Interacting Particle Models of Systemic Risk

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Systemic Risk: Motivation

Stochastic Model

Simulating Systemic Shocks

Examples of Rare Events
Systemic Risk in Banking Networks

- Banking networks pose the most significant source of systemic risk due to their high degree of interconnections and impact on the economy.
- These networks are highly complex, continuously evolve over time and are subject to many ongoing shocks.
- Stochastic modeling of this problem remains open with multiple proposed approaches.
- We propose a new view using methods of Interacting Particle Systems.
- A flexible framework that is convenient for large-scale qualitative analysis while allowing for much granularity.
Modeling Approach

- Focus on the dynamic evolution of the whole network.
- Think of banks as particles or *individuals* that interact with other banks.
- Banks are born, grow over time, and eventually die (i.e. default).
- These mechanisms, especially default, involve mean-field-type interactions that create inter-dependencies.
Related Literature

- Network models: Cont et al. (2011a,b)
- Scaling Limits of Large Portfolio Losses: Giesecke et al. (2011), Cvitanic et al. (2011)
- Large deviations approaches: Giesecke et al. (2011), Papanicolaou (2012)
Stochastic Model

- $X_i(t)$ – net assets of bank $i$ at date $t$.
- Each bank is born at epoch $g_i$ and dies at $d_i$.
- Let $I(t) = \{i : g_i \leq t < d_i\}$ be the subset of banks alive at $t$.
- $N(t) = |I(t)|$ is the size of the system.
- $\bar{X}(t) := \frac{1}{N(t)} \sum_{i \in I(t)} X_i(t)$ – average bank size.
- $X(t) \in \bigcup_n \mathbb{R}^n$ is the state process.
System Evolution

- $N(t)$ is a **birth-and-death process** modeled through corresponding event intensities.
- Birth rate $\lambda^+(N(t))$ – larger when $N(t)$ is small (less competition).
- Birth size $\xi_i \sim F$.
- Default rate $\lambda^-(X_i(t))$ – reduced-form credit model; decreases in $X_i$.
- Due to interconnections, defaults affect other banks: $X_i(d_j+) = [1 - \theta(X_j(d_j-), \bar{X}(d_j-), \omega)] \cdot X_i(d_j-)$, where $\theta(\cdot)$ is the random proportion of bank $i$ assets reduced due to default of bank $j$.
- Between defaults, assets grow at deterministic rate $r_i$.
- **Piecewise-deterministic** model.
**Individual dynamics:**

\[ X_i(t) = X_i(g_i) + \int_{g_i}^{t} r_i X_i(s) ds - \sum_{j=1}^{\infty} \theta(X_j(d_j-), \overline{X}(d_j-)) X_i(d_j-) \mathbf{1}\{g_i < d_j \leq t\} ; \quad g_i \leq \lambda^- (x) \]

**Collective dynamics:** 

\[ P(t) = \prod_{i \in I(t)} X_i(t) \]

\[ P(t) = P(s) \exp \left[ \sum_{j} (r_j(t - g_j) + \log \xi_j) \mathbf{1}\{s < g_j < t\} \right. \]

\[ + \sum_{\ell} \left\{ (N(d_\ell-) - 1) \log(1 - \theta(\overline{X}(d_\ell-))) - r_\ell(t - s) - \log X_\ell(s) \right\} \mathbf{1}\{s < d_\ell < t\} \].

**Illustrative example:**  

\[ r_i \equiv r, \quad \lambda^-(x) = \frac{\lambda^-}{1+x}, \quad \lambda^+(n) = \frac{\lambda^+}{1+n}, \]

\[ \theta(x_j, \overline{X}) \sim \text{Beta}(\frac{x_j}{\rho x}, b). \]
Sample Path of System Dynamics

Figure: Trajectory of mean asset size $\bar{X}(t)$ and normalized number of banks $N(t)/N(0)$. Bottom panel: default times and corresponding impact proportions $\theta(\cdot)$. We take $r = 0.02$, $\lambda^{-}(x) = 0.05/(1 + x)$, $\lambda^{+}(n) = 300/(1 + n)$, $\xi \sim \text{Exp}(1)$ and $N_0 = 100$. and $N_0 = 100$.
Model Features

- The system is **self-stabilizing**: when \( N(t) \) is large there are more defaults causing \( \overline{X}(t) \) to fall; when \( N(t) \) is small, birth rate is higher.

