On the convergence of Island particle models

C. Dubarry, P. Del Moral, E. Moulines

Institut Mines-Télécom, Télécom ParisTech/ Télécom SudParis, INRIA Bordeaux

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1 Introduction

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4 Extensions
Notations

- $(X_n, \mathcal{X}_n)_{n \geq 0}$ is a sequence of measurable sets.
- $\mathcal{B}_b(X_n, \mathcal{X}_n)$ is the Banach space of all bounded and measurable functions on $(X_n, \mathcal{X}_n)$.
- $(X_n)_{n \geq 0}$ is a non-homogenous Markov chain with initial distribution $\eta_0$, and Markov kernels $(M_n)_{n \geq 1}$.
- Feynman-Kac flow

\[
\begin{align*}
\eta_n(f_n) & \overset{\text{def}}{=} \frac{\gamma_n(f_n)}{\gamma_n(1)}, \\
\gamma_n(f_n) & \overset{\text{def}}{=} \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right].
\end{align*}
\]
Feynman-Kac flow

- Define by $\mathcal{P}(X_n, \mathcal{X}_n)$ the set of probabilities on $(X_n, \mathcal{X}_n)$.
- The sequence of probabilities $(\eta_n)_{n \geq 0}$ satisfies the following recursion:
  \[
  \eta_{n+1} = \Psi_n(\eta_n)M_{n+1},
  \]
  where $\Psi_n : \mathcal{P}(X_n, \mathcal{X}_n) \to \mathcal{P}(X_n, \mathcal{X}_n)$ is defined by:
  \[
  \Psi_n(\eta_n)(A_n) \overset{\text{def}}{=} \frac{1}{\eta_n(g_n)} \int_{A_n} g_n(x_n) \eta_n(dx_n), \quad A_n \in \mathcal{X}_n.
  \]
Outline

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4. Extensions
Particle approximation

- Let $N_1$ be an integer. For any integer $p$ we set $(X_p, x_p) \equiv (X_{pN_1}, x_p \otimes N_1)$.
- Define the Markov kernel $M_{n+1}$ from $(X_n, x_n)$ to $(X_{n+1}, x_{n+1})$ as the product measure

$$M_{n+1}(x_n, A_{n+1}) \equiv \prod_{1 \leq i \leq N_1} \Psi_n(m(x_n, \cdot)M_{n+1}(A_{n+1}^i),$$

where $m(x_n, \cdot)$ stands for the empirical measure of $x_n$ given for any $A_n \in \mathcal{X}_n$ by

$$m(x_n, A_n) \equiv \frac{1}{N_1} \sum_{i=1}^{N_1} \delta_{x_n^i}(A_n).$$

- The particles are multinomially resampled with probabilities proportional to their potential $\{g_n(x_n^i)\}_{i=1}^{N_1}$; new particle positions are then sampled from the Markov kernel $M_{n+1}$. 
Particle approximation

- Define a Markov chain \( \{X_n\}_{n \geq 0} \) where for each \( n \in \mathbb{N} \),
  \[
  X_n = (\xi_n^1, \ldots, \xi_n^{N_1}) \in \mathbb{X}_n
  \]
  with initial distribution \( \eta_0 \overset{\text{def}}{=} \eta_0^\otimes N_1 \) and transition kernel \( M_{n+1} \).
- \( N_1 \)-particle approximations

\[
\eta_n^{N_1}(f_n) \overset{\text{def}}{=} m(X_n, f_n)
\]
\[
\gamma_n^{N_1}(f_n) \overset{\text{def}}{=} \eta_n^{N_1}(f_n) \prod_{0 \leq p < n} \eta_p^{N_1}(g_p).
\]
The estimator $\gamma_{n}^{N_1}(f_n)$ is unbiased:

\[
\gamma_{n}(f_n) = \mathbb{E} \left[ \gamma_{n}^{N_1}(f_n) \right]
= \mathbb{E} \left[ \eta_{n}^{N_1}(f_n) \prod_{0 \leq p < n} \eta_{p}^{N_1}(g_p) \right] = \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right],
\]

where $g_{n}(x_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} g_{n}(x_{n}^i)$ and $f_{n}(x_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} f_{n}(x_{n}^i)$. 
The island Feynman-Kac model

