta = D - D_-$ is the jump process of $D$

- **Funding coefficient** ($dt$-cost) $g_t(\pi)$ of the bank (in excess over $r_t\pi$)

- **TVA process** $\Theta$ defined (rather than derived for simplicity in these slides) by: For $t \in [0, \bar{\tau}]$,
  \[ \beta_t \Theta_t = \mathbb{E}_t \left[ \beta_{\bar{\tau}} 1_{\tau < \bar{\tau}} \xi(P_\tau - \Theta_{\tau_-}) + \int_t^{\bar{\tau}} \beta_s g_s(P_s - \Theta_s)ds \right] \]

- **BSDE** in integral form over the random time interval $[0, \bar{\tau}]$

- **Overall price (cost-of-the-hedge)** $\Pi = P - \Theta$
  - A positive/negative $\Theta$ makes the deal easier/more difficult as it decreases/increases the bank’s cost-of-the-hedge
Outline

1. Generic TVA Setup
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4. Dynamized Gaussian Copula TVA Model
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\( \tau = \min_{e \in E} \tau_e \) endowed with a mark \( e \) in a finite set \( E \)

- Mark already implicitly here in the standard approach: default of the bank alone, of the client alone or of both simultaneously

- **Intensity** \( \lambda^e_t \) of \( \tau_e \)

- \( \xi(\pi) = \tilde{\zeta}^e_t(\pi) \) on \( \{ \tau = \tau_e \} \) for a predictable marked function \( \tilde{\zeta}^e_t(\pi) \) of \( \pi \)

- **Notation** \( \lambda_t \cdot f_t = \sum_{e \in E} \lambda^e_t f^e_t \) for every \( f_t = f^e_t \)

- **Intensity** \( \gamma_t = \sum_{e \in E} \lambda^e_t = \lambda_t \cdot 1_E \) of \( \tau \)

- **Compensated jump-to-default martingale** defined on \( [0, \bar{\tau}] \) as

\[
\xi(Y_{t-})dJ_t + \lambda_t \cdot \tilde{\zeta}_t(Y_t)dt
\]

for every price process (càdlàg semimartingale) \( Y \)
Assumption (A)

(A.1) A filtration $\mathcal{F}$ over $[0, T]$ such that $\mathcal{F}_t \subseteq \mathcal{G}_t$ on $[0, \bar{\tau}]$, and $\mathcal{F}$- semimartingales stopped at $\tau$ are $\mathcal{G}$-semimartingales.

(A.2) A probability measure $\mathbb{P}$ equivalent to $\mathbb{Q}$ on $\mathcal{F}_{\bar{\tau}}$ such that an $(\mathcal{F}, \mathbb{P})$-martingale “stopped at $\tau$” is a $(\mathcal{G}, \mathbb{Q})$-martingale,

- ~ Pseudo-stopping time $\tau$ in the “immersion” case $\mathbb{P} = \mathbb{Q}$
- But richer filtration $\mathcal{G}$ via the mark $e$ than in the standard case where $\mathcal{G}_t = \mathcal{F}_t \lor \sigma(\tau \land t)$
  - An $\mathcal{F}$-adapted càdlàg process can jump at $\tau$

(A.3) An $\mathcal{F}$-progressive random function $\tilde{g}_t(\vartheta)$ such that $\tilde{g}_t(\vartheta) dt = \tilde{g}_t(P_t - \vartheta) dt$ on $[0, \bar{\tau}]$ where for every real $\theta$

$$\tilde{g}_t(P_t - \vartheta) = g_t(P_t - \vartheta) + \lambda_t \cdot \tilde{x}_t(P_t - \vartheta) - \tilde{r}_t \vartheta$$

with $\tilde{r}_t = r_t + \gamma_t$

Conditional expectation given $(\mathcal{F}_t, \mathbb{P})$ denoted by $\tilde{E}_t$.
Proposition

