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Whittle's index in a multi-class queue with abandonments

M. Larrañaga[1,2,3], U. Ayesta[2], I.M. Verloop[1,3]

[1]IRIT, CNRS, 31071 Toulouse, France.
[2]LAAS, CNRS, F-31400 Toulouse, France.
[3]Univ. de Toulouse, INP, LAAS, F-314000 Toulouse, France.

1. Introduction

Abandonment or reneging takes place when customers, unsatisfied of their long waiting time, decide to voluntarily leave the system. It has a huge impact in various real life applications such as the Internet or call centers, where customers may abandon while waiting in the queue. In this paper we study the phenomena of abandonments in a single server queue with multiple classes of users with the objective of finding resource allocation strategies. In the presence of abandonments and/or convex holding cost, a characterization of the optimal control is out of reach. We therefore develop approximations to tackle the problem, in particular, we study Whittle's index.

Index Rules have enjoyed a great popularity, since a complex control problem whose solution might, a priori, depend on the entire state space turns out to have a strikingly simple structure. In a seminal work, Whittle introduced the so-called Restless Multi Armed Bandit Problems (RMABP), see [2]. In a RMABP all bandits (in our study a bandit is a class of customers) in the system incur a cost, the scheduler selects one bandit to be made active, but all bandits might evolve over time according to a stochastic kernel that depends on whether the bandit was active or *frozen*. The objective is to determine the control policy that, based on the entire state-space description, selects the bandit with the objective of optimizing the average performance criterion. Whittle introduced an approximate control policy of index-type, which is nowadays referred as Whittle's index.

The multi-class single server queue with abandonments can be modeled as a RMABP problem and Whittle's index can be derived.

The reminder of the paper is structured as follows : in Section 2 the model under study is described, and the objective of the study is presented, in Section 3 the Whittle index is derived and in Section 4 the performance of this index policy is measured. For additional results and proofs we refer to [1].

2. Model Description

We consider a multi-class single-server queue with K classes of customers. Class-k customers arrive according to a Poisson process with rate λ_k and have an exponentially distributed service requirement with mean $1/\mu_k$, $k = 1, \ldots, K$. We denote by $\rho_k := \lambda_k/\mu_k$ the traffic load of class k, and by $\rho := \sum_{k=1}^K \rho_k$ the total load to the system. We model abandonments of customers in the following way : any class-k customer in the system that has not completed service, abandons after an exponentially distributed amount of time with mean $1/\theta_k$, $k = 1, \ldots, K$, with $\theta_k > 0$.

The server has capacity 1 and can serve at most one customer at a time, where the service can be preemptive. At each moment in time, a policy φ decides which class is served. Because of the Markov property, we can focus on policies that only base their decisions on the current number of customers present in the various classes. For a given policy φ, $N_k^\varphi(t)$ denotes the number of class-k customers in the system at time t, (hence, including the one in service), and $\vec{N}^\varphi(t) = (N_1^\varphi(t), \ldots, N_K^\varphi(t))$. Let $S_k^\varphi(\vec{N}^\varphi(t)) \in \{0, 1\}$ represent the service capacity devoted to class-k customers at time t under policy φ. The constraint on the service amount devoted to each class is $S_k^\varphi(\vec{n}) = 0$ if $n_k = 0$ and $\sum_{k=1}^K S_k^\varphi(\vec{n}) \leq 1$. The above describes a birth and death process.

Let $C_k(n_k, a)$ denote the cost per unit of time when there are n_k class-k customers in the system when class k is not served (if $a = 0$), or when class k is served (if $a = 1$).

We further introduce a cost d_k for every class-k customer that abandoned the system when not being served. We then want to find the optimal scheduling policy φ under the average-cost criteria, that is,

$$\limsup_{T\to\infty} \sum_{k=1}^{K} \frac{1}{T}\mathbb{E}\left(\int_0^T \tilde{C}_k(\vec{N}^\varphi(t), S_k^\varphi(\vec{N}^\varphi(t)))\, dt\right), \tag{1}$$

where $\tilde{C}_k(n_k, a) := C_k(n_k, a) + d_k\theta_k n_k$ in state n_k and under action a.

