Averaging on Dynamic Networks

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1. Introduction

In dynamic networks, the topology of the nodes in a network or links connecting these nodes changes with time. This can be due to mobility, link failure, or nodes failure. We are interested in this paper to study consensus on dynamic networks. Most of the work on consensus in dynamic network settings consider fixed number of nodes that are trying to reach agreement in the presence of either mobility or non-robust links (so only the links are dynamic) [1]. Average consensus on these networks is modeled following a random adjacency matrix $A^{n\times n}(k)$ where n is the number of nodes and is fixed while the elements of this matrix are random (being for example i.i.d. at every iteration k). The study of consensus on that model is reduced to studying the convergence of the backward product of random matrices. Some papers give sufficient conditions on the weight matrices at every time iteration that guarantee convergence, others use coefficient of ergodicity as a tool to show the convergence of their system to consensus [2]. However, little study has been made on networks with dynamic number of nodes. In the latter case, the dimensions of the adjacency (and weight) matrices can be unbounded, and thus the traditional tools for studying the consensus are not applicable. We refer in this report to this type of networks : nodes arrive and leave as in a queuing system. We would like to study the effect of averaging in these networks, and to see if the nodes could actually reach consensus.

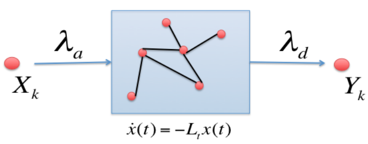

FIGURE 1 – The network model.

2. Model

Networks with dynamic nodes are characterized by nodes' arrival and nodes' departure (see Fig. 1). Each node arrives with a random value X_k (where k is the number of arrival and each node is labeled by its arrival number). We suppose that there is initially a node having a label 0 in the system that does not leave (it arrives at time $t = 0$ and does not depart). Let $X_0, X_1, X_2, \ldots$ be pairwise independent random variables following some distribution. Let A_k be the arrival time of node k and D_k its departure time. At any instance t, nodes within the system are connected to each other by some network topology. Let $V(t)$ be the set of all nodes in the system at time instant t, i.e., $V(t) = \{k \mid A_k \leq t < D_k\}$. Let $Y_k(t)$ be the value of node k that changes with time depending on its history of connection with other nodes, for example it should be clear that $Y_k(t)$ has a constant value before arrival ($Y_k(t) = X_k$ for $t < A_k$) and after departure ($Y_k(t) = Y_k(D_k)$ for $t \geq D_k$) but this value changes during the time it spends in the system because of interaction with other nodes. The nodes perform a continuous time averaging :

$$\dot{\mathbf{x}}(t) = -L_t \mathbf{x}(t), \tag{1}$$

where $\mathbf{x}(t)$ is the state vector of the nodes present in the system at time t, and $x_i(t) = Y_{\gamma(i)}(t)$ where $\gamma(i)$ is the arrival number of the i-th oldest node in the system (notice that $x_1(t) = Y_0(t)$ is always true for the node labeled 0 because it is the oldest node in system and does not depart). L_t is the Laplacian of the graph at time t, and $\dot{\mathbf{x}}(t) = \frac{\partial \mathbf{x}(t)}{\partial t}$. We call this model a *consensus queue* due to its similarity to queuing systems. If the inter-arrival and inter-leave times are exponentially distributed random variables with a FIFO discipline (the nodes first to come are the first to leave except for node 0), then the system is $M/M/1$ consensus queue with arrival rate λ_a and departure rate λ_d.

The Laplacian of the graph L_t is used to give a general model for different graph topologies. However, this work is just preliminary and we will give some simplifications in the next section : 1) we only consider two graph topologies, the complete graph and the tree, 2) the averaging is faster than the dynamics of the queue (i.e., equation (1) converges before the arrival or the departure of a new node).