Assume an \((\mathcal{F}, \mathbb{P})\)-semimartingale \(\tilde{\Theta}\) satisfies:

\[
\tilde{\Theta}_t = \tilde{E}_t \int_{t}^{T} \tilde{g}_s(\tilde{\Theta}_s)ds , \quad t \in [0, T]
\]

and define \(\Theta = \tilde{\Theta}\) on \([0, \bar{T})\) and \(\Theta_{\bar{T}} = 1_{\bar{T} < T} \xi(P_{\bar{T}-} - \tilde{\Theta}_{\bar{T}-})\). Then \(\Theta\) satisfies the TVA defining equation. Moreover the \((\mathcal{G}, \mathbb{Q})\)-martingale component

\[
d\mu_t = d\Theta_t - (r_t \Theta_t + g_t(P_t - \Theta_t))dt
\]

of \(\Theta\) satisfies for \(t \in [0, \bar{T}]\):

\[
d\mu_t = d\tilde{\eta}_t \wedge_{\bar{T}-} - \left((\xi^* - \tilde{\Theta}_{\bar{T}-})dJ_t + (\lambda_t \cdot \tilde{\xi}_t - \gamma_t \Theta_t)dt\right)
\]

where

\[
d\tilde{\eta}_t = d\tilde{\Theta}_t - \tilde{g}_t(\tilde{\Theta}_t)dt
\]

is the \((\mathcal{F}, \mathbb{P})\)-martingale component of \(\tilde{\Theta}\), and \(\xi^*\) and \(\tilde{\xi}_t \equiv \hat{\xi}_t^e\) are shorthands for \(\xi(P_{\bar{T}-} - \tilde{\Theta}_{\bar{T}-})\) and \(\hat{\xi}_t^e(P_t - \tilde{\Theta}_t)\).
Pre-default Markov setup $\dot{g}_t(\psi)dt = \dot{g}_t(t, X_t, \psi)dt$ with $\mathcal{(F, \mathbb{P})}$-generator $\mathcal{X}$ of $X$

\[ \tilde{\Theta}_t = \tilde{\Theta}(t, X_t) \] where

\[
\begin{cases}
\tilde{\Theta}(T, x) = 0, x \in \mathbb{R}^d \\
(\partial_t + \mathcal{X})\tilde{\Theta}(t, x) + \dot{g}(t, x, \tilde{\Theta}(t, x)) = 0, t < T, x \in \mathbb{R}^d
\end{cases}
\]

- Semilinear funding term
- Simulation schemes only viable computational alternative as soon as $d \geq 3$ or $4$
  - “Purely forward” branching particule schemes in vanilla cases
    - Explicit $\dot{g}$
- Non-linear regression “forward-backward” simulation schemes in exotic cases
  - Subject to Markovian tractability of $P_t$

Pre-default Markov setup $\tilde{g}_t(\psi)dt = \hat{g}_t(t, X_t, \psi)dt$ with $(\mathcal{F}, \mathbb{P})$-generator $\mathcal{X}'$ of $X$

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CSA close-out valuation process \( Q \) of the contract

CSA collateralization scheme \( \Gamma \)

Debts of the bank to her counterparty and to her external funder at time \( \tau \)

\[
\chi = Q_\tau + \Delta_\tau - \Gamma_\tau, \quad \mathcal{X}(\pi) = - (\pi - \Gamma_\tau^-)
\]

Close-out cash-flow from the bank \( R(\pi) = R^i - 1_{\tau=\theta} R^f(\pi) \) where

\[
R^i(\pi) = \Gamma_\tau + 1_{\tau=\theta} (\rho \chi^+ - \chi^-) - 1_{\tau=\bar{\theta}} (\bar{\rho} \chi^- - \chi^+) - 1_{\theta=\bar{\theta}} \chi
\]

\[
R^f(\pi) = (1 - \tau) \mathcal{X}^+(\pi)
\]

