Variance reduction by conditioning in the pricing problem where the underlying is a continuous-time finite state Markov process

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1based on joint work with J.M. Montes and V. Prezioso
Asset price evolutions are generally given by a geometric Brownian motion or an exponential Levy.

The latter includes the case of a continuous time Markov chain (CTMC), but for this case a direct approach is computationally more convenient.
A full theory of financial markets based on CTMC (prices, rates or, more generally, factors) is given in Norberg (2003).

For an underlying $X_t \in \{x^1, \cdots, x^N\}$ with a time homogeneous transition intensity matrix $Q$ and a simple claim of the form $H(X_T) = H_0 := [H(x^1), \cdots, H(x^N)]'$, the price $\Pi_i(t)$ at time $t$ when $X_t = x^i$ is given by

$$\Pi_i(t) = [\exp\{(Q - R)(T - t)\} H_0]_i$$

where $[z]_i$ denotes the $i$–th component of the vector $z$ and $R$ is the diagonal matrix with elements $r^i$ ($i = 1, \cdots, N$) having assumed that $r_t = r^i$ if $X_t = x^i$. 
The previous explicit formula may not be of much use if:

- The evolution of the underlying is not time homogeneous;
- the underlying is multivariate;
- the derivative is path dependent.
In all these more involved cases a Monte Carlo (MC) simulation is always possible:

- For the CTMC $X_t$ simulate the successive jump times $\tau_n$ and the values $X_n$ of $X_t$ at $\tau_n$.

- For an intensity matrix $Q = \{q_{i,j}\}$, putting $q_i = \sum_{i \neq j} q_{i,j}$ one has that, if $X_{\tau_n} = x^i$, the inter-jump times $\tau_{n+1} - \tau_n$ are exponentially distributed with parameter $q_i$ and the probability for $X_{\tau_{n+1}} = j \neq i$ is $p_{i,j} = \frac{q_{i,j}}{q_i}$.
Show that, conditionally on the number $\nu_{t,T}$ of jumps of $X_t$ in a given interval $[t, T]$, one can obtain an \textit{explicitly computable expression} also for \textit{exotic derivatives} and when the underlying is \textit{multivariate} and/or has a time \textit{non homogeneous} evolution.
Since

\[ \Pi_i(t) = E^{\tilde{P}} \left\{ e^{-\int_t^T r_s ds} H(X_T) \mid X_t = i \right\} \]

\[ = E^{\tilde{P}} \left\{ E^{\tilde{P}} \left\{ e^{-\int_t^T r_s ds} H(X_T) \mid \nu_{t,T}, X_t = i \right\} \mid X_t = i \right\} \]

where \( \tilde{P} \) is a (calibrated) martingale measure, then, since the inner expression allows for an explicit computation, one needs to simulate only the r.v. \( \nu_{t,T} \).

With respect to a full MC this allows to reduce the variance (variance reduction by conditioning).

→ Shall show how to compute the inner expression in various more general cases
For convenience of exposition we first present the procedure for the case of a simple claim on a time homogeneous underlying $X_t$ given by a CTMC.

Successively we show the extensions/changes for the more general case.

Finally we present numerical results and comparisons.
The model

The model (simple case first)

\( X_t \) a \textbf{CTMC} under a martingale measure \( \tilde{P} \)

- state space \( E = \{x^1, x^2, \ldots, x^N\}, \ N \in \mathbb{N} \) \textit{(identify } x^i \textit{ with } i \text{)}
- \( Q = (q_{i,j})_{1 \leq i,j \leq N} \) the \textit{transition intensity matrix}, homogeneous w.r. to time
- \( q_i := \sum_{j=1 \atop j \neq i}^{N} q_{i,j}, \ i = 1, \ldots, N \) the \textit{intensities} associated with the states \( x^i \).
The model

