Dichromatic number of surfaces

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ANR DIGRAPh meeting, June 15–18, 2021
Surfaces

**Classification Theorem for Surfaces**: Every surface is homeomorphic to

- either the $k$-torus – a sphere with $k$-handles $\mathbb{S}_k$

or the $k$-cross surface – a sphere with $k$-cross-caps $\mathbb{N}_k$.

**Euler characteristic**: $c(\mathbb{S}_k) = 2 - 2k$ and $c(\mathbb{N}_k) = 2 - k$. 
Graphs on surfaces

**G embeddable** on $\Sigma$:drawable on $\Sigma$ without edge crossing.

**Euler’s Formula:**
$G$ connected embedded on $\Sigma$. Then

$$n(G) - m(G) + f(G) \geq c(\Sigma)$$

**Corollary:** $G$ embeddable $\Sigma$:

$$m(G) \leq 3n(G) - 3c(\Sigma) \quad \text{and} \quad \text{Ad}(G) \leq 6 - \frac{6c(\Sigma)}{n(G)}.$$

with equality if $G$ is a triangulation.
Chromatic number of surfaces

\textbf{\textit{k}-colouring} of \( G \) : partition of \( V(G) \) into \( k \) \textbf{stable sets}.

\( G \) \textbf{\textit{k}-colourable} if it has a \( k \)-colouring.

\textbf{chromatic number} \( \chi(G) \) :
\( \chi(G) = \min\{k \mid G \text{ is } k\text{-colourable}\} \)

\textbf{chromatic number} \( \chi(\Sigma) = \max\{\chi(G) \mid G \text{ embeddable on } \Sigma\} \)

\( \text{Ad}(G) \leq 6 - \frac{6c(\Sigma)}{n(G)} < \max\{6; 6 - c(\Sigma)\} \).

So \( \chi(G) \leq \max\{6; 6 - c(\Sigma)\} \).

\( \chi(\Sigma) \leq \max\{6; 6 - c(\Sigma)\} \).
Chromatic number of surfaces

**Heawood 1890**: If $c(\Sigma) \leq 0$, then $\chi(\Sigma) \leq H(c) = \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$.

**Ringel and Youngs 1968**: If $\Sigma \neq \mathbb{N}_2$, then the complete graph of order $H(c(\Sigma))$ is embeddable on $\Sigma$.

**Four Colour Theorem 1977**: $\chi(S_0) = 4$. 
Colouring triangle-free graphs on surfaces

Grötzsch 1959: Every triangle-free planar graph is 3-colourable.

Kronk and White 1972: Every triangle-free graph embeddable on the torus is 4-colourable.

Gimbel and Thomassen 1997: \(\exists c_1\) and \(c_2\) such that:

(i) Every triangle-free graph embeddable on \(S_k\) has chromatic number at most \(c_1 \frac{3\sqrt{k}}{\log k}\).

(ii) for each \(k\), there exists a triangle-free graph which is embeddable on \(S_k\) and with chromatic number at least \(c_2 \frac{3\sqrt{k}}{\log k}\).
**Dichromatic number**

**k-dicollouring** of $D = \text{partition of } V(D) \text{ into } k$

**subsets inducing acyclic subdigraphs.**

$D$ is **k-dicolourable** if it has a k-dicolouring.

**dichromatic number** $\vec{\chi}(D)$ : least $k$ such that $D$ is k-dicolourable.

$$\vec{\chi}(D) \leq \chi(D)$$

**bidirected graph** $\leftrightarrow G$: digraph obtained from $G$ by replacing each edge by a **digon**.

$$\chi(G) = \vec{\chi}(\leftrightarrow G)$$
Dichromatic number of surfaces

**oriented graph**: graphs with no digon.

**dichromatic number** $\vec{\chi}(\Sigma)$

$$\vec{\chi}(\Sigma) = \max\{\vec{\chi}(\vec{G}) \mid \vec{G} \text{ oriented graph embeddable on } \Sigma\}$$

$$\vec{\chi}(\Sigma) \leq \chi(\Sigma)$$

**Neumann Lara 1982**: $\vec{\chi}(\mathbb{S}_0) \leq 3$.

**Conjecture** (Neumann Lara 1982): $\vec{\chi}(\mathbb{S}_0) = 2$.

**Problem**: Determine $\vec{\chi}(\Sigma)$ for every surface $\Sigma$. 
Dichromatic number and arboricity

**arboricity** of $G$ = partition of $V(G)$ into $k$

**subsets inducing forests** (acyclic subdigraphs).

$$\bar{\chi}(\vec{G}) \leq a(G) \leq \chi(G)$$

**Kronk 1969**: $\bar{\chi}(\Sigma) \leq a(\Sigma) \leq \left\lfloor \frac{9 + \sqrt{49 - 24c(\Sigma)}}{4} \right\rfloor$
Dichromatic number and cochromatic number

\textbf{\textit{k-cocolouring}} of \( G = \) partition of \( V(G) \) into \( k \) \textit{stable sets} or \textit{cliques}.

