Inversion number of an oriented graph and related parameters

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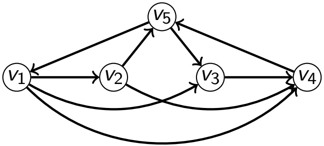
Inversion

inversion of X in D = reversing the direction of all arcs of $D\langle X\rangle$.



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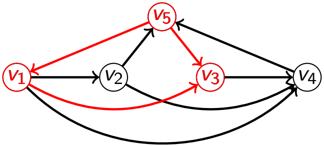

$X_1 = \{v_1, v_3, v_5\}$

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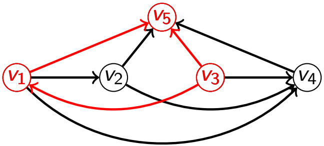

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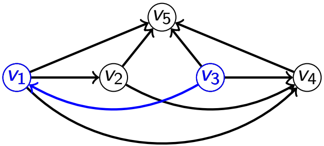

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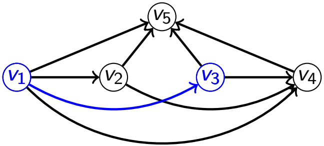

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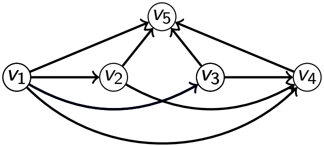

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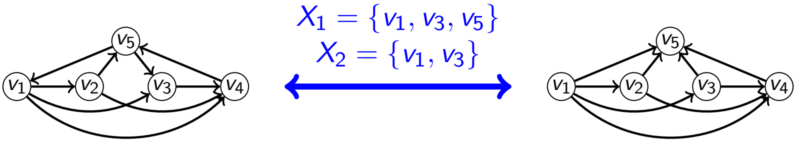
inversion number inv(D) : minimum number of inversions to make D acyclic.

Inversion of extensions

extension of D: tournament T such that $V(D) = V(T)$ and $A(D) \subseteq A(T)$.

Lemma: There is an extension T of D such that $\mathrm{inv}(T) = \mathrm{inv}(D)$.



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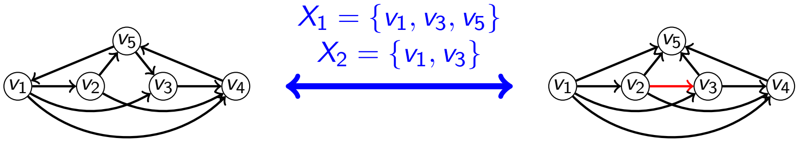

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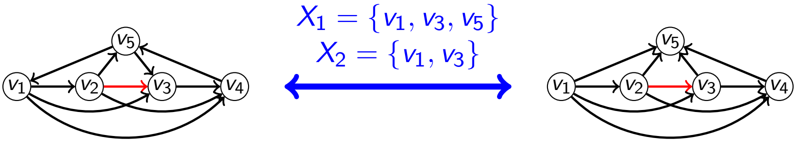


Inversion and cycle arc-transversal

cycle arc-transversal (feedback arc set) : set of arcs whose reversal results in an acyclic digraph.
cycle arc-transversal number $\tau'(D)$: minimum size of a cycle arc-transversal of D.

$$\text{inv}(D) \leq \tau'(D)$$

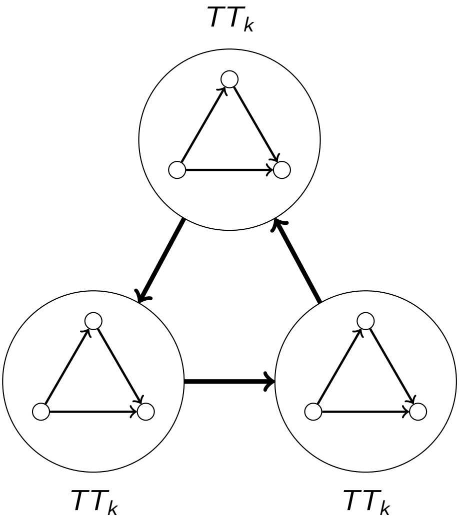
Inversion and cycle transversal

cycle transversal (feedback vertex set) : set of vertices whose deletion results in an acyclic digraph.
cycle transversal number $\tau(D)$: minimum size of a cycle transversal of D.