- \( \tau_n \): random time at which the \( n^{th} \) jump occurs,
- \( X_n := X_{\tau_n} \) and \( X_s \equiv X_n \) for \( s \in [\tau_n, \tau_{n+1}) \)
- \( r_{\tau_n} = r^i \) if \( X_{\tau_n} = x^i \) \((i = 1, \ldots, N)\), i.e. \( r^i = r(x^i) \)
  (write \( r_n := r_{\tau_n}; \ r_s = r_n \) for \( s \in [\tau_n, \tau_{n+1}) \))
- \( (\tau_{n+1} - \tau_n \mid X_{\tau_n} = x^i) \sim \exp(q_i) \)

- \( \nu_t := \sup\{n \mid \tau_n \leq t\} \) \((\#\text{of jumps up to time } t)\)

\[
\begin{array}{ccccccc}
& & t & & & & T \\
\tau_{\nu_t} & | & \tau_{\nu_t+1} & | & \tau_{\nu_T} & | & \tau_{\nu_T+1}
\end{array}
\]
Pricing of a derivative

\[ \Pi(t) = E^{\tilde{P}} \left\{ e^{- \int_t^T r_s ds} H(X_T) \mid \mathcal{F}_t \right\} \]

\[ = \sum_{i=1}^N E^{\tilde{P}} \left\{ e^{- \int_t^T r_s ds} H(X_T) \mid X_t = i \right\} \mathbf{1}_{\{X_t=i\}} \]

\[ \downarrow \]

\[ \Pi_i(t) = E^{\tilde{P}} \left\{ \exp[r(t - \tau_{\nu t})] \exp \left[ - \sum_{i=\nu t}^{\nu T-1} r_i(\tau_{i+1} - \tau_i) - r_T(T - \tau_{\nu T}) \right] H(X_T) \mid X_t = i \right\} \]

\[ = \exp[r(t - \tau_{\nu t})] E^{\tilde{P}} \left\{ \exp \left[ - \sum_{i=\nu t}^{\nu T-1} r_i(\tau_{i+1} - \tau_i) - r_T(T - \tau_{\nu T}) \right] H(X_T) \mid X_t = i \right\} \]

\[ \rightarrow \text{Not restrictive to assume } t = \tau_{\nu t} \]
Prototype product

Prototype product (analogue to Arrow-Debreu prices)

- Its price at time $t < T$ is

$$V_{H_0,t,T}(X_t) = \mathbb{E}^{\tilde{P}} \left\{ \exp \left[ - \sum_{i=\nu_t}^{\nu_T-1} r_i (\tau_{i+1} - \tau_i) - r_{\nu_T} (T - \tau_{\nu_T}) \right] H_0(X_T) \mid X_t \right\}$$

with

$$H_0(\cdot) = \sum_{i=1}^{N} w_i^0 1_{\{\cdot = x^i\}}, \quad x^i \in E, \quad w_i^0 \in \mathbb{R}$$

→ In the calculations to follow we shall drop the last factor, i.e. compute an upper bound (in general it is small and can be included in the simulations).
For given $n \in \mathbb{N}$ consider the recursions
\[
\begin{align*}
H_0(X_{\nu_t+n}) & \text{ given by the } \textbf{Prototype payoff} \quad (H_0(\cdot) = \sum_{i=1}^{N} w_i^0 1_{\{\cdot = x^i\}}) \\
H_h(X_{\nu_t+n-h}) & = \mathbb{E}_{\tilde{P}} \left\{ e^{-r_{\nu_t+n-h}(\tau_{\nu_t+n-h+1}-\tau_{\nu_t+n-h})} H_{n-1}(X_{\nu_t+n-h+1}) \mid X_{\nu_t+n-h} \right\} \\
& \forall h = 1, \ldots, n
\end{align*}
\]