\( \rho, \bar{\rho} \) Recovery rates between the two parties

\( \tau \) Recovery rate of the bank toward her funder

Exposure at default

\[
\xi(\pi) = P_\tau + \Delta_\tau - R(\pi)
\]

\[
= P_\tau - Q_\tau + 1_{\tau=\theta} ((1 - \rho) \chi^+ + (1 - \tau) \mathcal{X}^+(\pi)) - (1 - \bar{\rho}) 1_{\tau=\bar{\theta}} \chi^-
\]

Funding coefficient

\[
g_t(\pi) = (b_t \Gamma_t^+ - \bar{b}_t \Gamma_t^-) + \lambda_t (\pi - \Gamma_t)^+ - \bar{\lambda}_t (\pi - \Gamma_t)^-
\]

\( b, \bar{b}, \lambda, \bar{\lambda} \) Liquidity and/or credit bases
Assume

1. \( \theta = \min_{e \in E_\theta} \tau_e \), \( \bar{\theta} = \min_{e \in \bar{E}_\theta} \tau_e \) where \( E_\theta \cup \bar{E}_\theta = E \) (not nec. disj.)

2. For every process \( U = P, \Delta, Q, \Gamma \), one can find a predictable marked process \( U^e \) such that \( U_\tau = U^e_\tau \) on \( \{ \tau = \tau_e \} \)

\[ g_t(P_t - \vartheta) + r_t \vartheta = - \left( 1 - \bar{\rho} \right) \lambda_t \cdot (1_{E_\theta} (\tilde{Q}_t + \tilde{\Delta}_t - \tilde{\Gamma}_t)^-) \]

- costly Crebit Valuation Adjustment CVA

\[ + \left( 1 - \rho \right) \lambda_t \cdot (1_{E_\theta} (\tilde{Q}_t + \tilde{\Delta}_t - \tilde{\Gamma}_t)^+) \]

- beneficial Debit Valuation Adjustment DVA

\[ + b_t \Gamma_t^+ - \bar{b}_t \Gamma_t^- + \lambda_t (P_t - \vartheta - \Gamma_t)^+ - \tilde{\lambda}_t (P_t - \vartheta - \Gamma_t)^- \]

- excess-funding benefit/cost Funding Valuation Adjustment FVA

\[ + \lambda_t \cdot \left( \tilde{P}_t - 1_{E_\theta} \vartheta - \tilde{Q}_t \right) \]

- Replacement Cost RC

\[ \tilde{\lambda}_t := \bar{\lambda}_t - (1 - \tau) \lambda_t \cdot 1_{E_\theta} \]

External borrowing basis net of credit spread

Liquidity borrowing funding basis
Assume

1. \[ \theta = \min_{e \in E_\theta} \tau_e, \quad \bar{\theta} = \min_{e \in E_{\bar{\theta}}} \tau_e \text{ where } E_\theta \cup E_{\bar{\theta}} = E \text{ (not nec. disj.)} \]
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External borrowing basis net of credit spread

- Liquidity borrowing funding basis
Numerical Case Study: IR swap in an increasing term structure

Short rate $r_t$ following an IG-driven mean-reverting HJM model
Clean price process $P_t = P(t, r_t)$ of a receiver IR swap (fixed leg worth 100$ at inception).
Five CSA Specifications.

\[(r, \rho, \bar{\rho}) = (40, 40, 40)\% , \quad Q = P , \quad \Gamma = 0 \]
\[(r, \rho, \bar{\rho}) = (100, 40, 40)\% , \quad Q = P , \quad \Gamma = 0 \]
\[(r, \rho, \bar{\rho}) = (100, 100, 40)\% , \quad Q = P , \quad \Gamma = 0 \]
\[(r, \rho, \bar{\rho}) = (100, 100, 40)\% , \quad Q = \Pi , \quad \Gamma = 0 \]
\[(r, \rho, \bar{\rho}) = (100, 40, 40)\% , \quad Q = P , \quad \Gamma = Q = P . \]
Time-0 TVA and its decomposition. *Top Table*: Payer swap; *Bottom Table*: Receiver swap; *Rows*: Five CSA Specifications.