$$\text{inv}(D) \leq 2\tau(D)$$

Proposition: $\text{inv}(D) \leq \text{inv}(D-x)+2$.
[[*Proof:* Inverting $N^+(x)$ and $N^+[x]$ makes x a sink a leaves $D-x$ unchanged.]]

Small inv and large τ.



$\tau = k$

inv $= 1$

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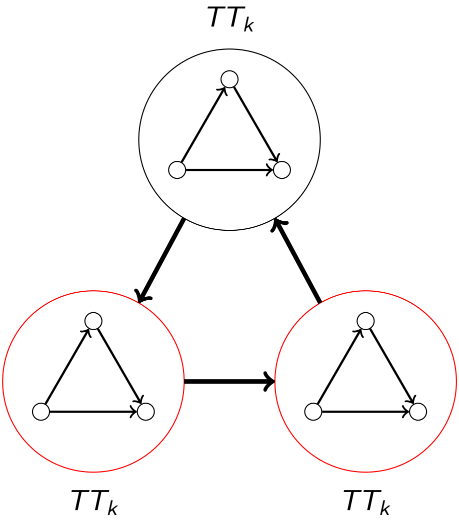

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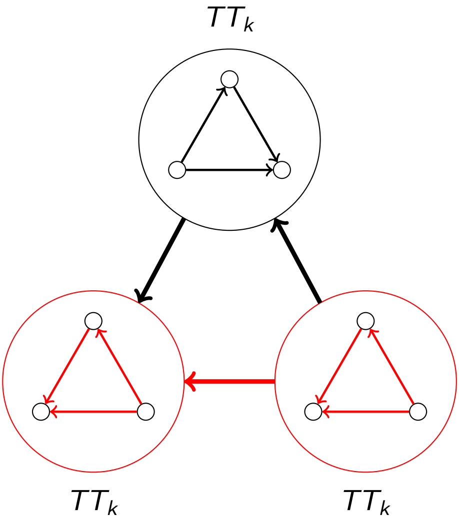
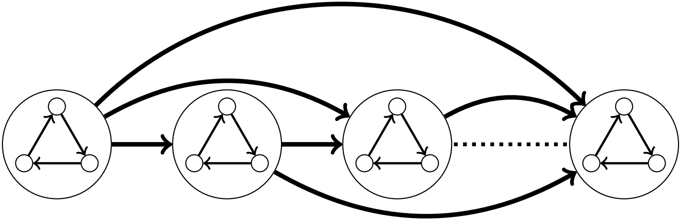


$\tau = k$

inv $= 1$

Tightness of inv $\leq \tau'$.

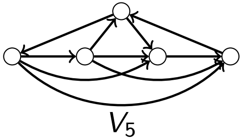
Pouzet et al. 2010:[1] $\text{inv}(TT_n[\vec{C_3}]) = n$.



$\tau'(TT_n[\vec{C_3}]) = n$.

[1] H. Belkhechine, M. Bouaziz, I. Boudabbous, and M. Pouzet, *Comptes Rendus Mathematique*, 348(13):703 – 707, 2010.

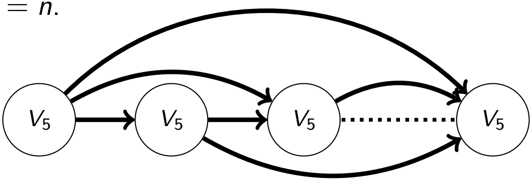
Tightness of $\text{inv} \leq 2\tau$.



$\text{inv}(V_5) = 2$ and $\tau(V_5) = 1$.

Conjecture: $\text{inv}(TT_n[\vec{V_5}]) = 2n$.