The model described here is interesting because it can be applied to different and diverse applications. Queuing consensus can be for example a model for human interactions and their behavior. Consider a system where people arrive at an open market and products' prices are not fixed. Each customer has an initial estimation of the price of the product that varies depending on the interaction process with other customers in the system. You can also consider this model as a representative of a wireless sensor network where sensors monitor some environmental measurement (as temperature or pressure) where nodes can fail according to a Poisson process and new nodes are added to the network. We are interested then by the average in the system, mainly by :

$$Z(t) = \frac{1}{N(t)} \sum_{i=1}^{N(t)} x_i(t), \quad (2)$$

where $N(t) = |V(t)|$ is the number of nodes that are present in the system at time t and $x_i(t)$ are their estimates. We note that $Z(t)$ is not a continuous process in general because with every arrival (or departure) with an estimate X_k (or Y_k) different from $Z(t)$, the process jumps to a different value.

3. Simple Network Topologies

There are two sources of randomness in this model, the first one is the input estimate X that follows some distribution, and the other one is the queuing system with random arrivals and departures. In the following sections, we will characterize the average in the system and the output process by considering several simplifications :

1. Complete Graph : the network is a full mesh network (all nodes in the system are connected to each others) and once a node enters a system, all nodes will have the average of nodes presented (instantaneous averaging), i.e., $x_i(t) = Z(t)$ for all i in the system.
2. Directed Tree : nodes arriving can only connect to one node chosen uniformly at random in the system, and their estimate changes only once till they leave the network depending on the chosen node's estimate and it's distance from the root.

3.1. Complete Graph

Let Z_k be the value of $Z(t)$ just after the k-th arrival and before the $k+1$-th arrival. Then Z_k can be written as a weighted average of the nodes in the network, i.e., $Z_k = \sum_{i=0}^{k} w_i X_i$, where w_i is the weight given to the value of node i and it is a random variable depending on the stochastic arrivals and departures. It is important to study these weights to see how the system preserves the history of old values. To do this we take two extreme cases. The first case of no departures (if $\lambda_d = 0$), then we have

$$w_i = \frac{1}{k+1} \text{ for } i = 0, \ldots, k. \quad (3)$$

The second case of very fast departures ($\lambda_d >> \lambda_a$), then we suppose each node that arrives, averages with node 0 and then leaves the system, so

$$w_i = \begin{cases} (\frac{1}{2})^{k+1-i} & \text{for } i = 1, \ldots, k, \\ (\frac{1}{2})^{k} & \text{for } i = 0. \end{cases} \quad (4)$$

The results are interesting as they show that if the system's departure rate is fast, then the weights for old values decrease exponentially, but if the departure is slow then the weights for old values decrease linearly in k. For future work, we would like to characterize the decrease in the average weight of the history as function of the performance parameter of the queue (as function of $\rho = \lambda_a/\lambda_d$).

3.2. Directed Tree

For this topology, we assume that there are no departures from the system. Each node arrives, connects uniformly at random to one node (we call it its parent) in the network. Let L be the distance from a newly connected node to the root (node 0). We suppose that each node i arrives at level L, connects to its parent j, and then averages as follows :

$$Y_i(t) = \frac{1}{L} X_i + \frac{L-1}{L} Y_j(t) \text{ for } t > A_i.$$

Notice that $Y_i(t) = \frac{1}{L} \sum_{s \in P} X_s$ is a constant for $t > A_i$ where P is the set of nodes on the path from the root to i. Since L converges to $\log n$ in probability as $n \to \infty$ [3], we conclude that if $X_1, X_2, \ldots$ are i.i.d random variables of mean μ, then the value $Y_i(t)$ for $t > A_i$ converges in probability to μ as $i \to \infty$.

References

1. Olfati-Saber, R. ; Murray, R.M., "Consensus problems in networks of agents with switching topology and time-delays", *IEEE Trans. on Automatic Control*, vol. 49, no. 9, pp. 1520-1533, Sept. 2004.
2. Tahbaz-Salehi, A. ; Jadbabaie, A., "A Necessary and Sufficient Condition for Consensus Over Random Networks" , *IEEE Trans. on Automatic Control*, vol. 53, no. 3, pp. 791-795, April 2008.
3. L. Devroye, "Applications of the theory of records in the study of random trees", *Acta Informatica*, vol. 26, no. 1-2, pp. 123-130, 1988.