**Proposition:** The price of the Prototype product can be computed as
\[
V_{H_0,t,T}(X_t) = \mathbb{E}_{\tilde{P}} \left\{ H_{\nu_t,T}(X_t) \mid X_t \right\} = \sum_{n=0}^{+\infty} H_n(X_t) \tilde{P}(\nu_{t,T} = n \mid X_t)
\]
where
- $\nu_{t,T} = \nu_T - \nu_t$ (number of jumps between $t$ and $T$)
- $H_n(X_t) = H_n(X_{\nu_t})$ is as obtained recursively above.
Setting \( \underline{x} = [x^1, \ldots, x^N]' \) we have the representations

\[
H_0(\underline{x}) := [w_1^0, \ldots, w_N^0]' \quad \rightarrow \quad H_n(\underline{x}) := [w_1^n, \ldots, w_N^n]'
\]

Putting, furthermore,

\[
\tilde{Q} = (\tilde{q}_{i,j})_{1 \leq i, j \leq N} \quad \text{with} \quad \tilde{q}_{i,j} = \begin{cases} \frac{q_{i,j}}{r^i + q_i} & i \neq j \\ 0 & i = j \end{cases}
\]

one obtains, at the generic \( \tau_n \), the following one-step evolution of \( H_n \),

\[
H_n(\underline{x}) = \tilde{Q} H_{n-1}(\underline{x}),
\]

\( \rightarrow \) It also follows that \( H_n(\underline{x}) = \tilde{Q}^n H_0(\underline{x}) \) by putting \( \tilde{Q}^0 = I_N \).
The actual derivative price is then given by

\[ \Pi_i(t) = V_{H_0,t,T}(X_t)|_{X_t=x^i} \]

\[ = \sum_{n=0}^{\infty} \left[ \tilde{Q}^n H_0(x) \right]_i \tilde{P} \left( \nu_{t,T} = n \mid X_t = x^i \right) \]

\[ = \mathbb{E} \tilde{P} \left\{ \left[ Q^{\nu_{t,T},H_0}(x) \right]_i \mid X_t = x^i \right\} \]

([z]_i is the i-th component of the vector z).

From here two possibilities for actual computation:

- **Explicit numerical computation** (middle term)
- **MC simulation by simulating just \( \nu_{t,T} \)** (rightmost term), i.e. **MC simulation by conditioning.**
CIR AFFINE TERM
STRUCTURE MODEL
\[
\begin{align*}
    dr(t) &= \theta(k - r(t)) \, dt + \sigma \sqrt{r(t)} \, dW_t \\
    r(0) &= \tilde{r}
\end{align*}
\]

RECOMBINING BINOMIAL TREE (RBT)

CONTINUOUS-TIME AFFINE DIFFUSION MODEL (closed formula) (CF)

KUSHNER’S APPROXIMATION (K-A)

PROTOTYPE PRODUCT METHOD (PPM)
 prototype product

Bond prices with **CF, RBT, PPM(MC1)+K-A and PPM(MC2)+K-A**

(*stepsMC=stepsRBT=500*)

<table>
<thead>
<tr>
<th>$T$(years)</th>
<th>0.5</th>
<th>0.5</th>
<th>0.5</th>
<th>0.5</th>
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<tr>
<td>$\tilde{r}(=r^i)$</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>$k$</td>
<td>0.8</td>
<td>0.5</td>
<td>1.1</td>
<td>1.2</td>
</tr>
<tr>
<td>$\theta$</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
<td>0.02</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.1</td>
<td>0.05</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>CF</td>
<td>0.995014</td>
<td>0.990051</td>
<td>0.985116</td>
<td>0.990052</td>
</tr>
<tr>
<td>RBT</td>
<td>0.995042</td>
<td>0.99007</td>
<td>0.985146</td>
<td>0.990072</td>
</tr>
<tr>
<td>PPM(MC1)+K-A</td>
<td><strong>0.995024</strong></td>
<td><strong>0.990143</strong></td>
<td><strong>0.985128</strong></td>
<td><strong>0.990059</strong></td>
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<tr>
<td>PPM(MC2)+K-A</td>
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<td>0.989963</td>
<td>0.984903</td>
<td><strong>0.990049</strong></td>
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</table>
**Prototype product**