- The estimator $\gamma_n^{N_1} (f_n)$ is unbiased:

$$\gamma_n(f_n) = \mathbb{E} \left[ \gamma_n^{N_1} (f_n) \right]$$

$$= \mathbb{E} \left[ \eta_n^{N_1} (f_n) \prod_{0 \leq p < n} \eta_p^{N_1} (g_p) \right] = \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right],$$

where $g_n(x_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} g_n(x^i_n)$ and $f_n(x_n) = \frac{1}{N_1} \sum_{i=1}^{N_1} f_n(x^i_n)$.

- Consequently, we can define the island Feynman-Kac model

$$\eta_n(f_n) \overset{\text{def}}{=} \frac{\gamma_n(f_n)}{\gamma_n(1)},$$

$$\gamma_n(f_n) \overset{\text{def}}{=} \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right].$$

- We get $\gamma_n(f_n) = \gamma_n(f_n)$ and $\eta_n(f_n) = \eta_n(f_n)$. 
The island Feynman-Kac model

- From now on, a population of particles $X_n$ is called an island.
- Idea: we may apply the interacting particle system approximation of the Feynman-Kac semigroups both within each island but also across island.
- To be more specific, we will now describe the so-called double bootstrap algorithm where the bootstrap algorithm is applied both within an island but also across the islands.
- Of course, many other options are available (more to come !)
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4 Extensions
Feynman-Kac at the island level

- Define by $\mathcal{P}(\mathcal{X}_n, \mathcal{X}_n)$ the set of probabilities measures on $(\mathcal{X}_n, \mathcal{X}_n)$.
- The sequence of measures $(\eta_n)_{n \geq 0}$ satisfies the following recursion

$$\eta_{n+1} = \Psi_n(\eta_n) M_{n+1},$$

where $\Psi_n : \mathcal{P}(\mathcal{X}_n, \mathcal{X}_n) \rightarrow \mathcal{P}(\mathcal{X}_n, \mathcal{X}_n)$ is defined by

$$\Psi_n(\eta_n)(A_n) \overset{\text{def}}{=} \frac{1}{\eta_n(g_n)} \int_{A_n} g_n(x) \eta_n(dx), \quad A_n \in \mathcal{X}_n.$$
The double bootstrap algorithm

- Let $N_2$ be the number of interacting islands.

- During the selection stage, we select randomly $N_2$ islands $\left(\hat{\xi}_n^i\right)_{1 \leq i \leq N_2}$ among the current islands $\left(\xi_n^i\right)_{1 \leq i \leq N_2} \in \mathbb{X}_n^{N_2}$ with probability proportional to the empirical mean of the individuals in each island

$$g_n(\xi_n^i) = N_1^{-1} \sum_{j=1}^{N_1} g_n(\xi_n^{i,j}), \ 1 \leq i \leq N_2.$$

- During the mutation transition, selected islands $\left(\hat{\xi}_n^i\right)_{i=1}^{N_2}$ evolve randomly to a new configuration $\xi_{n+1}$ according to the Markov transition $M_{n+1}$. 
One step of the algorithm
One step of the algorithm

\[
\xi_{n1}^1 \quad 1 \quad \sum_{j=1}^{N_1} \frac{1}{N_1} g_n(\xi_{n1}^1, j)
\]

\[
\xi_{n2}^2 \quad 1 \quad \sum_{j=1}^{N_1} \frac{1}{N_1} g_n(\xi_{n2}^2, j)
\]

\[
\xi_{n3}^3 \quad 1 \quad \sum_{j=1}^{N_1} \frac{1}{N_1} g_n(\xi_{n3}^3, j)
\]
Algorithm description

