Decomposition theorem for locally out-transitive tournaments

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TALGO/IRIF

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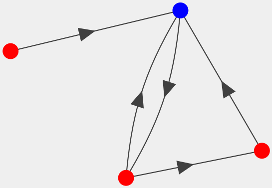
Definition

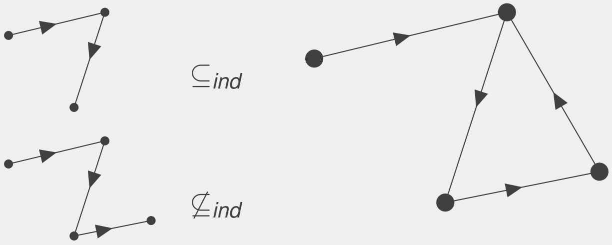
$G = (V, E)$ k-colourable iff $V = \bigcup_{i=1}^{k} V_i$ and $G[V_i]$ are acyclic.



$$\vec{\chi}(G) = \min_k \{k \mid G\ k\text{-colourable}\} = 2$$

$$\vec{\chi}(\mathcal{G}) = \max\{\vec{\chi}(G) \mid G \in \mathcal{G}\}$$

Induced subgraphs



$$\mathrm{Forb}\left(\mathcal{G}\right) = \{H | \forall G \in \mathcal{G}, G \not\subseteq_{ind} H\}$$

- Tournaments : $\mathrm{Forb}(\{\overline{K_2}\})$
- Transitive tournaments : $\mathrm{Forb}(\{\overline{K_2}, C_3\})$
- Hero : H such that $\overrightarrow{\chi}(\mathrm{Forb}(\{\overline{K_2}, H\}))$ bounded

Conjecture (Aboulker, Charbit, Naserasr, 2020)

H hero, F oriented forest
$\overrightarrow{\chi}(\mathrm{Forb}(\{H, F\}))$ bounded if and only if:

- F is a disjoint union of stars or
- H is transitive

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$$H \in \{K_1, \overrightarrow{K_2}\}, F \in \{K_1, \overrightarrow{K_2}\} \implies \text{Trivial}$$

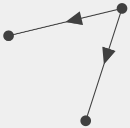
The conjecture

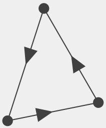
Conjecture (Aboulker, Charbit, Naserasr, 2020)

H hero, *F* oriented forest
$\overrightarrow{\chi}(\text{Forb}(\{H, F\}))$ bounded if and only if:

- *F* is a disjoint union of stars or
- *H* is transitive

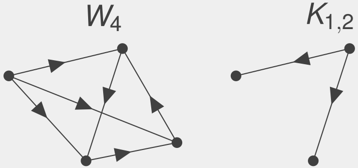
Thus, easiest remaining case :





Theorem (Aboulker, Aubian, Charbit / Steiner, 2021)

$\vec{\chi}(\text{Forb}(\{K_{1,2}, W_4\})) \leq 2$



- $\mathrm{Forb}(\{K_{1,2}\}) \implies$ out-neighbourhood tournament
- $\mathrm{Forb}(\{W_4\}) \implies$ out-neighbourhood C_3-free

Thus $\mathrm{Forb}(\{W_4, K_{1,2}\}) \implies$ out-neighbourhood transitive
We call these *locally out-transitive*

1. Decomposition theorem for $\text{Forb}(\{W_4, K_{1,2}\})$
2. - Use this to prove $\vec{\chi}(\text{Forb}(\{K_{1,2}, W_4\})) \leq 2$
   - Prove Caccetta-Häggkvist for $\text{Forb}(\{K_{1,2}, W_4\})$

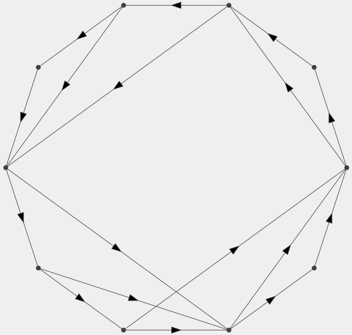
These results are thus implied for $\text{Forb}(\{K_{1,2}, C_3\})$

STRUCTURE OF LOCALLY OUT-TRANSITIVE

Definition

An oriented graph $D = (V, A)$ is in-round if there exists a cyclic order of V such that :

$$\forall xy \in A, \forall z \in]x, y[, zy \in A$$



Theorem

Let D be a strong oriented graph such that for every vertex x, x^+ induces a tournament and x^- induces an acyclic digraph. Then D is in-round.