<table>
<thead>
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<th>TVA</th>
<th>CVA</th>
<th>DVA</th>
<th>LVA</th>
<th>RC</th>
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<td>0.00</td>
<td>0.32</td>
<td>0.00</td>
</tr>
<tr>
<td>0.44</td>
<td>0.00</td>
<td>0.00</td>
<td>0.72</td>
<td>-0.29</td>
</tr>
</tbody>
</table>
Receiver swap’s “Expected Exposures” – **Columns**: CVA/DVA/LVA/RC; **Rows**: 5 CSA Specifications.
Uncollateralized payer swap TVA process (CSA specif. 2).
Collateralized payer swap TVA process (CSA specif. 5).

![Graph showing collateralized payer swap TVA process](image-url)
Counterparty Risk on Credit Derivatives

- Let $N = \{-1, 0, 1, \ldots, n\}$, $N^* = \{1, \ldots, n\}$.
- Netted portfolio of credit derivatives on $n$ underlying credit names $(\tau_1, \ldots, \tau_n) = (\tau_i)_{i \in N^*}$
- Full credit dependence (wrong-way risk) model of $(\tau_{-1}, \tau_0, \tau_1, \ldots, \tau_n) = (\tau_i)_{i \in N}$ with $\tau_{-1} \equiv \theta$ and $\tau_0 \equiv \bar{\theta}$
- In the following two sections we develop reduced-form TVA approaches in DGC (dynamized Gaussian copula) and DMO (dynamized Marshall-Olkin) models of the $(\tau_i)_{i \in N}$
Outline

1. Generic TVA Setup
2. Pre-default Intensity Setup
3. Wrong Way Risk Setup
4. Dynamized Gaussian Copula TVA Model
5. Dynamized Marshall-Olkin TVA Model
Static Gaussian Copula

Exponential-λ_i default times \( \tau_i = -\lambda_i^{-1} \log(\Phi(\varepsilon_i)) \)

\( \varepsilon_i \) standard Gaussian r.v. with pairwise correlation \( \varrho \geq 0 \)

\( \Phi \) Standard normal survival function

“The formula that killed Wall Street”

- Conditional independence
- \( \mathbb{P}(\tau_i > \theta_i, \ i \in N) = \int_{\mathbb{R}} \left( \prod_{i=1}^{n} \Phi\left( \frac{\Phi^{-1}(\mathbb{P}(\tau_i > \theta_i)) - \sqrt{\varrho} y}{\sqrt{1-\varrho}} \right) \right) \phi(y)dy > 0 \)

\( \phi \) Standard normal density
Dynamized Gaussian Copula

- Static setup vs dynamic model needed for counterparty credit risk modeling
- “Informational dynamization”
  - \( \varepsilon_i = \int_0^{+\infty} \varsigma(t) dB_t^i \)
  - \( \varsigma(\cdot) \) function with unit \( L_2 \)-norm
  - \( B_t^i \) standard BM with pairwise correlation \( \varrho \)
- Full model filtration

\[
G_t = \mathcal{F}_t^B \vee \left( \bigvee_{i \in N} \sigma(t_i \wedge t) \right)
\]
Markovian Structure

\[ m_t = (m^i_t)_{i \in \mathbb{N}} \text{ with } m^i_t = \int_0^t \zeta(u) dB^i_u \]
\[ k_t = (k^i_t)_{i \in \mathbb{N}} \text{ with } k^i_t = \tau_i 1_{\tau_i \leq t} \]