One step of the algorithm

\[ \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^1, j) \]

\[ \hat{\xi}_n^1 = \xi_n^2 \]

\[ \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^2, j) \]

\[ \hat{\xi}_n^2 = \xi_n^3 \]

\[ \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^3, j) \]

\[ \hat{\xi}_n^3 = \xi_n^3 \]
One step of the algorithm

\[
\frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_{n,j}^1) \quad \Rightarrow \quad \hat{\xi}_{n}^1 = \xi_{n}^2
\]

\[
\frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_{n,j}^2) \quad \Rightarrow \quad \hat{\xi}_{n}^2 = \xi_{n}^3
\]

\[
\frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_{n,j}^3) \quad \Rightarrow \quad \hat{\xi}_{n}^3 = \xi_{n}^3
\]
One step of the algorithm

\[ \xi_n = \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^j) \]

\[ \hat{\xi}_n^1 = \xi_n^1, \hat{\xi}_n^2, \hat{\xi}_n^3 \]

\[ \xi_n^{1+1} = \hat{\xi}_n^1, \xi_n^{2+1}, \xi_n^{3+1} \]

\[ \xi_n^{1+1} = \xi_n^{1+1}, \xi_n^{2+1}, \xi_n^{3+1} \]

\[ \hat{\xi}_n^{1+1} = \xi_n^{1+1}, \xi_n^{2+1}, \xi_n^{3+1} \]

\[ \hat{\xi}_n^{2+1} = \xi_n^{1+1}, \xi_n^{2+1}, \xi_n^{3+1} \]

\[ \hat{\xi}_n^{3+1} = \xi_n^{1+1}, \xi_n^{2+1}, \xi_n^{3+1} \]

\[ g_n(\xi_n^j) \]

\[ g_n(\xi_n^j) \]

\[ g_n(\xi_n^j) \]

\[ g_n(\xi_n^j) \]

\[ g_n(\xi_n^j) \]

\[ g_n(\xi_n^j) \]
One step of the algorithm

\[
\begin{align*}
\hat{\xi}_n^1 &= \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^1,j) \\
\hat{\xi}_n^2 &= \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^2,j) \\
\hat{\xi}_n^3 &= \frac{1}{N_1} \sum_{j=1}^{N_1} g_n(\xi_n^3,j)
\end{align*}
\]

\[
\begin{align*}
\hat{\xi}_n &= \xi_n \\
\hat{\xi}_n^1 &= \hat{\xi}_n^2 = \hat{\xi}_n^3 \\
\hat{\xi}_n^1 &= \hat{\xi}_n^2 = \hat{\xi}_n^3 \\
\hat{\xi}_n^1 &= \hat{\xi}_n^2 = \hat{\xi}_n^3
\end{align*}
\]
One step of the algorithm

\[ \hat{\xi}_n = \xi_n + \sum_{j=1}^{N_1} g_n(\xi_n, j) \]

\[ \hat{\xi}_n^{(1)} = \xi_n^{(1)} + \sum_{j=1}^{N_1} g_n(\xi_n^{(1)}, j) \]

\[ \hat{\xi}_n^{(2)} = \xi_n^{(2)} + \sum_{j=1}^{N_1} g_n(\xi_n^{(2)}, j) \]

\[ \hat{\xi}_n^{(3)} = \xi_n^{(3)} + \sum_{j=1}^{N_1} g_n(\xi_n^{(3)}, j) \]

\[ M_{n+1} = \{ \xi_{n+1}^{(1)}, \xi_{n+1}^{(2)}, \xi_{n+1}^{(3)} \} \]
Bias and variance of the double bootstrap

Bootstrap approximation: bias and variance

Theorem

For any time horizon \( n \geq 0 \) and any bounded function \( f_n \in B_b(X_n, \mathcal{X}_n) \), we have

\[
\lim_{N_1 \to \infty} N_1 \mathbb{E} \left[ \eta_{N_1}^n (f_n) - \eta_n (f_n) \right] = B_n(f_n),
\]

\[
\lim_{N_1 \to \infty} N_1 \text{Var} \left( \eta_{N_1}^n (f_n) \right) = V_n(f_n),
\]

where \( B_n(f_n) \) and \( V_n(f_n) \) can be computed explicitly.
Double bootstrap approximation: bias and variance

