Twin-width for digraphs

Édouard Bonnet

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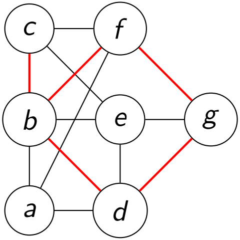
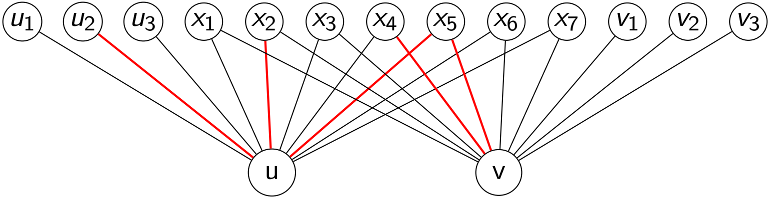
meeting ANR DIGRAPHS

Trigraphs



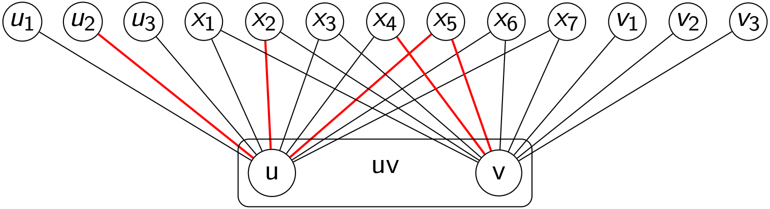
Three outcomes between a pair of vertices:
edge, or non-edge, or red edge (error edge)

Contractions in trigraphs



Identification of two non-necessarily adjacent vertices

Contractions in trigraphs



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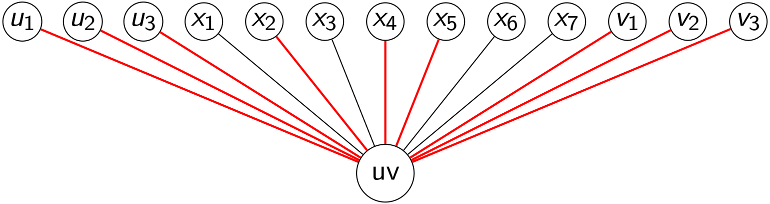
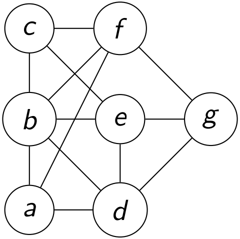


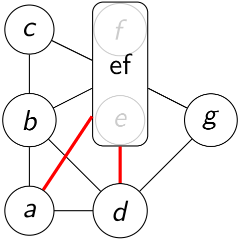
edges to $N(u) \triangle N(v)$ turn red, for $N(u) \cap N(v)$ red is absorbing

Contraction sequence



A contraction sequence of G:
Sequence of trigraphs $G = G_n, G_{n-1}, \ldots, G_2, G_1$ such that G_i is obtained by performing one contraction in G_{i+1}.

Contraction sequence



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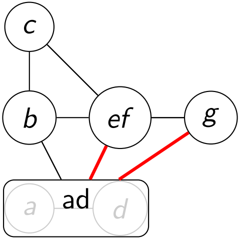

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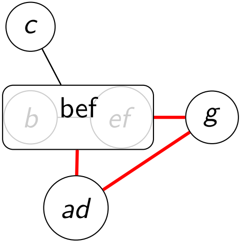

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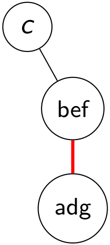

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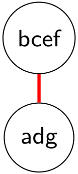

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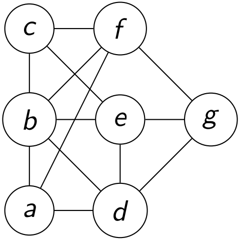
Twin-width

tww(G): Least integer d such that G admits a contraction sequence where all trigraphs have *maximum red degree* at most d.



Maximum red degree $= 0$

overall maximum red degree = 0

Twin-width

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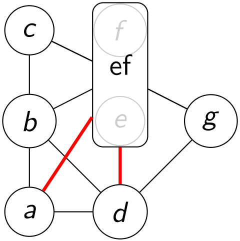

Maximum red degree $= 2$

overall maximum red degree = 2

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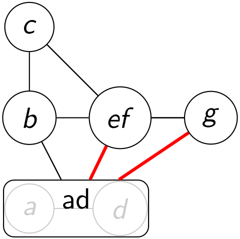

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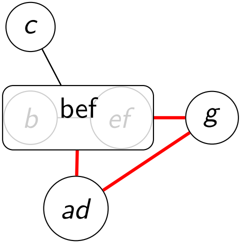

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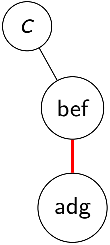

Maximum red degree $= 1$

overall maximum red degree = 2

Twin-width

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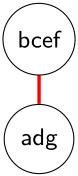

Maximum red degree $= 1$

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Twin-width

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Maximum red degree $= 0$
overall maximum red degree = 2

Twin-width of digraphs

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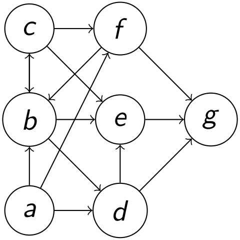

Maximum red degree $= 0$

overall maximum red degree = 0

Twin-width of digraphs

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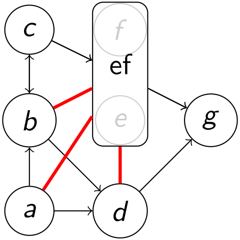