\( D \) is \textit{k-cocolorable} if it has a \( k \)-cocolouring.

\textbf{cochromatic number} \( \text{co}\chi(D) \) : least \( k \) s. t \( D \) is \( k \)-cocolorable.

\[
\bar{\chi}(D) \leq \text{co}\chi(F(D))
\]

\textbf{Gimbel and Thomassen 1997:}

\[
a_1 \frac{\sqrt{-c}}{\log(-c)} \leq \bar{\chi}(\Sigma) \leq \text{co}\chi(\Sigma) \leq a_2 \frac{\sqrt{-c}}{\log(-c)}
\]
Our results

<table>
<thead>
<tr>
<th>Σ</th>
<th>$c(Σ)$</th>
<th>Bounds for $\overrightarrow{χ}(Σ)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sphere $N_0 = S_0$</td>
<td>2</td>
<td>$2 \leq \overrightarrow{χ} \leq 3$</td>
</tr>
<tr>
<td>Projective plane $N_1$</td>
<td>1</td>
<td>$\overrightarrow{χ} = 3$</td>
</tr>
<tr>
<td>Klein bottle $N_2$</td>
<td>0</td>
<td>$\overrightarrow{χ} = 3$</td>
</tr>
<tr>
<td>Torus $S_1$</td>
<td>0</td>
<td>$\overrightarrow{χ} = 3$</td>
</tr>
<tr>
<td>Dyck’s surface $N_3$</td>
<td>−1</td>
<td>$\overrightarrow{χ} = 3$</td>
</tr>
<tr>
<td>$S_2, N_4$</td>
<td>−2</td>
<td>$3 \leq \overrightarrow{χ} \leq 4$</td>
</tr>
<tr>
<td>$N_5$</td>
<td>−3</td>
<td>$3 \leq \overrightarrow{χ} \leq 4$</td>
</tr>
<tr>
<td>$S_3, N_6$</td>
<td>−4</td>
<td>$3 \leq \overrightarrow{χ} \leq 4$</td>
</tr>
<tr>
<td>$N_7$</td>
<td>−5</td>
<td>$3 \leq \overrightarrow{χ} \leq 4$</td>
</tr>
<tr>
<td>$S_4, N_8$</td>
<td>−6</td>
<td>$3 \leq \overrightarrow{χ} \leq 4$</td>
</tr>
<tr>
<td>$N_9$</td>
<td>−7</td>
<td>$3 \leq \overrightarrow{χ} \leq 4$</td>
</tr>
<tr>
<td>$S_5, N_{10}$</td>
<td>−8</td>
<td>$\overrightarrow{χ} = 4$</td>
</tr>
</tbody>
</table>
Dichromatic number of tournaments

tournament: orientation of a complete graph.

\[
\max \vec{\chi}(n) = \max\{\vec{\chi}(\vec{G}) \mid \vec{G} \text{ oriented graph of order } n\} \\
= \max\{\vec{\chi}(T) \mid T \text{ tournament of order } n\}
\]

Erdős and Moser; Harutyunyan:

\[
\frac{n}{2 \log(n) + 1} \leq \max \vec{\chi}(n) \leq \frac{3n}{\log n}
\]

Conjecture: \( \vec{\chi}(\Sigma) = \max \vec{\chi}(H(c(\Sigma))) \) when \(-c(\Sigma)\) is large enough.
Dichromatic number of tournaments

- $\max \vec{\chi}(n) = 1$ for $n \leq 2$.
- $\max \vec{\chi}(n) = 2$ for $3 \leq n \leq 6$.
- $\max \vec{\chi}(n) = 3$ for $7 \leq n \leq 10$.
- $\max \vec{\chi}(n) = 4$ for $11 \leq n \leq 15$. 

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Our results

- If $\Sigma \neq S_0$, then $\vec{\chi}(\Sigma) \geq 3$.
- $\vec{\chi}(N_3) \leq 3$.
- $\vec{\chi}(S_5) = \vec{\chi}(N_{10}) = 4$. 
Dichromatic number of the projective plane

\[ \chi(\mathbb{N}_1) \geq 3 \]
Dichromatic number of the projective plane

$$\bar{\chi}(\mathbb{N}_1) \geq 3$$
Dicritical digraphs

\( D \ k\text{-dicritical} : \begin{cases} 
\vec{\chi}(D) = k \text{ and} \\
\vec{\chi}(H) < k \text{ for every proper subdigraph } H \text{ of } D.
\end{cases} \)

**Property:** \( \vec{G} \ k\text{-dicritical oriented graph embeddable in } \Sigma. \)

- \( d^+(v), d^-(v) \geq k - 1 \text{ for all } v \in V(\vec{G}). \)

- If \( k \geq 5 \), then \( n(\vec{G}) \leq \frac{-3c}{k - 4} \).