**Theorem**

*For any time horizon $n \geq 0$ and any $f_n \in \mathcal{B}_b(\mathcal{X}_n, \mathcal{X}_n)$, we have*

\[
\lim_{N_1 \to \infty} \lim_{N_2 \to \infty} N_1 N_2 \mathbb{E} \left[ \eta_n^{N_2}(m(\cdot, f_n)) - \eta_n(m(\cdot, f_n)) \right] = B_n(f_n) + \tilde{B}_n(f_n),
\]

\[
\lim_{N_1 \to \infty} \lim_{N_2 \to \infty} N_1 N_2 \text{Var} \left( \eta_n^{N_2}(m(\cdot, f_n)) \right) = V_n(f_n) + \tilde{V}_n(f_n),
\]

*where $B_n(f_n)$, $\tilde{B}_n(f_n)$, $V_n(f_n)$, $\tilde{V}_n(f_n)$ can be computed explicitly.*
Bias and variance of the double bootstrap

Independent islands

Theorem

For any time horizon \( n \geq 0 \) and any \( f_n \in \mathcal{B}_b(\mathcal{X}_n, \mathcal{X}_n) \), we have

\[
\lim_{N_1 \to \infty} N_1 \left\{ \mathbb{E} \left[ \tilde{\eta}^{N_2}_n (m(\cdot, f_n)) \right] - \eta_n (f_n) \right\} = B_n (f_n) ,
\]

\[
\lim_{N_1 \to \infty} N_1 N_2 \text{Var} \left( \tilde{\eta}^{N_2}_n (m(\cdot, f_n)) \right) = V_n (f_n) ,
\]

where \( B_n (f_n) \) and \( V_n (f_n) \) are the same than for the single island model.
How to choose between interacting and independent islands?

<table>
<thead>
<tr>
<th></th>
<th>Independent islands</th>
<th>Interacting islands</th>
</tr>
</thead>
<tbody>
<tr>
<td>Squared bias</td>
<td>$\frac{B_n(f_n)^2}{N_1^2}$</td>
<td>$(B_n(f_n) + \tilde{B}_n(f_n))^2$</td>
</tr>
<tr>
<td>Variance</td>
<td>$\frac{V_n(f_n)}{N_1N_2}$</td>
<td>$\frac{V_n(f_n) + \tilde{V}_n(f_n)}{N_1N_2}$</td>
</tr>
<tr>
<td>Sum</td>
<td>$\frac{V_n(f_n)}{N_1N_2} + \frac{B_n(f_n)^2}{N_1^2}$</td>
<td>$\frac{V_n(f_n) + \tilde{V}_n(f_n)}{N_1N_2}$</td>
</tr>
</tbody>
</table>

\[
\frac{V_n(f_n)}{N_1N_2} + \frac{B_n(f_n)^2}{N_1^2} < \frac{V_n(f_n) + \tilde{V}_n(f_n)}{N_1N_2} \iff N_1 > \frac{B_n(f_n)^2}{\tilde{V}_n(f_n)}N_2.
\]
Numerical application: Linear Gaussian Model

- The model is defined by

  - $X_{p+1} = \phi X_p + \sigma_u U_p$ ,
  - $Y_p = X_p + \sigma_v V_p$.

- Computing the predictive distribution of the state $X_n$ given the observations $Y_{0:n-1} = y_{0:n-1}$ up to time $n - 1$ can be cast into the framework of Feynman-Kac model by setting for all $p \geq 0$

  \[ M_{p+1}(x_p, dx_{p+1}) = \frac{1}{\sqrt{2\pi\sigma_u}} \exp \left[ -\frac{(x_{p+1} - \phi x_p)^2}{2\sigma_u^2} \right] dx_{p+1} , \]

  \[ g_p(x_p) = \frac{1}{\sqrt{2\pi\sigma_v}} \exp \left[ -(y_p - x_p)^2 / 2\sigma_v^2 \right] . \]