Maximum red degree $= 3$

overall maximum red degree = 3

Twin-width of digraphs

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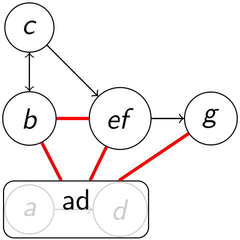

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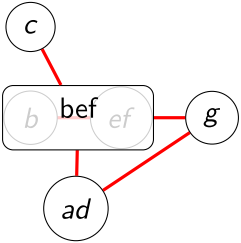

Maximum red degree = 3

overall maximum red degree = 3

Twin-width of digraphs

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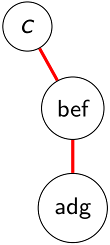

Maximum red degree $= 2$

overall maximum red degree = 3

Twin-width of digraphs

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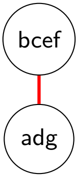

Maximum red degree = 1
overall maximum red degree = 3

Twin-width of digraphs

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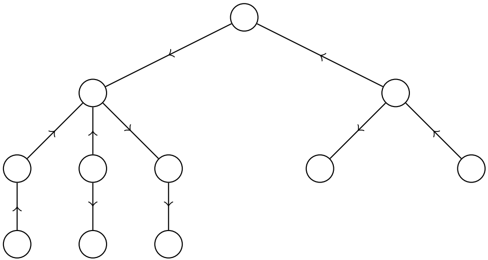
Maximum red degree $= 0$

overall maximum red degree = 3

Classes with bounded twin-width

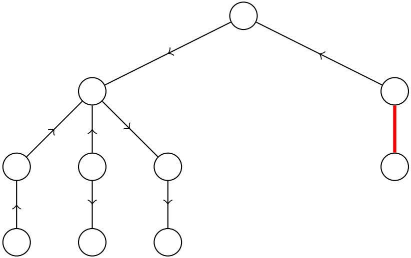
- cographs = twin-width 0
- trees, bounded treewidth, clique-width/rank-width
- grids
- ...

Trees



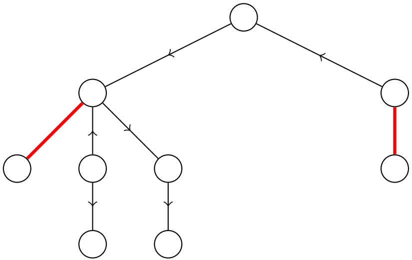
If possible, contract two leaves with the same parent

Trees



If not, contract a deepest leaf with its parent

Trees



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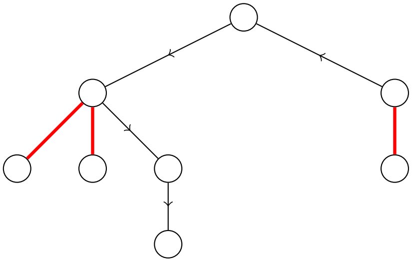

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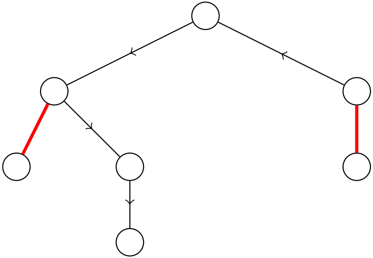
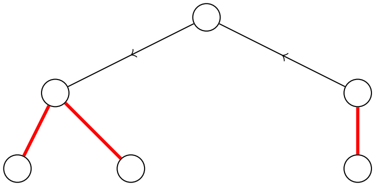
Trees



Cannot create a red degree-3 vertex

Trees



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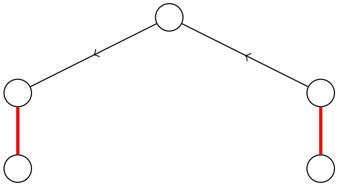

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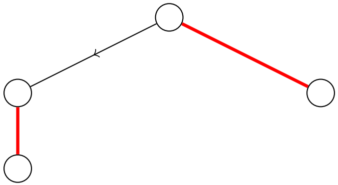

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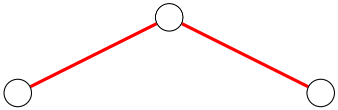

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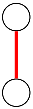
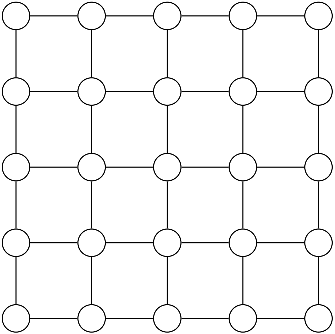


Cannot create a red degree-3 vertex

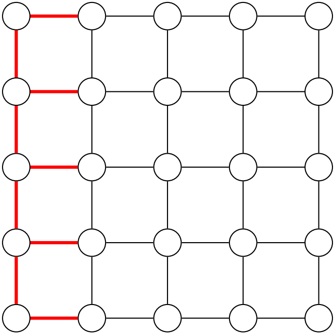
Trees

Generalization to orientations of bounded *treewidth* graphs,
and to undirected bounded *rank-width* graphs