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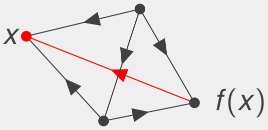
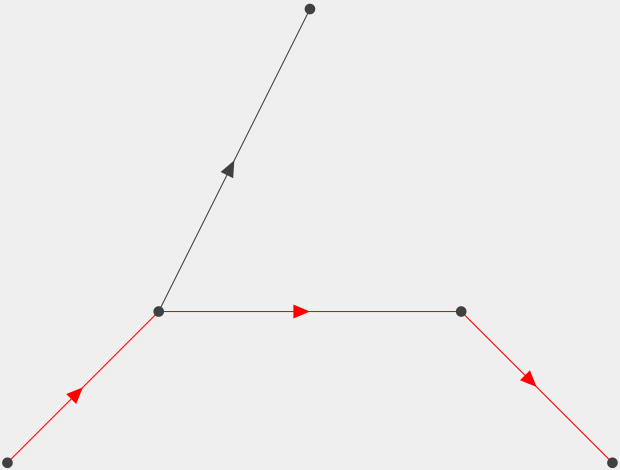
Proof :



In-round theorem proof



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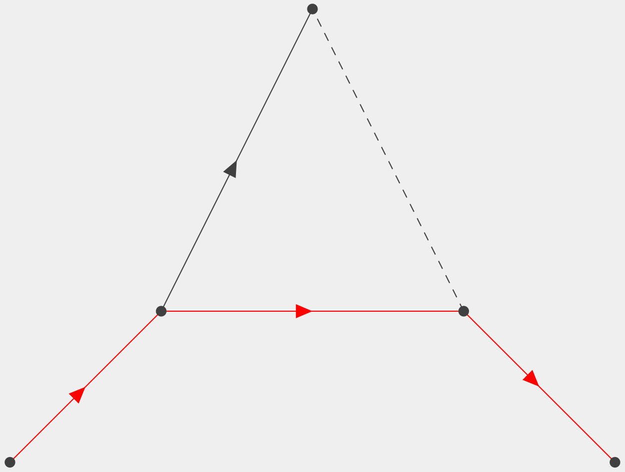