3. Whittle's index

The approach by Whittle is based on relaxing the original problem, allowing one bandit being made active on average, that is,

$$\limsup_{T\to\infty} \frac{1}{T}\int_0^T \sum_{k=1}^{K} S_k^\varphi(\vec{N}^\varphi(t))dt \leq 1. \tag{2}$$

This allows to decompose the original control problem into individual problems for each bandit. Whittle's index can then be interpreted as the Lagrange

multiplier of the constraint such that a given state joins the passive set.
The objective is now to determine the policy that solves (1) under Constraint (2). This can be solved by considering the unconstrained control problem

$$\limsup_{T\to\infty} \sum_{k=1}^{K} \frac{1}{T}\mathbb{E}\left(\int_0^T \tilde{C}_k(\vec{N}^\varphi(t), S_k^\varphi(\vec{N}^\varphi(t))) - W(1 - S_k^\varphi(\vec{N}^\varphi(t)))\, dt\right), \quad (3)$$

where W is the Lagrange multiplier and $\tilde{C}(\cdot,\cdot)$ is assumed to be convex and increasing. We observe that the multiplier W can be interpreted as a subsidy for passivity.
In summary, the relaxed optimization problem can be written as K independent one-dimensional Markov decision problems, namely :

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}\left(\int_0^T \tilde{C}(N^\varphi(t), S^\varphi(N^\varphi(t))) - W(1 - S^\varphi(N^\varphi(t)))\, dt\right). \quad (4)$$

Under an indexability property (established in [1] for the present model), one can define class-based indices (the so-called Whittle's index) such that the solution to (1) is obtained by activating in every moment those classes whose current index is larger than the Lagrange multiplier.
We further prove that the optimal policy is monotone, it is fully characterized by a threshold n such that the passive action is prescribed for states $m \leq n$, and the active for states $m > n$. We will then denote this policy as $\varphi = n$. We can now state the main result.

Proposition 1 *The Whittle index for problem (4) is*

$$W(n) = \frac{\mathbb{E}(\tilde{C}(N^n, S^n(N^n))) - \mathbb{E}(\tilde{C}(N^{n-1}, S^{n-1}(N^{n-1})))}{\sum_{m=0}^{n} \pi^n(m) - \sum_{m=0}^{n-1} \pi^{n-1}(m)} \quad (5)$$

with $\pi^n(m)$ the steady-state probability under threshold policy n.

The heuristic for the original K-dimensional problem known as Whittle's index policy, prescribes to serve the class k having currently the highest non-negative Whittle's index $W_k(n_k)$.
We now present Whittle's index for linear cost.

Proposition 2 *Assume linear holding cost $C_k(n_k, a) = c_k n_k$. Then, Whittle's index for class k, with $\tilde{c}_k = c_k + d_k\theta_k$, is*

$$W_k(n_k) = \frac{\tilde{c}_k \mu_k}{\theta_k}, \textit{ for all } n_k. \quad (6)$$

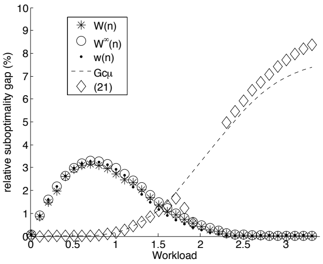

FIGURE 1 – Performance of Whittle's index policy.

Whittle's index for a more general setting (with convex holding cost) as well as limiting properties are analyzed in [1]. This study allows to recover indices that have been obtained in the literature. Observe that (6) is independent on the number of class-k customers, which is not the case for convex holding costs.

4. Numerical Results

In this section we numerically analyze the performance of Whittle's index policy. In Figure 1 we plot the relative sub optimality gap of the $W(n)$ index as well as other indices obtained in [1]. The optimal policy is computed using a Value Iteration algorithm. We carried out extensive simulations and were able to conclude that Whittle's Index policy is optimal as the workload increases, whereas indices that have been computed for multi-class systems without abandonments behave poorly.

References

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