Grids

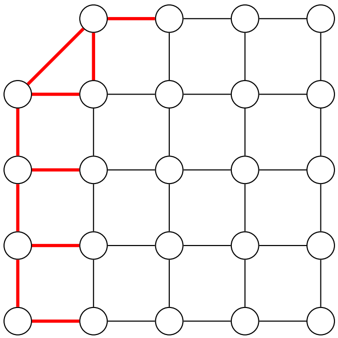


Grids



The following sequence works for any orientation

Grids



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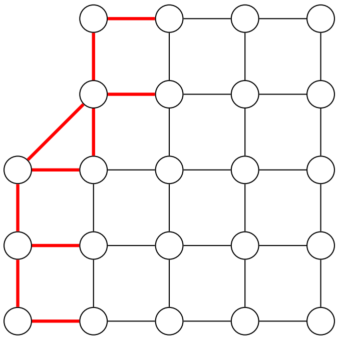
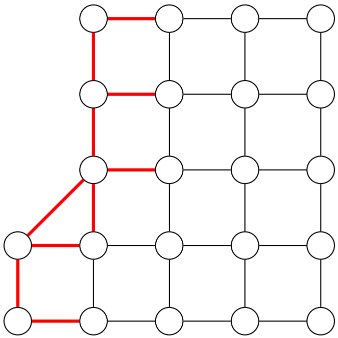


Grids



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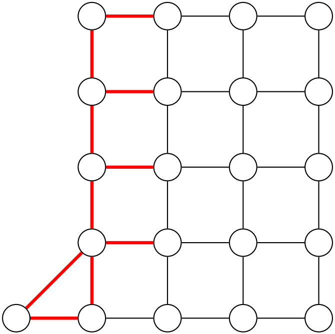

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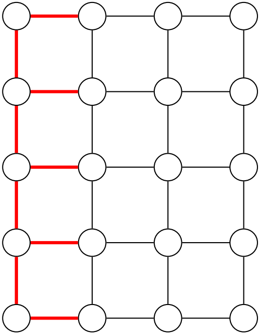


4-sequence for orientations of planar grids

Orientations of bounded twin-width classes

Perhaps every "sparse" class of bounded twin-width has an orientation closure of bounded twin-width?

Orientations of bounded twin-width classes

Perhaps every "sparse" class of bounded twin-width has an orientation closure of bounded twin-width?

Theorem

The class of all orientations of graphs from a $K_{t,t}$-free class of bounded twin-width has itself bounded twin-width.

We will see later why

Simple operations preserving twin-width

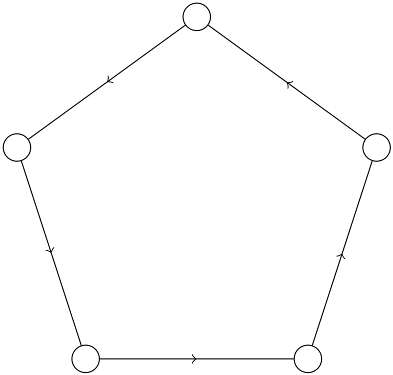
For graphs:

- complementation: remains the same
- taking induced subgraphs: may only decrease
- adding one apex: at most "doubles"
- substitution $G(v \leftarrow H)$: max of the twin-width of G and H

For digraphs:

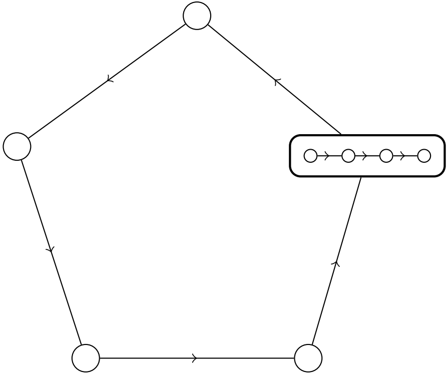
- any map $\{\rightarrow, \leftrightarrow, \cdots\} \rightarrow \{\rightarrow, \leftarrow, \leftrightarrow, \cdots\}$: may only decrease
- taking induced subdigraphs: may only decrease
- adding one apex: at most "quadruples"
- substitution $G(v \leftarrow H)$: max of the twin-width of G and H

Substitution and lexicographic product



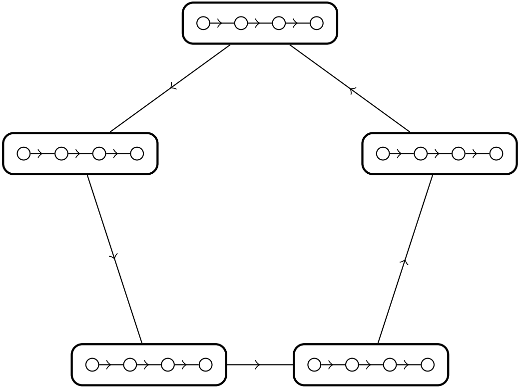
$G = \vec{C}_5$

Substitution and lexicographic product



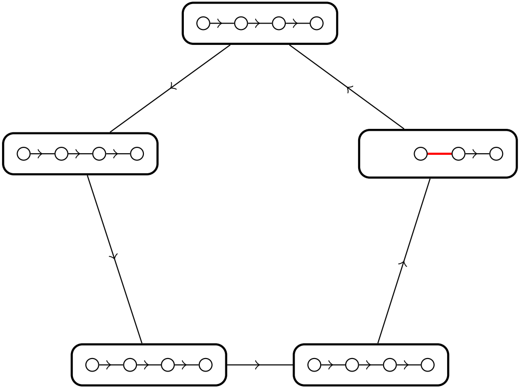
$G = \vec{C}_5$, $H = \vec{P}_4$, substitution $G[v \leftarrow H]$