- Lifetime of each individual bank is finite.

- High turnover microscopically; stable macroscopically.

- If \( \lambda^-(x) > 0 \) is bounded away from zero then with positive probability the system will eventually completely collapse and regenerate. Large shocks are intrinsic.

- Numerous **add-ons** are possible: diffusion terms in \( X_i(t) \); more correlations; default cascades; inhomogeneous dynamics, etc.
Recurrence

Proposition

*Under technical conditions on $\lambda^-$ and $\theta$, the system has an invariant distribution on $\bigcup_n \mathbb{R}^n$.***

- Rule out explosion (intrinsic defaults + interaction); births allow regeneration.
- Sufficient condition: $\int_0^\infty \lambda^- (x_0 e^{rt}) \, dt = +\infty$ for all $x_0$. 
Systemic Shocks

- While the system will eventually collapse and regenerate, large shocks are very rare.
- We are interested in understanding the mechanism/frequency of such shocks in terms of model ingredients.
- Focus on the case of moderately large $N(0)$ – still far from mean-field limit.
- Large shock $= N(t)$ or $X(t)$ "small".
- e.g. let $\tau = \inf \{ t : N(t) < n \}$. Wish to compute:
  - $P(N(T) < n)$;
  - $P(\tau < T)$;
  - Path-distribution of $X(\cdot)$ on $[0, \tau)$.
- Monte Carlo methods seem to be the only feasible tool in this direction.
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- **Monte Carlo** methods seem to be the only feasible tool in this direction.
Generating Large Shocks

- Think of one realization of $X(\cdot)$ as a "scenario".
- Naive use of Monte Carlo to generate a lot of scenarios to understand behavior of big shocks is very inefficient when the corresponding probabilities are small.
- Variance Reduction is a must.
- Importance Sampling is hard to implement as it’s not clear how to choose a new measure for $X$.
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- Importance Sampling is hard to implement as it’s not clear how to choose a new measure for $X$.
- Use **Empirical IS** Method originally proposed in Del Moral and Garnier (2005); elaborated in Carmona et al (2009).
Empirical Importance Sampling

- Think of each scenario as a "path particle".
- Scenarios are propagated forward using a genetic-type algorithm: \textit{resampling-mutation} steps.
- Particles that get "closer" to events of interest are assigned higher weights and are more likely to spawn children.
- Particles with low weights get culled.
- Work under the original system dynamics.
Feynman Kac Potentials

- Consider a collection of $J$ scenario path-particles $X^{(j)}$, $j = 1, \ldots, J$.
- The particles evolve according to a Feynman-Kac measure change:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \frac{1}{Z_T} \prod_{k=1}^{T} G_k(X)$$

where $G_k$ are the Feynman-Kac potentials on the increasing path spaces of length $k$: $G_k(X) \equiv G_k(X(t_0), X(t_1), \ldots, X(t_k))$.

- Sequential algorithm: At dates $t_k$ re-sample particles using weights $w_j := \frac{G_k(X^{(j)})}{\sum_\ell G_k(X^{(\ell)})}$.
- Propagate particles independently using $\mathbb{P}$-dynamics.
IS Approximation

 Approximate:

\[
\mathbb{E}[f(\mathbf{X}(T))] = \mathbb{E} \left[ f(\mathbf{X}(T)) \prod_{k=1}^{T} \mathbf{G}^{-1}_{k}(\mathbf{X}) \prod_{k=1}^{T} \mathbf{G}_{k}(\mathbf{X}) \right]
\]

\[
= \eta_{T}(\tilde{f}(\mathbf{X})) \prod_{k=1}^{T} \eta_{k}(\mathbf{G}_{k}), \quad \tilde{f}(\mathbf{X}) := f(\mathbf{X}(T)) \prod_{k=1}^{T} \mathbf{G}^{-1}_{k}(\mathbf{X})
\]

\[
\eta_{k}(g) = \gamma_{k}(g)/\gamma_{k}(1) \quad \gamma_{k}(g) := \mathbb{E} \left[ g(\mathbf{X}) \prod_{\ell=1}^{k} \mathbf{G}_{\ell}(\mathbf{X}) \right].
\]