- We have $\mathbb{E} \left[ X_n | Y_{0:n-1} = y_{0:n-1} \right] = \eta_n(\text{Id})$.
- $n + 1 = 11$ observations were generated with $\phi = 0.9$, $\sigma_u = 0.6$ and $\sigma_v = 1$. 
Numerical application

Results for the LGM

\[ N_1 = 100 \]

\[ N_2 = 100 \]

\[ N_2 = 250 \]

\[ N_2 = 500 \]

\[ N_2 = 1000 \]

(1) Bootstrap

(2) Independent

Real value
Outline

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4. Extensions
Effective Sample Size Interaction

- For any integer $p$ we set
  \[(X_p, \mathcal{X}_p) \overset{\text{def}}{=} ((X_p \times \mathbb{R}^+)^{N_1}, (\mathcal{X}_p \otimes \mathcal{B}(\mathbb{R}^+)) \otimes N_1).\]
- Define

\[\Theta_{n, \alpha} = \left\{ (x_1^n, w_1^n, \ldots, x_{N_1}^n, w_{N_1}^n) \in \mathbb{X}_n \left| \frac{\sum_{i=1}^{N_1} w_i^n g_n(x_i^n)^2}{\sum_{i=1}^{N_1} (w_i^n g_n(x_i^n))^2} \geq \alpha N_1 \right. \right\}\]

- Define $m(x_n, \cdot)$ as the empirical measure of $x_n$ given for any $A_n \in \mathcal{X}_n$ by

\[m(x_n, A_n) \overset{\text{def}}{=} \frac{1}{\sum_{i=1}^{N_1} w_i^n} \sum_{i=1}^{N_1} w_i^n \delta_{x_i^n}(A_n).\]
Effective Sample Size Interaction

- Consider the Markov kernel $M_{n+1}$

$$M_{n+1}(x_n, A_{n+1}) = \begin{cases} \prod_{i=1}^{N_1} \delta_{w_n(x_n)}(B_{n+1}^i)M_{n+1}(x_n, A_{n+1}^i) & x_n \in \Theta_{n,\alpha} \\ \prod_{i=1}^{N_1} \delta_1(B_{n+1}^i)\Psi_n(m(x_n, \cdot))M_{n+1}(A_{n+1}^i) & x_n \not\in \Theta_{n,\alpha} \end{cases}$$

- Define a Markov chain $\{X_n\}_{n \geq 0}$ where for each $n \in \mathbb{N}$,

$$X_n = [(\xi_{n,1}^1, \omega_{n,1}^1), \ldots, (\xi_{n,N_1}^{N_1}, \omega_{n,N_1}^{N_1})] \in X_n$$
ESS: particle approximation

$N_1$-particle approximations of the measures $\eta_n$ and $\gamma_n$

$$\eta_{n}^{N_1}(f_n) \overset{\text{def}}{=} m(X_n, f_n) = \frac{1}{\sum_{i=1}^{N_1} \omega_n^i} \sum_{i=1}^{N_1} \omega_n^i f_n(\xi_n) ,$$

$$\gamma_{n}^{N_1}(f_n) \overset{\text{def}}{=} \eta_{n}^{N_1}(f_n) \prod_{0 \leq p < n} \eta_{p}^{N_1}(g_p) .$$

Theorem

For any $f_n \in B_b(X_n, X_n)$, $\gamma_{n}^{N_1}(f_n)$ is an unbiased estimator of $\gamma_n(f_n)$:

$$\mathbb{E} \left[ \gamma_{n}^{N_1}(f_n) \right] = \mathbb{E} \left[ \eta_{n}^{N_1}(f_n) \prod_{0 \leq p < n} \eta_{p}^{N_1}(g_p) \right]$$

$$= \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right] .$$
ESS: Feynman-Kac approximation

- For $x_n = (x_n^1, w_n^1, \ldots, x_n^{N_1}, w_n^{N_1}) \in \mathbb{X}_n$ we set

$$g_n(x_n) \overset{\text{def}}{=} m(x_n, g_n) = \frac{1}{\sum_{i=1}^{N_1} w^i_n} \sum_{i=1}^{N_1} w^i_n g_n(x_n^i).$$