Substitution and lexicographic product



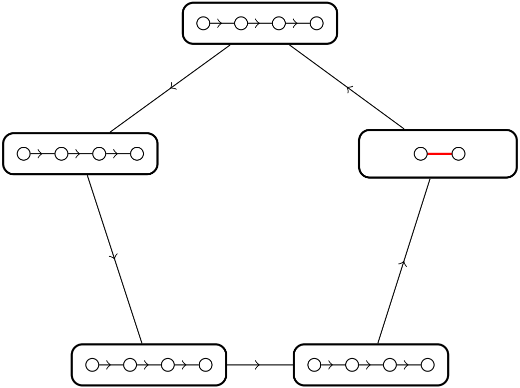
$G = \vec{C}_5$, $H = \vec{P}_4$, lexicographic product $G[H]$

Substitution and lexicographic product



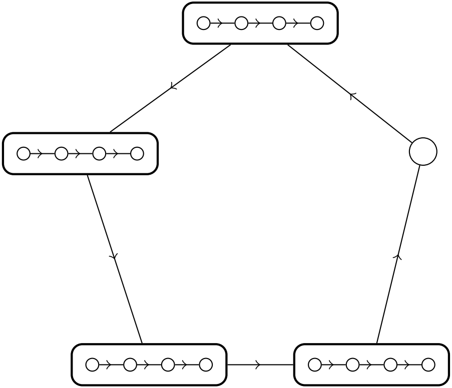
More generally any modular decomposition

Substitution and lexicographic product



More generally any modular decomposition

Substitution and lexicographic product



$$\text{tww}(G[H]) = \max(\text{tww}(G), \text{tww}(H))$$

Theorem (B., Geniet, Kim, Thomassé, Watrigant '20 & '21)

The following classes have bounded twin-width, and $O(1)$-sequences can be computed in polynomial time.

- *Bounded rank-width, and even, boolean-width graphs,*
- *every hereditary proper subclass of permutation graphs,*
- ***posets of bounded antichain size,***
- *unit interval graphs,*
- *K_t-minor free graphs,*
- *map graphs with embedding,*
- *d-dimensional grids,*
- *K_t-free unit d-dimensional ball graphs,*
- $\Omega(\log n)$*-subdivisions of all the n-vertex graphs,*
- *cubic expanders defined by iterative random 2-lifts from K_4,*
- *flat classes,*
- *subgraphs of every $K_{t,t}$-free class above,*
- *first-order transductions of all the above.*

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Encode a bipartite graph (or, if symmetric, any graph)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Contraction of two columns (similar with two rows)

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How is the twin-width (re)defined?

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

How to tune it for non-bipartite graph?

Twin-width in the language of matrices

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & r & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & r & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & r & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Digraph encoding:

- $i \to j$: 1 at (i, j), -1 at (j, i),
- $i \leftrightarrow j$: 2 at (i, j) and (j, i),
- otherwise: 0 at (i, j) and (j, i).

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Maximum number of non-constant "zones" per column or row part
= **error value**

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Maximum number of non-constant "zones" per column or row part
. . . until there are a single row part and column part

Partition viewpoint

Matrix partition: partitions of the row set and of the column set
Matrix division: same but all the parts are *consecutive*

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Twin-width as maximum error value of a contraction sequence

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: not full of 0 entries

$$
\left[\begin{array}{cc|cc|cc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

4-grid minor

Grid minor

t-grid minor: $t \times t$-division where every cell is non-empty
Non-empty cell: not full of 0 entries

$$\left[\begin{array}{cc|cc|cc|cc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

4-grid minor

A matrix is said t-**grid free** if it does not have a t-grid minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$
\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]
$$

3-mixed minor

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

Every mixed cell is witnessed by a 2×2 square $=$ **corner**

Mixed minor

Mixed cell: not horizontal nor vertical

$$\left[\begin{array}{cc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 1
\end{array}\right]$$

3-mixed minor

A matrix is said t-**mixed free** if it does not have a t-mixed minor

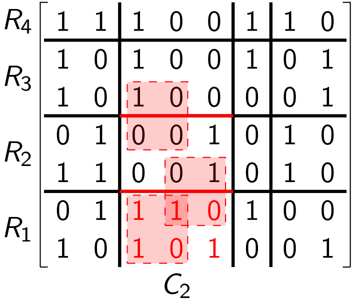
Mixed value

$$
\begin{array}{r}
R_4 \\ \\ R_3 \\ \\ R_2 \\ \\ R_1 \\
\end{array}
\left[\begin{array}{cc|ccc|c|cc}
1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
\hline
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
$$

C_2

$\approx$ (maximum) number of cells with a corner per row/column part

Mixed value



But we add the number of *boundaries* containing a corner

Mixed value

R_4	1	1	1	0	0	1	1	0
R_3	1	0	1	0	0	1	0	1
	1	0	1	0	0	0	0	1
$\cup$								
R_2	0	1	0	0	1	0	1	0
	1	1	0	0	1	0	1	0
R_1	0	1	1	1	0	1	0	0
	1	0	1	0	1	0	0	1
				C_2				

$\therefore$ merging row parts do not increase mixed value of column part

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If G admits ***a*** *t-mixed free adjacency matrix, then* $tww(G) = 2^{2^{O(t)}}$.