- If \( k = 4 \), then
  \[
  3m(\vec{G}) \geq 10n(\vec{G}) - 4. \quad \text{(Kostochka and Stiebitz, 2020)}
  \]
  So \( n(\vec{G}) \leq 4 - 9c \) if \( k = 4 \).
Dichromatic number of Dyck’s surface

\[ \chi'(\mathbb{N}_3) \leq 3 \]

Assume \( \vec{G} \) 4-dicritical oriented graph embeddable on \( \mathbb{N}_3 \).

▷ \( G \) is not a triangulation of \( \mathbb{N}_3 \).

\[ 10n(\vec{G}) - 4 \leq 3m(\vec{G}) \leq 9n(\vec{G}) + 6. \] (Kostochka-Stiebitz + Euler’s Formula)
So \( n(\vec{G}) \leq 10 \), a contradiction.

▷ \( G \) is a triangulation of \( \mathbb{N}_3 \). Then \( n(\vec{G}) \leq 13 \).

**Use of computer**: Generation of all triangulations of \( \mathbb{N}_3 \) of order 11, 12 and 13. They all have arboricity 3, so their orientations are 3-dicolourable, contradiction.
\( \bar{\chi}(S_5) = \bar{\chi}(N_{10}) = 4 \)

\( \vec{G} \) 5-dicritical oriented graph in \( S_5 \) or \( N_{10} \).

\[ T : \quad d^+ = d^- = 4 \]

\[ H = \vec{G} - T \]
\( \chi(S_5) = \chi(N_{10}) = 4 \)

\( \vec{G} \) 5-dicritical oriented graph in \( S_5 \) or \( N_{10} \).

Harutyunyan and Mohar:
\( T \) directed cactus
\[ \chi(T) \leq 2 \; ; \; m(T) \leq \frac{3}{2}(n(T) - 1) \]

\( T : d^+ = d^- = 4 \)

\( H = \vec{G} - T \)
\[ \chi(\mathbb{S}_5) = \chi(\mathbb{N}_{10}) = 4 \]

\[ \vec{G} \text{ 5-dicritical oriented graph in } \mathbb{S}_5 \text{ or } \mathbb{N}_{10}. \]

Harutyunyan and Mohar:

\[ T \text{ directed cactus } \]

\[ \chi(T) \leq 2 ; \ m(T) \leq \frac{3}{2}(n(T) - 1) \]

\[ \chi(H) \geq 3 ; \ m(H) \geq 20 \]

\[ T : d^+ = d^- = 4 \]

\[ H = \vec{G} - T \]
\[ \overrightarrow{\chi}(S_5) = \overrightarrow{\chi}(N_{10}) = 4 \]

\[ \overrightarrow{G} \text{ 5-dicritical oriented graph in } S_5 \text{ or } N_{10}. \]

Harutyunyan and Mohar:

- \( T \) directed cactus
  \[ \overrightarrow{\chi}(T) \leq 2 \; ; \; m(T) \leq \frac{3}{2}(n(T) - 1) \]
  \[ \overrightarrow{\chi}(H) \geq 3 \; ; \; m(H) \geq 20 \]
- \( m(\overrightarrow{G}) = m(H) + 8n(T) - m(T) \)
- \( 13n(T) \leq 2m(\overrightarrow{G}) - 43 \)
\[ \chi(S_5) = \chi(N_{10}) = 4 \]

\( \vec{G} \) 5-dicritical oriented graph in \( S_5 \) or \( N_{10} \).

Harutyunyan and Mohar:

\( T \) directed cactus

\[ \chi'(T) \leq 2 ; \ m(T) \leq \frac{3}{2} (n(T) - 1) \]

\[ \chi'(H) \geq 3 ; \ m(H) \geq 20 \]

\[ m(\vec{G}) = m(H) + 8n(T) - m(T) \]

\[ 13n(T) \leq 2m(\vec{G}) - 43 \]

By Euler’s formula:

\[ 3(n(\vec{G}) - 16) \leq n(T) \leq \frac{6n(\vec{G}) + 5}{13} \]

\[ n(\vec{G}) \leq 19 \]
\( \chi(S_5) = \chi(N_{10}) = 4 \)

\( \tilde{G} \) 5-dicritical oriented graph in \( S_5 \) or \( N_{10} \). Assume \( n(\tilde{G}) = 19 \).