- \((\mathcal{G}, \mathbb{Q})\)-fundamental martingales

\[ dW^i_t = dB^i_t - \gamma^i_t dt, \quad dM^i_t = 1_{\tau_i \leq t} - \lambda^i_t dt \]

for explicit (but combinatorially involved) processes

\[ \gamma^i_t = \gamma_i(t, m_t, k_t), \quad \lambda^i_t = \lambda_i(t, m_t, k_t) \]

- Process \((t, m_t, k_t)\) is \((\mathcal{G}, \mathbb{Q})\)-Markov
DGC TVA Model

- DMO setup now used as a TVA wrong-way risk model on credit derivatives
  - $E_\theta = \{-1\}$, $E_{\bar{\theta}} = \{0\}$
  - No joint defaults in this (Lebesgue-density) model $\Rightarrow$
    $$\Delta_\tau = 0, \quad \chi = Q_\tau - \Gamma_\tau$$

- Assume for every process $U = P$, $Q$ and $\Gamma$ and $i = -1, 0$, there exists a function $\tilde{U}_i$ such that
  $$U_\tau = U_i(\tau, m_\tau, k_{\tau-}) = \tilde{U}_i^\tau \text{ on } \{\tau = \tau_i\}$$

- Holds on $P$ (for every $i \in N$) for standard credit derivatives
  - Single-name CDSs or multi-name CDOs
- Holds or not on $Q$ and $\Gamma$ depending on the CSA
  - Collateral path-dependent specifications can be dealt with by augmentation of the state space
Reduced-Form DGC TVA Approach

- Reduced DGC model \((t, X_t) = (t, m_t, \tilde{k}_t)\) with \(\tilde{k}_t = (\tilde{k}_i^t)_{i \in \mathbb{N}^*}\) and

\[
F_t = F_t^B \vee \left( \bigvee_{i \in \mathbb{N}^*} \sigma(\tau_i \land t) \right)
\]

- Fundamental \((\mathcal{F}, \mathbb{Q})\)-martingales

\[
d\tilde{W}_t^i = dB_t^i - \tilde{\gamma}_t^i dt, \quad i \in \mathbb{N};
\]

\[
d\tilde{M}_t^i = d1_{\tau_i \leq t} - \tilde{\chi}_t^i dt, \quad i \in \mathbb{N}^*
\]

- Fundamental \((\mathcal{F}, \mathbb{P})\)-martingales

\[
dW_t^i = dB_t^i - \gamma_t^i dt, \quad i \in \mathbb{N};
\]

\[
dM_t^i = d1_{\tau_i \leq t} - \chi_t^i dt, \quad i \in \mathbb{N}^*
\]

for a suitably changed measure \(\mathbb{P}\)

- Connection with


- \((A)\) satisfied by \((\mathcal{F}, \mathbb{P})\)

\(\mathbb{P} \neq \mathbb{Q} \rightarrow\) No immersion case

- Pre-default setup Markov in terms of \((t, X_t) = (t, m_t, \tilde{k}_t)\)
DGC: To Sum-up

- Simplicity and Consistency of a dynamized Gaussian copula set-up
  - Decoupled calibration methodology
    - Automatically calibrated marginals
  - Fast single-name and portfolio credit derivatives pricing schemes
  - Model simulation very fast

- A reduced-form approach is possible

- No immersion: Good or bad??
  - Makes financial sense
  - Combinatorially involved formulas

- Poor Gaussian copula dependence structure
Outline

1. Generic TVA Setup
2. Pre-default Intensity Setup
3. Wrong Way Risk Setup
4. Dynamized Gaussian Copula TVA Model
5. Dynamized Marshall-Olkin TVA Model
Let \( \mathcal{I} = \{I_1, \ldots, I_m\} \) denote a family of pre-specified subsets of \( N \)

- Sets of obligors susceptible to default simultaneously
- Defaults are the consequence of triggering-events affecting simultaneously pre-specified groups of obligors