Holds for binary structures in general

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists\sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

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If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Merge consecutive parts greedily

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0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

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0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

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0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Question

For every k, is there a c_k such that every $n \times m$ $0,1$-matrix with at least c_k 1 per row and column admits a k-grid minor?

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Stanley-Wilf conjecture / Marcus-Tardos theorem

Auxiliary 0,1-matrix with one entry per cell: a 1 iff the cell is mixed

Conjecture (reformulation of Füredi-Hajnal conjecture '92)

For every k, there is a c_k such that every $n \times m$ $0,1$-matrix with at least $c_k \max(n, m)$ 1 entries admits a k-grid minor.

Conjecture (Stanley-Wilf conjecture '80s)

Any proper permutation class contains only $2^{O(n)}$ n-permutations.

Klazar showed Füredi-Hajnal $\Rightarrow$ Stanley-Wilf in 2000
Marcus and Tardos showed Füredi-Hajnal in 2004

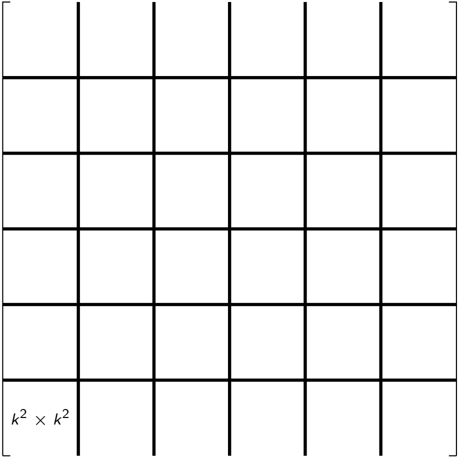
Marcus-Tardos one-page inductive proof

$$M = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Let M be an $n \times n$ $0,1$-matrix without k-grid minor

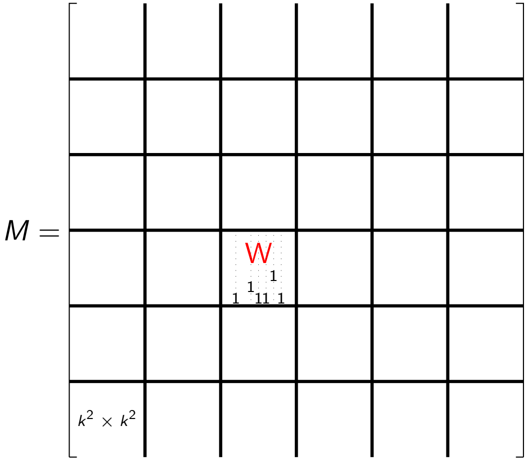
Marcus-Tardos one-page inductive proof

$M =$



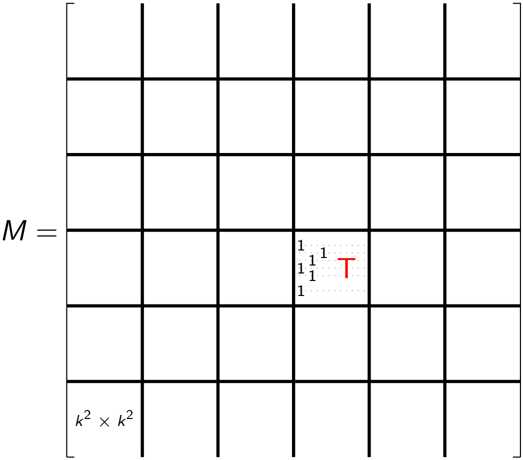
Draw a regular $\frac{n}{k^2} \times \frac{n}{k^2}$ division on top of M

Marcus-Tardos one-page inductive proof



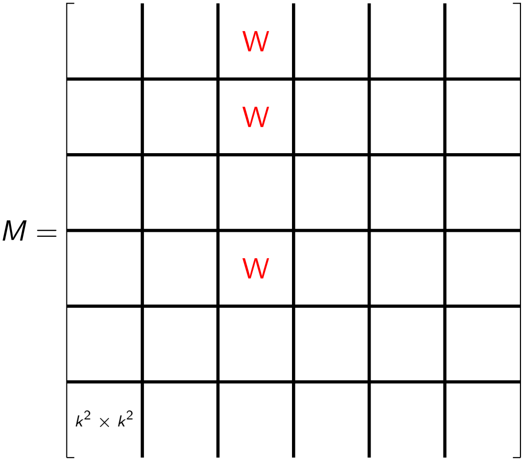
A cell is *wide* if it has at least k columns with a 1

Marcus-Tardos one-page inductive proof



A cell is *tall* if it has at least k rows with a 1

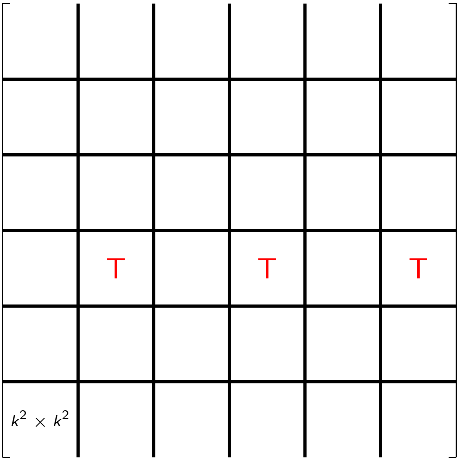
Marcus-Tardos one-page inductive proof



There are less than $k\binom{k^2}{k}$ wide cells per column part. Why?