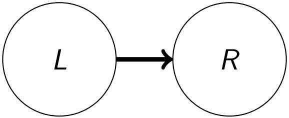
$\tau(TT_n[\vec{V_5}]) = n$.



The conjecture holds for $n = 2$.

Inversion number and dijoin

dijoin from L to R: $L \to R$ obtained from the disjoint union of L and R by adding all arcs from L to R.



Proposition: $\text{inv}(L \to R) \leq \text{inv}(L) + \text{inv}(R)$.

Dijoin Conjecture: $\text{inv}(L \to R) = \text{inv}(L) + \text{inv}(R)$.

This conjecture is equivalent to its restriction to tournaments.

It holds when $\text{inv}(L) = 1$ and $\text{inv}(R) \leq 2$
and when $\text{inv}(L) = \text{inv}(R) = 2$ and L and R are strongly connected.

Inversion and cycle packing

cycle packing : set of vertex disjoint cycles.
cycle packing number, $\nu(D)$: max. size of a cycle packing in D.

$$\nu(D) \leq \tau(D)$$

Reed et al.: There is a function f such that $\tau(D) \leq f(\nu(D))$.

Corollary: There is a function g such that $\text{inv}(D) \leq g(\nu(D))$, and $g \leq 2f$.

McCuaig: If $\nu(D) = 1$, then $\tau(D) \leq 3$. $f(1) = 3$.

Theorem: If $\nu(D) = 1$, then $\text{inv}(D) \leq 4$. $g(1) \leq 4$.

Conjecture: $g(k) = O(k \log k)$.
There is C such that $\text{inv}(D) \leq C \cdot \nu(D) \log(\nu(D))$ for all D.

Complexity

k-INVERSION.
Input: An oriented graph D.
Question: $\text{inv}(D) \leq k$?

k-TOURNAMENT-INVERSION.
Input: A tournament.
Question: $\text{inv}(T) \leq k$?

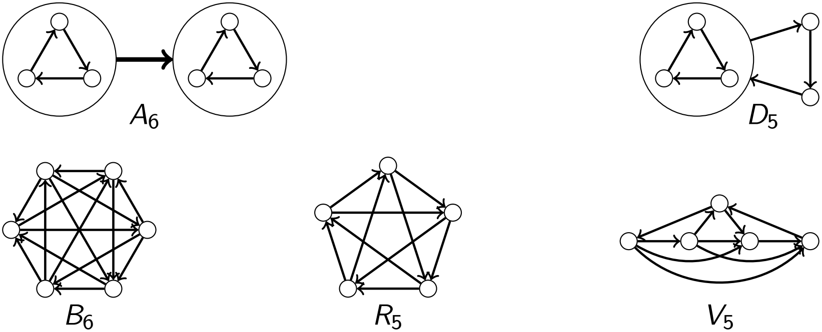
Complexity for tournaments

T is k-**inversion-critical** if $\text{inv}(T) = k$ and $\text{inv}(T - x) = k - 1$ for all $x \in V(T)$.

Pouzet et al. 2010:
For all k, there is a finite number of k-inversion-critical tournaments.

Corollary: k-Tournament-Inversion is polynomial time solvable for tournaments.

Complexity for tournaments

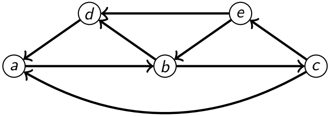


1-TOURNAMENT-INVERSION can be solved in $O(n^3)$-time.
2-TOURNAMENT-INVERSION can be solved in $O(n^6)$-time.

Complexity for oriented graphs

Theorem: 1-Inversion *is NP-complete.*

Gadget J



Only the inversion of $\{a, b, e\}$ or $\{b, c, d\}$. can make J acyclic

Conjecture: k-Inversion is NP-complete for all $k \geq 1$.

This conjecture is implied by the Dijoin Conjecture.

Dijoin Conjecture when $\text{inv}(L) = \text{inv}(R) = 1$, we get that 2-Inversion is NP-complete.