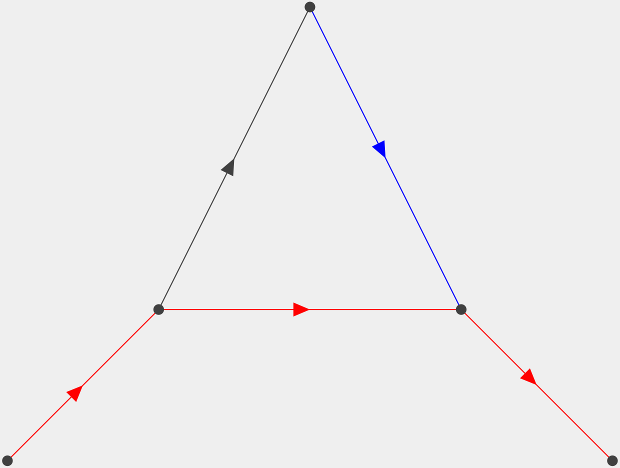

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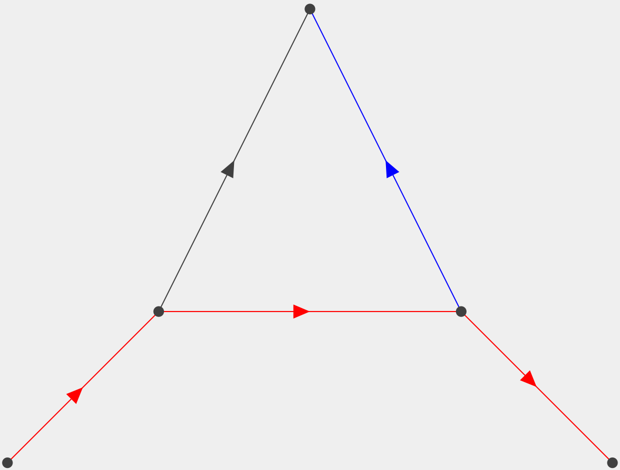
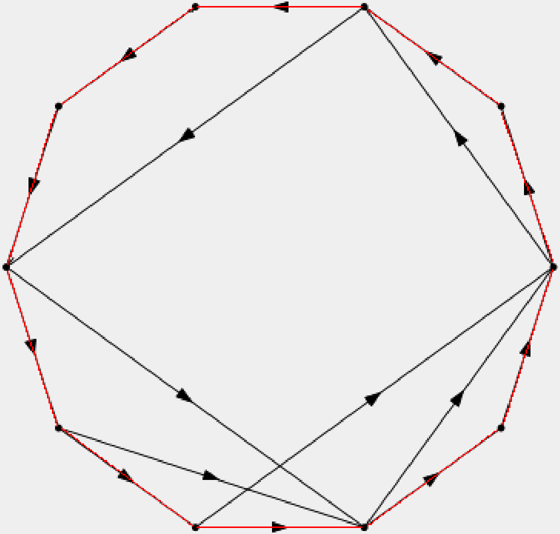
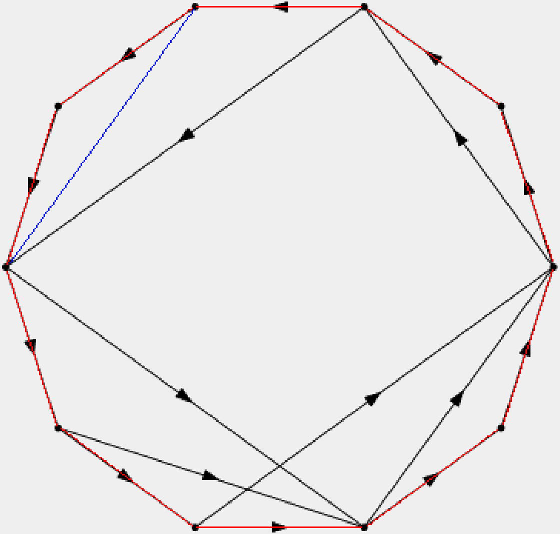


THE HAMILTONIAN CYCLE INDUCES AN IN-ROUND



The hamiltonian cycle induces an in-round



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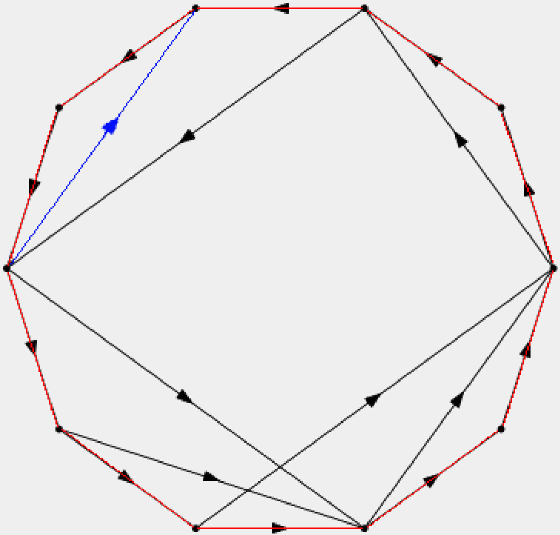

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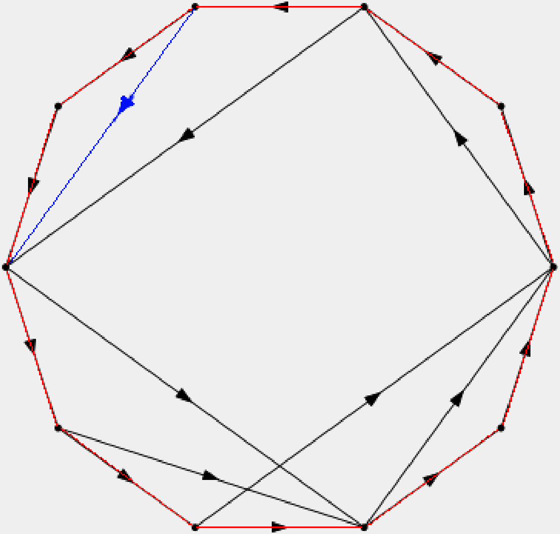
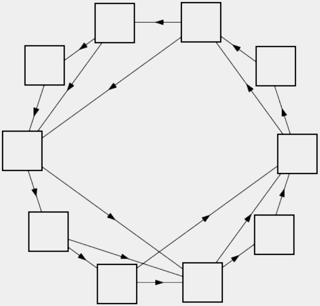


DECOMPOSITION THEOREM

Theorem

Let D a strong locally out-transitive oriented graph. There exists a partition of its set of vertices into strong subdigraphs whose contraction yields a strong in-round oriented graph.