Bond prices with **CF**, **RBT**, **PPM(MC1)+K-A** and **PPM(EF)+K-A**  
(\(stepsMC=stepsRBT=500\))

<table>
<thead>
<tr>
<th></th>
<th>(T) (years)</th>
<th>(\tilde{r} (= r^i))</th>
<th>(k)</th>
<th>(\theta)</th>
<th>(\sigma)</th>
<th><strong>CF</strong></th>
<th><strong>RBT</strong></th>
<th><strong>PPM(MC1)+K-A</strong></th>
<th><strong>PPM(EF)+K-A</strong></th>
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<td>0.861394</td>
<td>0.861104</td>
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</table>
Extensions

- Time inhomogeneous case
  - Knock-in/knock-out options
    (may include credit risky derivatives)

- $X_t$ multivariate
  - Path dependent derivatives/claims
    - lookback options
    - Asian options
Consider e.g. \((X_t, Y_t)\) with

\[
X_t \in \{x^1, \ldots, x^N\} \quad \text{and} \quad Y_t \in \{y^1, \ldots, y^M\}
\]

and put

\[
r_{\tau_n} = r^{i,h} \quad \text{if} \quad (X_{\tau_n}, Y_{\tau_n}) = (x^i, y^h)
\]

\rightarrow (in the more general time-inhomogeneous case)

\[
Q(n) = \left\{ q^h_{(i,h),(j,k)} \right\} \begin{bmatrix}
i, j & = 1, \ldots, N \\
h, k & = 1, \ldots, M
\end{bmatrix}
\]
With \( z = (x, y)' \) where \( x = (x^1, \cdots, x^N), \ y = (y^1, \cdots, y^M) \) and

\[
H_0(z) = H_0(x, y) = [w_1, \cdots, w_{N\cdot M}]'
\]

also

\[
H_n(z) = \tilde{Q}(n)H_{n-1}(z)
\]

where

\[
\tilde{Q}(n) = \left\{ \frac{q^n_{(i,h),(j,k)}}{r^{i,h} + q^n_{i,j}} \right\}
\]

\[
\begin{bmatrix}
    i, j & = 1, \cdots, N \\
    h, k & = 1, \cdots, M 
\end{bmatrix}
\]

with \( q^n_{i,j} = \sum_{j \neq i, k \neq h} q^n_{(i,h),(j,k)} \).
Lookback call options

- For an underlying CTMC $X_t$ consider a claim of the form

$$ H_T = \left( X_T - g(X^T_0) \right)^+ $$

- Put $Y_t := g(X^t_0)$ assuming $g(\cdot)$ to take a given finite number of values.

  - For $t \leq T$, the process $Y_t$ then takes a finite number of values (w.l.o.f $g$. we can identify them with $h = 1, \ldots, M$)
  - it jumps only at jump times of $X_t$.

- Assume, furthermore,

$$ g(X^{\tau_n}_0) = G(X_{\tau_n}, g(X^{\tau_{n-1}}_0)) \quad \text{for some measurable } G(\cdot, \cdot) $$

  - $(X_t, Y_t)$ is a CTMC and $H_T = (X_T - Y_T)^+$.
  - Need only to derive the $Q$–matrix for $(X_t, Y_t)$. 
Recall that, if for a scalar CTMC $X_t$ the $Q$–matrix is $q = \{q_{i,j}\}$, then the transition probabilities of the embedded chain $X_n$ are

$$p_{i,j} = \frac{q_{i,j}}{q_i} \quad \text{with} \quad q_i = \sum_{j \neq i} q_{i,j} \quad (q_{i,i} = p_{i,i} = 0)$$