Define, for \( Y \in \mathcal{Y} = \{-1\}, \{0\}, \{1\}, \ldots, \{n\} \cup \mathcal{I} \), affine intensity processes \( X_t^Y \) driven by independent BM \( W_t^Y \) under a pricing measure \( \mathbb{Q} \), and

\[
\tau_Y = \inf\{t > 0; \int_0^t X_s^Y \, ds \geq \epsilon_Y\}
\]

for independent standard exponential random variables \( \epsilon_Y \)

Set, for every \( i \in N \),

\[
\tau_i = \bigwedge_{Y \in \mathcal{Y}; \exists i} \tau_I = \tau_{\{i\}} \wedge \bigwedge_{I \in \mathcal{I}; \exists i} \tau_I
\]
Example: \( n = 5 \) and 
\[
\mathcal{Y} = \{\{-1\}, \{0\}, \{1\}, \{2\}, \{3\}, \{2, 3\}, \{0, 1, 2\}, \{-1, 0\}\}.
\]
Dynamized Marshall-Olkin

- “Informational dynamization” through

\[ G_t = \mathcal{F}_t^W \lor (\bigvee_{i \in N} \sigma(\tau_i \land t)) \]

where \( W = (W_t^Y)_{Y \in \mathcal{Y}} \)

- \((\mathcal{G}, \mathbb{Q})\)-fundamental martingales \( W^Y, Y \in \mathcal{Y} \) and \( M^Z, Z \subseteq N \) where

\[ dM_t^Z = d1_{\tau_Z \leq t} - \lambda^Z_t dt \]

\( \tau_Z \) Time of a joint default of names in set \( Z \) and only in \( Z \)

\( \lambda^Z_t = \sum_{Y \in \mathcal{Y}: Y_t = Z} X_t^Y = \lambda_Z(t, X_t, H_{t-}) \)

\( Y_t \) Set of survivors of set \( Y \) right before \( t \)

\( X_t = (X_t^Y)_{Y \in \mathcal{Y}} \)

\( H_t = (H_t^i)_{i \in N} \) with \( H_t^i = 1_{\tau_i \leq t} \)

- Process \((X_t, H_t)\) is \((\mathcal{G}, \mathbb{Q})\)-Markov
Toy Example: Two Names with Constant Intensities

Continuous-time Markov chain $H = (H^1, H^2)$ with state space $E = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$

Generator-matrix of $H = (H^1, H^2)$

$$\mathcal{A} = \begin{bmatrix} -l & l_1 & l_2 & l_3 \\ 0 & -q_1 & 0 & q_1 \\ 0 & 0 & -q_2 & q_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with $l = l_1 + l_2 + l_3$, $q_1 = l_1 + l_3$, $q_2 = l_2 + l_3$

Marshall-Olkin Copula

$$\mathbb{P}(\tau_1 > s, \tau_2 > t) = MOC(\mathbb{P}(\tau_1 > s), \mathbb{P}(\tau_2 > t))$$
DMO TVA Model

- DMO setup now used as a TVA wrong-way risk model on credit derivatives
- \( E_\theta = Z_\theta := \{ Z \subseteq N; -1 \in Z \} \) , \( E_{\bar{\theta}} = Z_{\bar{\theta}} := \{ Z \subseteq N; 0 \in Z \} \)
- Assume for every process \( U = P, \Delta \) [there are joint defaults in this model], \( Q \) and \( \Gamma \), there exists a function \( \tilde{U} \) such that
  \[
  U_\tau = \tilde{U}(\tau, X_{\tau}, H_{\tau-}, H_{\tau-}^{Z}) = \tilde{U}_{\tau}^{Z}
  \]
on every event \( \{ \tau = \tau_Z \} \) with \( Z \in E = Z_\theta \cup Z_{\bar{\theta}} \)
  - Holds on \( P \) and \( \Delta \) (for every \( Z \subseteq N \)) for standard credit derivatives
    - Single-name CDSs or multi-name CDOs
  - Holds or not on \( Q \) and \( \Gamma \) depending on the CSA
    - Collateral path-dependent specifications can be dealt with by augmentation of the state space
  - \( \tilde{U}(\tau, X_{\tau}, H_{\tau-}, H_{\tau-}^{Z}) \) better than “\( \tilde{U}_{Z}(\tau, X_{\tau}, H_{\tau-}) \)”
    - Reduction of dimensionality from “\( 2^n \) to \( n + m \)”
    - Sums over \( Z \subseteq N \) rewritten as sums over \( Y \in \mathcal{Y} \) (except in martingale representations but this does not harm numerically)
Reduced-Form DMO TVA Approach