Marcus-Tardos one-page inductive proof

$M =$



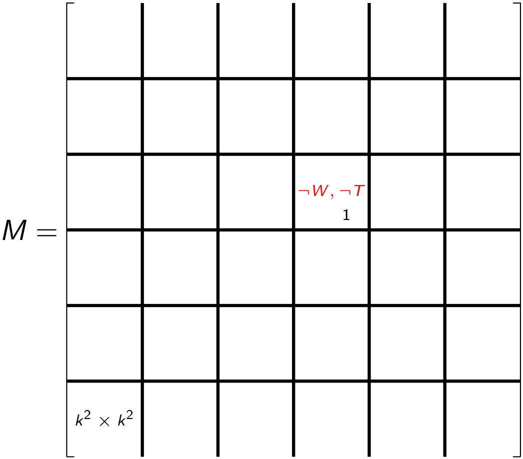
There are less than $k\binom{k^2}{k}$ tall cells per row part

Marcus-Tardos one-page inductive proof

$$M = \left[\begin{array}{c|c|c|c|c|c}
 & & W & & & \\ \hline
 & W & W & & & T \\ \hline
 & & & & & \\ \hline
 & T & W & T & & T \\ \hline
 & & T & & & \\ \hline
{\scriptstyle k^2 \times k^2} & & & & & W
\end{array}\right]$$

In W and T, at most $2 \cdot \frac{n}{k^2} \cdot k\binom{k^2}{k} \cdot k^4 = 2k^3\binom{k^2}{k}n$ entries 1

Marcus-Tardos one-page inductive proof



There are at most $(k-1)^2 c_k \frac{n}{k^2}$ remaining 1. Why?

Marcus-Tardos one-page inductive proof

$M =$

		W			
	W	W			T
			$\neg W, \neg T$ 1		
	T	W	T		T
		T			
$k^2 \times k^2$					W

Choose $c_k = 2k^4\binom{k^2}{k}$ so that $(k-1)^2 c_k \frac{n}{k^2} + 2k^3\binom{k^2}{k} n \leqslant c_k n$

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists\sigma$ s.t. $Adj_\sigma(G)$ is t-mixed free, then $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

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Step 1: find a division sequence $(\mathcal{D}_i)_i$ with mixed value $f(t)$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Stuck, removing every other separation $\rightarrow \frac{f(t)}{2}$ mixed cells per part

Impossible!

Twin-width and mixed freeness

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If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Step 1: find a division sequence $(\mathcal{D}_i)_i$ **with mixed value** $f(t)$
Step 2: find a contraction sequence with error value $g(t)$

1	1	1	1	1	1	1	0
0	1	1	0	0	1	0	1
0	0	0	0	0	0	0	1
0	1	0	0	1	0	1	0
1	0	0	1	1	0	1	0
0	1	1	1	1	1	0	0
1	0	1	1	1	0	0	1

Refinement of $\mathcal{D}_i$ where each part coincides on the non-mixed cells

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$*.*

Now to bound the twin-width of a class $\mathscr{C}$:

1) Find a *good* vertex-ordering procedure

2) Argue that, in this order, a t-mixed minor would conflict with $\mathscr{C}$

Twin-width and mixed freeness

Theorem (B., Kim, Thomassé, Watrigant '20)

If $\exists \sigma$ *s.t.* $Adj_\sigma(G)$ *is* t*-mixed free, then* $tww(G) = 2^{2^{O(t)}}$.

Theorem

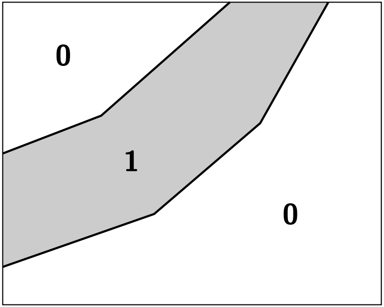
The following are equivalent.

- (i) $\mathscr{C}$ *has bounded twin-width.*
- (ii) $\mathscr{C}$ *has bounded "oriented twin-width."*
- (iii) $\mathscr{C}$ *is* t*-mixed free.*

Oriented twin-width: put red arcs from contracted vertices, and consider the red out-degree.

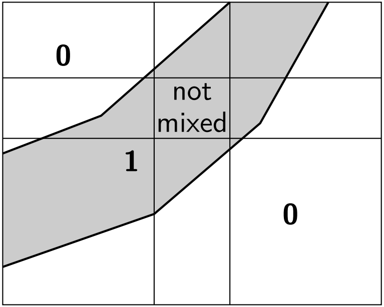
(i) $\Rightarrow$ (ii): immediate.
(ii) $\Rightarrow$ (iii): same simple proof as (i) $\Rightarrow$ (iii).
(iii) $\Rightarrow$ (i): what we just saw.