Viceversa, given $p_{i,j}$, there are various possible $q_{i,j}$ that lead to the same $p_{i,j}$. They differ by the choice of $q_i$ since we have $q_{i,j} = q_i p_{i,j}$. 
Since in our case $Y_t$ can jump only when $X_t$ does, we may put

$$q_{(i,h)} \quad (= \sum_{j,k} q_{(i,h),(j,k)}) = q_i \quad \forall h = 1, \ldots, M$$

where $q_i$ is the intensity of leaving state $i$ for the chain $X_t$. (At a generic $\tau_n$ the process $X_t$ actually leaves the current state, while $Y_t$ may jump to itself)

$\rightarrow$ Start thus from constructing $p_{(i,h),(j,k)}$. 
We have (recall $X_n = X_{\tau_n}$, $Y_n = Y_{\tau_n}$)

$$p(i,h),(j,k) := P\{X_{n+1} = j, Y_{n+1} = k \mid X_n = i, Y_n = h\}$$

$$= P\{X_{n+1} = j, G(X_{n+1}, Y_n) = k \mid X_n = i, Y_n = h\}$$

$$= P\{X_{n+1} = j \mid X_n = i\} 1\{G(j,h)=k\} = p_{i,j} 1\{G(j,h)=k\}$$

$$\rightarrow q(i,h),(j,k) = p(i,h),(j,k) \cdot q_i = q_{i,j} 1\{G(j,h)=k\}$$
Example

Let \( Y_t = g(X_t) := \min_{s \leq t} X_s \)

(Yₜ has the same finite number of possible values as Xₜ)

\[
G(X_{\tau_n}, g(X_{0}^{\tau_n-1})) = \min \left[ X_{\tau_n}, \min_{s \leq \tau_n-1} X_s \right]
\]

In this case (states in increasing order of magnitude)

\[
p_{(i,h),(j,k)} = \begin{cases} 
    p_{ik} & \text{if } k < h \\
    p_{ij} & \text{if } k = h, j \geq k \\
    0 & \text{if } k > h 
\end{cases}
\]

and, consequently,

\[
q_{(i,h),(j,k)} = p_{(i,h),(j,k)} \cdot q_i = \begin{cases} 
    q_{ik} & \text{if } k < h \\
    q_{ij} & \text{if } k = h, j \geq k \\
    0 & \text{if } k < h 
\end{cases}
\]
Asian options

For Asian options consider the two processes

\[
\begin{align*}
  & X_t \quad \text{a CTMC, and} \\
  & Y_t := \int_0^t X_s ds = \sum_{\tau_n \leq t} X_{\tau_{n-1}}(\tau_n - \tau_{n-1}) + X_{\tau_n}(t - \tau_n)
\end{align*}
\]

The claim of a **standard Asian option** can then be represented as (assume \( T = \tau_{\nu_T} \))

\[
H_T = \left( \frac{1}{T - t} \int_t^T X_s ds - K \right)^+ = \left( \frac{1}{T - t} (Y_T - Y_t) - K \right)^+
\]
Assuming that *(increasing order of magnitude)*

\[ X_t \in \{ x^1, \cdots, x^N \} , \quad \text{(denote them by} \ i = 1, \cdots, N) \]

the range for the values of \( Y_t \) is

\[
\left[ 0, T \max_{t \leq T} X_t \right] = \left[ 0, T x^N \right]
\]

Partition now the interval \([0, T x^N]\) into intervals of equal length \( \Delta \) assuming that \( T x^N = K \Delta \) for a suitable positive integer \( K \). The generic \( k-\)th interval of the partition is then

\[
A^k = [a^{k-1}, a^k) = [(k - 1) \Delta, k \Delta) , \quad k = 1, \cdots, K
\]
Denote by $y^k$ the midpoint of $A^k$ (other choices are possible) and let $Y_t = y^k$ if $Y_t \in A^k$ (in what follows denote this value simply by $k$).

