Extending the Gyárfás-Sumner conjecture to digraphs

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Non-oriented world

• $\chi(G)$: chromatic number

• Let $\mathcal{F}$ be a set of graphs. $G \in Forb(\mathcal{F})$ if G does not contains any member of $\mathcal{F}$ as an induced subgraph.

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Gyárfás-Sumner conjecture (1987)
For every integer k and every forest F, $Forb(K_k, F)$ has bounded chromatic number.

χ-boundedness

A hereditary class of graphs is χ-**bounded** if $\chi(G) \leq f(\omega(G))$ for every G in the class.

Gyárfás-Sumner conjecture (1987)
$Forb(F)$ is χ-bounded if and only if F is a forest.

It is enough to prove it for trees:

Directed world, dichromatic number

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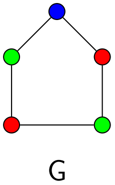
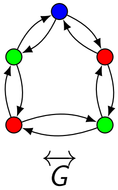
- Coloring a digraph D: no monochromatic cycle.
- $\vec{\chi}(D)$: the *dichromatic number* of D.

In other words: **partition** D **in acyclic induced subdigraphs** instead of stable sets.

Dichromatic number generalises chromatic number

Property: For every graph G, $\chi(G) = \vec{\chi}(\overleftrightarrow{G})$.





Heroic sets

Let $\mathcal{F}$ be a set of oriented graphs.

Forb $(\mathcal{F})$ is the class of oriented graphs containing no member of $\mathcal{F}$ as an induced subdigraph.

Problem: What are the finite sets $\mathcal{F}$ for which *Forb* $(\mathcal{F})$ has bounded dichromatic number?

Such sets are heroic.

Oriented graphs that must be contained in all heroic sets

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Harutyunyan and Mohar (2012): there is oriented graph with large dichromatic number and such that its underlying graph has large girth.

Tournaments and Heroes

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Theorem: [Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour and Thomassé, 2015]

- A strong tournament is a hero if and only if it is equal to $\Delta(H, TT_k, TT_1)$ or $\Delta(H, TT_1, TT_k)$, where H is a hero.
- A tournament H is a hero if and only if all its strong connected components are heroes.

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Conjecture [Aboulker, Charbit, Naserasr, 2020]: The set $Forb(H, F)$ has bounded dichromatic number if and only if:

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Conjecture [Aboulker, Charbit, Naserasr, 2020]: The set $Forb\,(H, F)$ has bounded dichromatic number if and only if:

- H is a hero and F is the disjoint union of stars or
- H is a transitive tournament and F is any oriented forest.

Some partial answers

Theorem [Chudnovsky, Scott, Seymour, 2019]
For every integer k and disjoint unions of stars F, $\mathit{Forb}\,(TT_k, F)$ has bounded chromatic number.

Theorem [Harutyunyan, Le, Newman, Thomassé, 2019]
For every integer t and every hero H, $\mathit{Forb}\,(H, \overline{K}_t)$ has bounded dichromatic number.

$\mathit{Forb}\,(\overline{K}_2)$ is the class of tournaments.

First part of the conjecture

Conjecture: for every hero H and every disjoint union of stars F, $Forb(H, F)$ has bounded dichromatic number.

What about forest on three vertices.

Complete multipartite oriented graphs

$Forb\,(\overrightarrow{K}_2 + K_1)$ is the class of complete multipartite oriented graphs.

Conjecture: for every hero H, H-free complete multipartite graph has bounded dichromatic number.

Theorem: $\vec{\chi}(Forb\,(\overrightarrow{C}_3, \overrightarrow{K}_2 + K_1)) = 2$ (Aboulker, Charbit, Naserasr, 2021)

Quasi-transitive graphs

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Corolary: for every hero H, H-free quasi-transitive graphs have bounded dichromatic number.

Local out-tournament

G is a **local out-tournament** if for every vertex x, $N^+(x)$ is a tournament.

It corresponds to $Forb(S_2^+)$.

Theorem: $\vec{\chi}(Forb(\overrightarrow{C}_3, S_2^+)) = 2$ [Steiner / Aboulker, Aubian, Charbit, 2021]

Conjecture: for every hero H, H-free local out-tournament have bounded dichromatic number.

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So we get a notion of $\vec{\chi}$-boundedness!

Conjecture: for every oriented tree T, $Forb(T)$ is $\vec{\chi}$-bounded

i.e. there is a function f such that for all $G \in Forb(T)$, $\vec{\chi}(G) \leq f(\omega(G))$.

Forbidding a path

Theorem [Gyárfás, 80's]: $Forb\,(P_k)$ is χ-bounded.

Proof that in a triangle-free (connected) graph with sufficiently large chromatic number, every vertex is the starting point of a long induced path.

Directed path

Conjecture: $\mathit{Forb}(\overrightarrow{P}_k)$ is $\vec{\chi}$-bounded.

► In a triangle-free (strongly connected) oriented graph with large $\vec{\chi}$, it is not true that every vertex is the starting point of a long induced path.

► Even if an oriented graph is strongly connected, there does need to be induced directed path between any pair of vertices.

Forbidding $\overrightarrow{P}_4$

First open case:

Conjecture: $Forb(\overrightarrow{P_4})$ is $\vec{\chi}$-bounded.

- $Forb(K_3, \overrightarrow{P_4})$ has dichromatic number at most 2.
- $Forb(K_4, \overrightarrow{P_4})$ has dichromatic number at most 414

The levelling technic

Let x be a vertex.

Let L_i the set of vertices at distance i from x.

If $\vec{\chi}(L_i) \leq k$ for every i, then $\vec{\chi}(G) \leq 2k$.

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Theorem: If $G \in Forb\,(K_3, \overrightarrow{P_4})$, then $\vec{\chi}(G) \leq 2$ because every L_i is a stable set.

Nice sets

Theorem: If $G \in Forb(K_4, \overrightarrow{P}_4)$, then $\vec{\chi}(G) \leq 414$.

Proof: G has a **nice set** with bounded dichromatic number.

Definition: A nonempty set of vertices S is nice if each vertex in S either has no out-neighbor in $V(D) \setminus S$ or has no in-neighbor in $V(D) \setminus S$.

Recap

Conjecture: For every hero H and every disjoint union of stars F, digraphs in $Forb(H, F)$.

Conjecture: or every integer k and every oriented tree T, $Forb(K_k, T)$ has bounded dichromatic number.

- $Forb(K_k, \overrightarrow{P_4})$ has bounded dichromatic number $(k \geq 5)$?
- $Forb(K_3, \overrightarrow{P_t})$ has bounded dichromatic number $(t \geq 5)$?

THANK YOU FOR YOUR ATTENTION