\[ T : d^+ = d^- = 4 \]

\[ H = \tilde{G} - T \]
\( \chi(S_5) = \chi(N_{10}) = 4 \)

\( \tilde{G} \) 5-dicritical oriented graph in \( S_5 \) or \( N_{10} \). Assume \( n(\tilde{G}) = 19 \).

\[ T : d^+ = d^- = 4 \]

\[ H = \tilde{G} - T \]

\[ n(T) = 9, \text{ so } n(H) = 10 \]
$\bar{\chi}(S_5) = \bar{\chi}(N_{10}) = 4$

$\vec{G}$ 5-dicritical oriented graph in $S_5$ or $N_{10}$. **Assume** $n(\vec{G}) = 19$.

$T : d^+ = d^- = 4$  

$H = \vec{G} - T$

$n(T) = 9$, so $n(H) = 10$  

$\bar{\chi}(H) \leq 3$
$\chi(S_5) = \chi(N_{10}) = 4$

$\vec{G}$ 5-dicritical oriented graph in $S_5$ or $N_{10}$. Assume $n(\vec{G}) = 19$.

$T : d^+ = d^- = 4$

$n(T) = 9$, so $n(H) = 10$

$\chi(H) \leq 3$

Pick $x$ s.t. $d_T^+(x) = d_T^-(x) = 1$.

$H = \vec{G} - T$
\( \chi(\mathbb{S}_5) = \chi(\mathbb{N}_{10}) = 4 \)

\( \vec{G} \) 5-dicritical oriented graph in \( \mathbb{S}_5 \) or \( \mathbb{N}_{10} \). Assume \( n(\vec{G}) = 19 \).

\[ T : d^+ = d^- = 4 \]

\[ H = \vec{G} - T \]

\( n(T) = 9 \), so \( n(H) = 10 \)

\( \chi(H) \leq 3 \)

Pick \( x \) s.t. \( d^+_T(x) = d^-_T(x) = 1 \).

Recolour two of its outneighbours in red.
\( \chi(S_5) = \chi(N_{10}) = 4 \)

\( \vec{G} \) 5-dicritical oriented graph in \( S_5 \) or \( N_{10} \). Assume \( n(\vec{G}) = 19 \).

\( n(T) = 9 \), so \( n(H) = 10 \)

\( \chi(H) \leq 3 \)

Pick \( x \) s.t. \( d_T^+(x) = d_T^-(x) = 1 \).
Recolour two of its outneighbours in red.

Extend the colouring to \( T \) s. t. a vertex of \( T \) has a colour distinct from its outneighbours.
Complexity of colouring graphs on surfaces

**Dirac ’57, Thomassen ’97:** For \( k \geq 5 \), there are only finitely many \((k + 1)\)-critical graphs embeddable on \( \Sigma \).

**Corollary:** For any \( k \geq 5 \), \( k \)-**COLOURABILITY** is **polynomial** for graphs embeddable on any fixed surface \( \Sigma \).

**Fisk ’78:** For \( k \in \{2, 3, 4\} \), there are infinitely many \((k + 1)\)-critical graphs embeddable on \( \Sigma \neq S_0 \).

**Theorem:** \( 2 \)-**COLOURABILITY** is **polynomial**.

**Stockmeyer ’73:** **Planar 3-COLOURABILITY** is **NP-complete**.

**THE QUESTION**
Complexity of \( 4 \)-**COLOURABILITY** for graphs on \( \Sigma \neq S_0 \) ?
Complexity of dicolouring digraphs on surfaces

**Theorem:** For $k \geq 6$, there are only finitely many $(k + 1)$-dicritical digraphs embeddable on $\Sigma$.

**Corollary:** For any $k \geq 6$, $k$-Dicolourability is polynomial for graphs on any fixed surface $\Sigma$.

**Theorem:** Planar 2-Dicolourability is NP-complete.

**Theorem:** Planar 3-Dicolourability is NP-complete.

**Problem:** Complexity of $k$-Dicolourability for digraphs on $\Sigma$ for $k \in \{4, 5\}$?
Complexity of dicolouring oriented graphs on surfaces

**Theorem:** For $k \geq 3$, there are only finitely many $(k + 1)$-dicritical oriented graphs embeddable on $\Sigma$.

**Corollary:** For any $k \geq 3$, $k$-Dicolourability is polynomial for oriented graphs on $\Sigma$.

**Problem:** Complexity of 2-Dicolourability for oriented graphs on $\Sigma$?

Neuman Lara Conjecture implies that it is trivially polynomial for $\Sigma = S_0$. But if this conjecture fails, then it is NP-complete for $\Sigma = S_0$. 