- The associated Feynman-Kac model $\{(\eta_n, \gamma_n)\}_{n \geq 0}$ is

$$\eta_n(f_n) = \gamma_n(f_n) / \gamma_n(1)$$

$$\gamma_n(f_n) = \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right],$$
ESS: Feynman-Kac approximation

Since $g_n(X_n) = \eta^{N_1}_n(g_n)$, for any $f_n$ of the form

$$f_n(x_n) = \left(\sum_{i=1}^{N_1} w_n^i\right)^{-1} \sum_{i=1}^{N_1} w_n^i f_n(x_n^i)$$

where $f_n \in \mathcal{B}_b(X_n, \mathcal{X}_n)$,

$$\mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right] = \mathbb{E} \left[ f_n(X_n) \prod_{0 \leq p < n} g_p(X_p) \right],$$

Therefore

$$\gamma_n(f_n) = \gamma_n(f_n) \text{ and } \eta_n(f_n) = \eta_n(f_n).$$
1: for $p$ from 0 to $n - 1$ do
2:  Selection step and weight actualization between islands:
3:  Set $N_{2}^{\text{eff}} = \left( \sum_{i=1}^{N_{2}} \Omega_{p}^{i} g_{p}(\xi_{p}^{i}, \omega_{p}^{i}) \right)^{2} / \sum_{i=1}^{N_{2}} \left( \Omega_{p}^{i} g_{p}(\xi_{p}^{i}, \omega_{p}^{i}) \right)^{2}$.
4:  if $N_{2}^{\text{eff}} \geq \alpha_{\text{Islands}} N_{2}$ then
5:    For $1 \leq i \leq N_{2}$, set $\Omega_{p+1}^{i} = \Omega_{p}^{i} g_{p}(\xi_{p}^{i}, \omega_{p}^{i})$.
6:    Set $I_{p} = (I_{p}^{i})_{i=1}^{N_{2}} = (1, 2, \ldots, N_{2})$.
7:  else
8:    Set $\Omega_{p+1} = (\Omega_{p+1}^{i})_{i=1}^{N_{2}} = (1, \ldots, 1)$.
9:    Sample $I_{p} = (I_{p}^{i})_{i=1}^{N_{2}}$ multinomially with proba. prop. to $(\Omega_{p}^{i} g_{p}(\xi_{p}^{i}, \omega_{p}^{i}))_{i=1}^{N_{2}}$.
10: end if
11: Island mutation step:
12: for $i$ from 1 to $N_{2}$ do
13:  Particle selection and weight actualization within each island:
14:    same business as usual
15: end for
16: end for
17: Approximate $\eta_{n}(f_{n})$ by $\frac{1}{\sum_{i=1}^{N_{2}} \Omega_{n}^{i}} \sum_{i=1}^{N_{2}} \Omega_{n}^{i} \frac{1}{\sum_{j=1}^{N_{1}} \omega_{n}^{i,j}} \sum_{j=1}^{N_{1}} \omega_{n}^{i,j} f_{n}(\xi_{n}^{i,j})$. 
Results for the ESS model

- \( N_2 = 100 \)
- \( N_2 = 250 \)
- \( N_2 = 500 \)
- \( N_2 = 1000 \)

- \( N_1 = 100 \)
- \( N_1 = 250 \)
- \( N_1 = 500 \)
- \( N_1 = 1000 \)

(1) ESS  (2) Bootstrap  (3) Independent

- Real value
Number of interactions between islands using ESS as a percentage of the one using Bootstrap

<table>
<thead>
<tr>
<th>$N_1$</th>
<th>100</th>
<th>250</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>4.32</td>
<td>4.76</td>
<td>4.92</td>
<td>4.98</td>
</tr>
<tr>
<td>250</td>
<td>0.88</td>
<td>0.60</td>
<td>0.34</td>
<td>0.32</td>
</tr>
<tr>
<td>500</td>
<td>0.04</td>
<td>0.02</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1000</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>