- Let $\tilde{\mathcal{Y}} = \{\{1\}, \ldots, \{n\}\} \cup \tilde{\mathcal{I}}$, where $\tilde{\mathcal{I}}$ consists of those $I_j$ in $\mathcal{I}$ which do not contain $-1$ nor $0$.
- Let for every obligor $i \in N^*$
  $$\tilde{\tau}_i = \min_{\{Y \in \tilde{\mathcal{Y}}; i \in Y\}} \tau_Y, \quad \tilde{H}_i^t = 1_{\tilde{\tau}_i \leq t}$$
- Reduced DMO model $X_t = (X_t, \tilde{H}_t)$ where $\tilde{H} = (\tilde{H}_i^t)_{i \in N^*}$, relatively to the filtration $\mathcal{F}$ such that for every $t$
  $$\mathcal{G}_t = \mathcal{F}_t^W \vee \left( \bigvee_{i \in N^*} \sigma(\tilde{\tau}_i \wedge t) \right),$$
  and to the unchanged probability measure $P = Q$.
- Markov copula DMO features $\rightarrow$ Fundamental $(\mathcal{F}, Q)$-martingales $W^Y, Y \in \mathcal{Y}$ and $\tilde{M}^Z, Z \subseteq N^*$
- (A) satisfied by $(\mathcal{F}, P = Q)$
  - “Immersion case”
- Pre-default setup Markov in terms of $X_t = (X_t, \tilde{H}_t)$
DMO: To Sum-up

- Simplicity and consistency of a dynamized Marshall-Olkin copula set-up
  - Decoupled calibration methodology
    - Automatically calibrated marginals
    - Model dependence parameters calibrated independently
    - Fast single-name and portfolio credit derivatives pricing schemes
    - The top-down versus bottom-up credit portfolio puzzle solved
  - Model simulation very fast
- A reduced-form approach is possible
DMO: To Sum-up

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- A reduced-form approach is possible
Devising a Cross – Asset Classes Model of Counterparty Risk

- Standard reduced form approach assuming immersion possible on most markets
- $\Delta D_\tau \neq 0$ needed for counterparty credit risk and $\Delta \Gamma_\tau \neq 0$ needed for collateral posted in other currencies
- Modeling a $\mathcal{G}$-default time $\tau$ with a mark
  - Reduced-form approaches still possible assuming all pre-default data in a sub-filtration $\mathcal{F}$ of $\mathcal{G}$ such that $(\mathcal{F}, \mathbb{P})$-martingales stopped at $\tau$ are $(\mathcal{G}, \mathbb{Q})$-martingales, for some “pre-default” reference filtration $\mathcal{F}$ and equivalent probability measure $\mathbb{P}$
  - Branching solution particle solutions to the reduced (but still nonlinear and high-dimensional) TVA equations?
    - Reduced but very high-dimensional
- Dynamized copulas: a general approach?
Facing the simulation challenge of TVA computations on real-life portfolios with tens of thousands of contracts

- More intensive than (Credit-)VaR or other risk measure computations
  - Value the portfolio at every time point of every simulated trajectory
- Devise appropriate variance reduction techniques
  - Importance Sampling exploiting the Markovian structure of the models
  - Sequential Monte Carlo (so particles again)