At the generic jump time $\tau_n < T$ of the chain $X_t$ we then have

$$Y_{n+1} \in A^k \iff Y_n + X_n(\tau_{n+1} - \tau_n) \in A^k$$

$$\iff \tau_{n+1} - \tau_n \in \left[ \frac{a^{(k-1)} - Y_n}{X_n}, \frac{a^k - Y_n}{X_n} \right]$$

$$= \left[ \frac{(k-1) \Delta - Y_n}{X_n}, \frac{k \Delta - Y_n}{X_n} \right]$$
For $\tau_n \leq T$ it turns out that

$$P\{X_{n+1} = j, Y_{n+1} = k \mid X_n = i, Y_n = h\}$$

$$= P\{Y_{n+1} = y^k \mid X_{n+1} = x^j, X_n = x^i, Y_n = y^h\} p_{i,j}$$

$$= P\{Y_{n+1} \in A^k \mid X_n = x^i, Y_n = y^h\} p_{i,j}$$

$$= P \left\{ \frac{(k-1)\Delta - y^h}{x^i} \leq \tau_{n+1} - \tau_n < \frac{k\Delta - y^h}{x^i} \right\} p_{i,j}$$

$$= e^{-q_i \frac{(k-1)\Delta - y^h}{x^i}} \left[ 1 - e^{-q_i \frac{\Delta}{x^i}} \right] \frac{q_{ij}}{q_i}$$

It follows that

$$q_{(i,h),(j,k)} = q_{ij} e^{-q_i \frac{(k-1)\Delta - y^h}{x^i}} \left[ 1 - e^{-q_i \frac{\Delta}{x^i}} \right]$$
Comparing Plain MC and MC + Variance Reduction for Lookback Call pricing.

\( E = [0.8, 0.9, 1.0, 1.1, 1.2], \ x_0 = 3, \ T = 2 \text{ years} \)

- **Q-matrix for Test 1**

\[
Q = \begin{bmatrix}
-1200 & 300 & 300 & 300 & 300 \\
0.6 & -2.4 & 0.6 & 0.6 & 0.6 \\
6 & 6 & -24.0 & 6 & 6 \\
21 & 21 & 21 & -84 & 21 \\
400 & 400 & 400 & 400 & -1600
\end{bmatrix}
\]

- **Q-matrix for Test 2**

\[
Q = \begin{bmatrix}
-0.12 & 0.03 & 0.03 & 0.03 & 0.03 \\
0.3 & -1.2 & 0.3 & 0.3 & 0.3 \\
0.6 & 0.6 & -2.3 & 0.5 & 0.6 \\
0.9 & 0.8 & 1 & -3.7 & 1 \\
1.1 & 1 & 0.9 & 0.8 & -3.8
\end{bmatrix}
\]
Running Mean of Price vs. Iteration Number (Test 1)

(Red) Plain MC;  (Blue) MC+Variance Reduction

Diagram Width = 3 empirical standard deviations
Running Mean of Price vs. Iteration Number (Test 2)

(Red) Plain MC;  (Blue) MC+Variance Reduction

Diagram Width = 3 empirical standard deviations
(Left) Empirical Distribution of Jump Counts for Test 1 samples
(Right) Empirical Distribution of Jump Counts for Test 2 samples
Price vs. Jump Count:
Test 1 samples (Left); Test 2 samples (Right)
red - sample price; green - theoretical price
Weighted Price vs. Jump Count:
Test 1 samples (Left); Test 2 samples (Right)

red - sample price; green - theoretical price
We have considered a specific market model where the underlying evolves as a continuous time finite state Markov chain (CTMC).

For those cases where an explicit analytic pricing formula is not available (i.e. most of the cases) we have presented a hybrid MC simulation method which, with respect to a plain MC allows to:

i) reduce the variance

ii) obtain more precise results

We have presented numerical results and comparisons for the case of lookback call options.
Thank you for your attention