 Obtain an unbiased estimator based on

\[
\eta_{k}^{(j)}(\mathbf{G}_{k}) = \frac{1}{J} \sum_{j=1}^{J} \mathbf{G}_{k}(\mathbf{X}^{(j)}).
\]

 Typical potential: \( \mathbf{G}_{k}(\mathbf{X}^{(j)}(t_{0}), \mathbf{X}^{(j)}(t_{1}), \ldots \mathbf{X}^{(j)}(t_{k})) = \exp \left( \alpha \left( \min_{\ell \leq k-1} N(t_{\ell}) - \min_{\ell \leq k} N(t_{\ell}) \right) \right) \) (multiplicative).

 Preference to particles where \( N(t) \) is setting new lows.
Importance Sampling

**Figure:** Histogram of 400 MC simulations: under original measure (independent scenarios) and using a F-K potential \( G_k(X) = \exp(-\alpha(\min_{\ell\leq k} N(\ell) - \min_{\ell<k} N(\ell))) \).
Algorithm Details

- Sufficient to have just a few hundred particles.
- The crux of the method is in choosing a good potential $G_k$ (in particular the potential strength $\alpha$) – must be adapted to the problem at hand.
- Resample for example every 1 period (in between use exact simulation of the birth-and-death process $N(t)$ and corresponding $X^{(i)}$, $t_k = k$.
- By storing the path genealogies of the particles surviving at $T$ have access to (unbiased) conditional distribution of $X(\cdot)$ before the shock (note: path degeneracy).
- The method dynamically twists the measure while keeping the simulation part very simple.
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Examples of Rare Events

- Probability of a very low number of banks (or very low total bank assets) at a fixed date $T$.
- Probability of a major shock in the next $T$ periods.
- Relationship between low $N(T)$ and low $\overline{X}(T)$.

Example:

\[
\begin{align*}
\mathbb{P}(\min_{s \leq 50} N(s) < 80) & \approx 0.011 \\
\mathbb{P}(\min_{s \leq 50} N(s) < 70) & \approx 7.3 \cdot 10^{-5} \\
\mathbb{P}(\min_{s \leq 50} N(s) < 60) & \approx 1.2 \cdot 10^{-7}.
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Observed Effects

- Little correlation between low $N(T)$ versus low $\overline{X}(T)$.
- Low $\overline{X}(T)$ is caused by a single major default ("Lehman").
- Low $N(T)$ is due to ongoing small defaults = "recession".
- Conditional on many defaults: $\overline{X}(T)$ is low, but $N(T)$ remains moderate.
- These properties are sensitive to the assumed default-size distribution.
Genealogy of the path-particles

Figure: Three particle genealogies conditional on a high number of defaults.
Numerical Challenges

- For estimating rare event probabilities we have a well-developed theory based on large deviations.
- The method provides an estimate of the CoV that can be used to guide potential function selection and evaluation of the performance.
- In practice, these estimates are quite stable.
- We are also interested in the conditional trajectories to rare events. Here it is hard to check accuracy/efficiency. Numerical evidence suggests that different potentials are more efficient for this task.
- Shape of the potential strongly affects the shape of the ancestral tree (reward the fit vs. cull the weak).
- Splitting Methods vs. Exponential Tilting.
Work in Progress

- Have a framework for simulating dynamic defaults in an interacting network.
- Understanding the relationships and mechanisms of rare events unlocks many interesting phenomena (a cascade of small defaults vs. a single large event).
- Another analytic tool is the LLN scaling limit as $N(0) \to \infty$ (average out the noise from the defaults and births).

THANK YOU!
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THANK YOU!
Toy Example

- $X_t = W_t - cN_t$ jump-diffusion model ($c \gg 1$)
- Rare event is: $A = \{ X_T \in -da \}$
- $G(x_t, x_{t-1}) = \exp(-\alpha(x_t - x_{t-1}))$ gives a strong preference to particles that jump.
- $G(x_t, x_{t-1}) = \exp(-\alpha(x_t - x_{t-1})) \land M$ is more "democratic" and creates a more diverse ancestral tree with minimal efficiency loss.

**Figure**: Ancestral Trees for 2 different FK potentials.
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