Bounded twin-width – unit interval graphs



Warm-up with unit interval graphs: order by left endpoints

Bounded twin-width – unit interval graphs

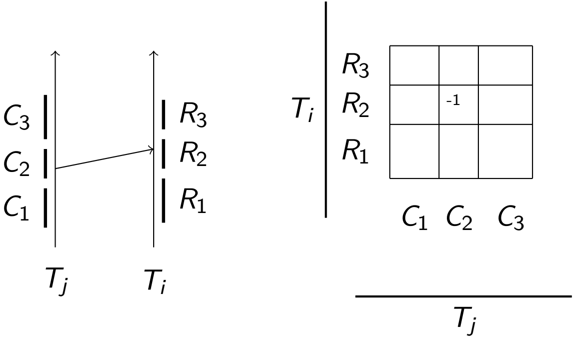


No 3-by-3 grid has all 9 cells crossed by two non-decreasing curves

Bounded twin-width – posets of bounded antichain

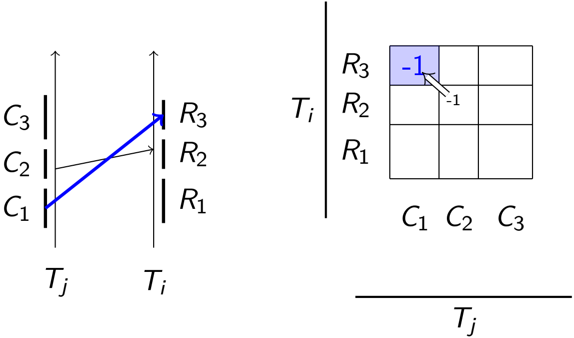
Put the k chains in order one after the other

Bounded twin-width – posets of bounded antichain



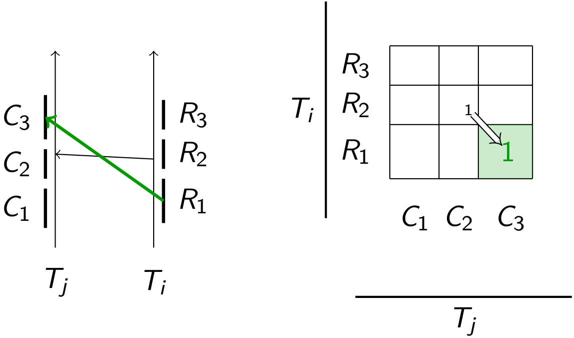
A $3k$-mixed minor implies a 3-mixed minor between two chains

Bounded twin-width – posets of bounded antichain



Transitivity implies that a zone is constant

Bounded twin-width – posets of bounded antichain



And symmetrically

Sparse twin-width

Theorem (B., Geniet, Kim, Thomassé, Watrigant 21)

If $\mathscr{C}$ is a hereditary class of bounded twin-width, tfae.

- *(i) $\mathscr{C}$ is $K_{t,t}$-free.*
- *(ii) $\mathscr{C}$ is d-grid free.*
- *(iii) Every n-vertex graph $G \in \mathscr{C}$ has at most gn edges.*
- *(iv) The subgraph closure of $\mathscr{C}$ has bounded twin-width.*
- *(v) $\mathscr{C}$ has bounded expansion.*

Sparse twin-width

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- *(v) $\mathscr{C}$ has bounded expansion.*

d-grid freeness is preserved by turning some 1 into -1 or 2

Theorem

The class of all orientations of graphs from a $K_{t,t}$-free class of bounded twin-width has itself bounded twin-width.

Sparse twin-width (2)

In the sparse setting d-mixed minor are replaced by d-grid minor

Theorem

If $\mathscr{C}$ is a hereditary $K_{t,t}$-free class, tfae.

- *(i) $\mathscr{C}$ has bounded twin-width.*
- *(ii) $\mathscr{C}$ is d-grid free.*

First-order model checking

FO MODEL CHECKING($\{E_2\}$)	**Parameter:** $\lvert\varphi\rvert$

Input: A digraph G and a first-order sentence $\varphi \in FO(\{E\})$
Question: $G \models \varphi$?

First-order model checking

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Input: A digraph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Example:

$$\varphi = \exists x_1 \exists x_2 \cdots \exists x_k \forall x \forall y \ (E(x,y) \Rightarrow \bigvee_{1 \leqslant i \leqslant k} x = x_i \vee y = x_i)$$

$G \models \varphi? \Leftrightarrow k$-VERTEX COVER

First-order model checking

FO MODEL CHECKING($\{E_2\}$) **Parameter:** $|\varphi|$
Input: A digraph G and a first-order sentence $\varphi \in FO(\{E\})$
Question: $G \models \varphi$?

Example:

$$\varphi = \bigvee_{1 \leqslant q \leqslant k,\ q \text{ is odd}} \exists x_1 \notin \{s\}\ E(s, x_1) \wedge (\forall x_2 \notin \{s, x_1\}\ \neg E(x_1, x_2) \vee$$

$$(\exists x_3 \notin \{s, x_1, x_2\}\ E(x_2, x_3) \wedge (\forall x_4 \cdots (\exists x_q \notin \{s, x_1, \ldots, x_{q-1}\} E(x_{q-1}, x_q)$$

$$\wedge\ (\forall x_{q+1}\ \neg E(x_q, x_{q+1}) \vee x_{q+1} \in \{s, x_1, \ldots, x_q\})) \cdots)))$$

$G \models \varphi$? $\Leftrightarrow$

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$$(\exists x_3 \notin \{s, x_1, x_2\}\ E(x_2, x_3) \wedge (\forall x_4 \cdots (\exists x_q \notin \{s, x_1, \ldots, x_{q-1}\} E(x_{q-1}, x_q)$$

$$\wedge\ (\forall x_{q+1}\ \neg E(x_q, x_{q+1}) \vee x_{q+1} \in \{s, x_1, \ldots, x_q\})) \cdots)))$$

$G \models \varphi$? $\Leftrightarrow$ Short Generalized Geography

First-order model checking

FO Model Checking($\{E_2\}$)	**Parameter:** $\|\varphi\|$
Input: A digraph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Also expressible in FO: k-Independent Set, k-Clique, k-Dominating Set, "transitive", etc.

First-order model checking

FO MODEL CHECKING($\{E_2\}$)	**Parameter:** $\lvert\varphi\rvert$
Input: A digraph G and a first-order sentence $\varphi \in FO(\{E\})$	
Question: $G \models \varphi$?	

Not expressible in FO: "k-colorable" for any $k \geqslant 2$, "cyclic", "Eulerian", "Hamiltonian", etc.

FO interpretations and transductions

FO simple interpretation: redefine the edges by a first-order formula
$\varphi(x, y) = \neg E(x, y)$ (complement)
$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

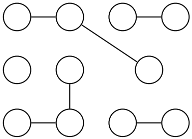
FO interpretations and transductions

FO simple interpretation: redefine the edges by a first-order formula

$\varphi(x, y) = \neg E(x, y)$ (complement)

$\varphi(x, y) = E(x, y) \vee \exists z E(x, z) \wedge E(z, y)$ (square)

FO transduction: color by $O(1)$ unary relations, interpret, delete



FO interpretations and transductions

FO simple interpretation: redefine the edges by a first-order formula

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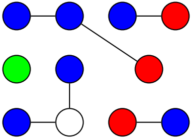

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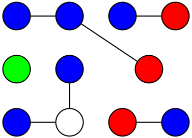

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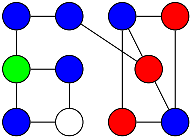

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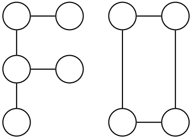

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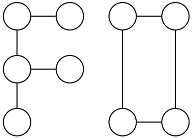

Theorem (B., Kim, Thomassé, Watrigant '20)

Transductions of bounded twin-width classes have bounded twin-width.

Dependence and monadic dependence

A class $\mathscr{C}$ is

dependent, if the hereditary closure of every interpretation of $\mathscr{C}$ misses some graph

monadically dependent, if every transduction of $\mathscr{C}$ misses some graph [Baldwin, Shelah '85]

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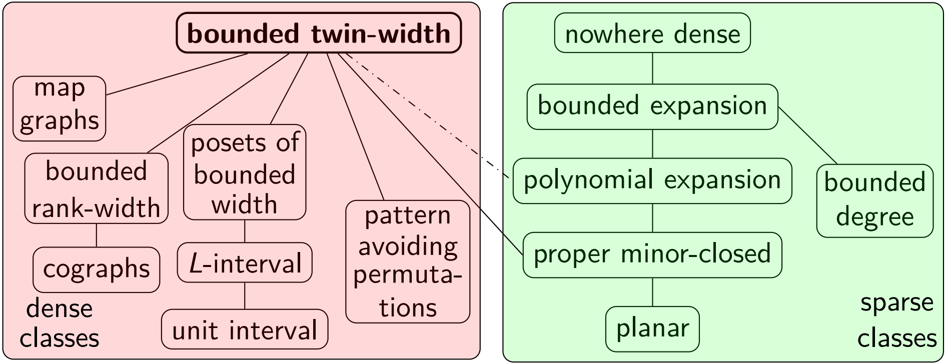
Theorem (Downey, Fellows, Taylor '96)

FO model checking is $AW[*]$-complete on general graphs,

thus unlikely FPT on independent classes

Could it be that on every dependent class, it is FPT?

Known FPT FO model checking –tractable classes



Theorem (B., Kim, Thomassé, Watrigant '20)

FO MODEL CHECKING *solvable in* $f(|\varphi|, d)n$ *on graphs with a* d*-sequence.*

Equivalences for **ordered** graphs

Theorem (B., Giocanti, Ossona de Mendez, Toruńczyk, Thomassé, Simon '21+)

Let $\mathscr{C}$ be a hereditary class of ordered graphs, the following are equivalent.

(*i*) $\mathscr{C}$ *has bounded twin-width.*

(*ii*) $\mathscr{C}$ *is tractable.*

(*iii*) $\mathscr{C}$ *is dependent.*

(*iv*) $\mathscr{C}$ *is monadically dependent.*

(*v*) $\mathscr{C}$ *has subfactorial growth.*

(*vi*) $\mathscr{C}$ *has exponential growth.*

Other settings where bounded twin-width $\Leftrightarrow$ tractable $\Leftrightarrow$ dependent?

Open question:
Let $\mathscr{T}$ be a hereditary class of tournaments.
$\mathscr{T}$ bounded twin-width $\Leftrightarrow \mathscr{T}$ tractable?

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Let $\mathscr{T}$ be a hereditary class of tournaments.
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Large transitive subtournaments $\rightarrow$ totally ordered pieces but no global order...

Caccetta-Häggkvist conjecture

CH: Every n-vertex oriented graph without directed cycles of length at most ℓ has minimum out-degree at most $(n-1)/\ell$.

"$\ell = 3$" has received the most attention

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The (assumed exhaustive list of) extremal configurations are built with lexicographic products so have bounded twin-width

Recap and open questions

We have seen that:

(1) Oriented twin-width is functionally equivalent to twin-width.

(2) Orientations of $K_{t,t}$-free bounded twin-width classes have bounded twin-width.

(3) Maximum "delimiting power" of twin-width on *ordered* graphs.

Open questions

- Marcus-Tardos-free proof of (1).
- Bounded twin-width $\Leftrightarrow$ tractable among hereditary classes of tournaments.
- Revisiting conjectures like CH with a bounded/unbounded twin-width win-win argument.