Decomposition theorem

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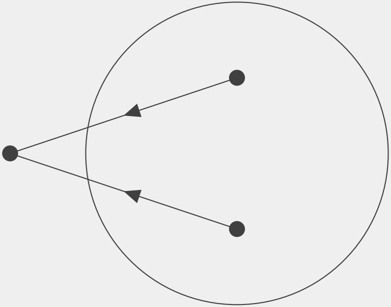
Let D a strong locally out-transitive oriented graph. There exists a partition of its set of vertices into strong subdigraphs whose contraction yields a strong in-round oriented graph.

Proof :

Definition

hub : a strongly connected in-dominated subdigraph

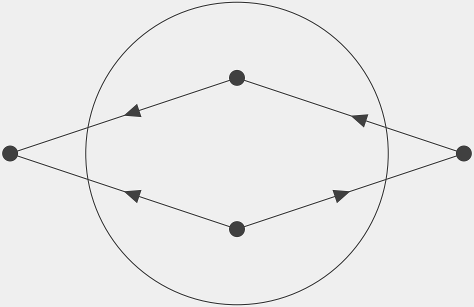
Decomposition proof



Decomposition proof

Claim

No vertex is mixed to a maximal hub



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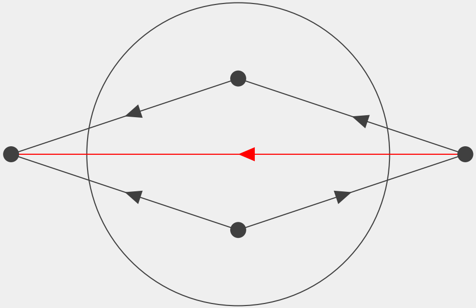

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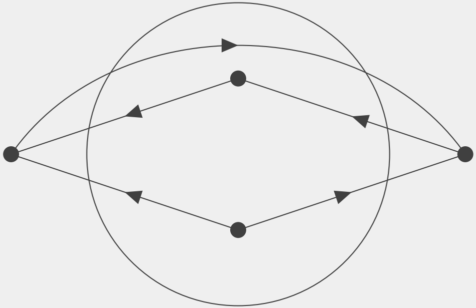

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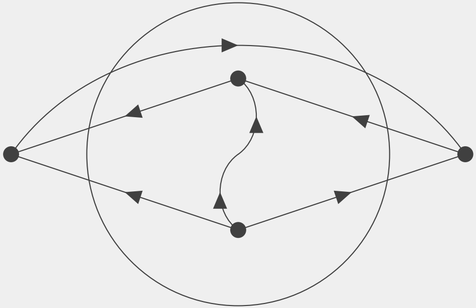

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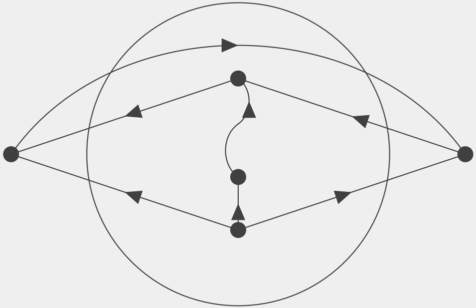

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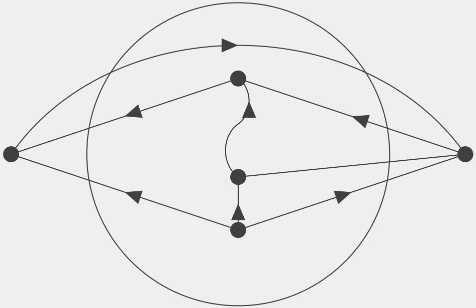

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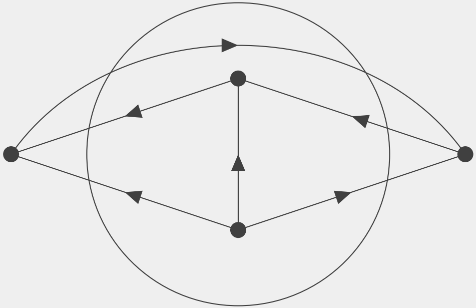

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No vertex is mixed to a maximal hub

Claim

Contracting a maximal hub cannot create a $K_{1,2}$ nor a W_4.

Thus contracting a maximal hub yields a locally out-transitive oriented graph.

Decomposition theorem's proof

Theorem

Let D a strong locally out-transitive oriented graph. There exists a partition of its set of vertices into strong subdigraphs whose contraction yields a strong in-round oriented graph.

- if there exists a non-trivial hub, contract it and recurse
- otherwise, inneighbourhoods are acyclic, thus D is in-round

COLOURING LOCALLY OUT-TRANSITIVE

Theorem

In-round oriented graphs are 2-dicolourable.

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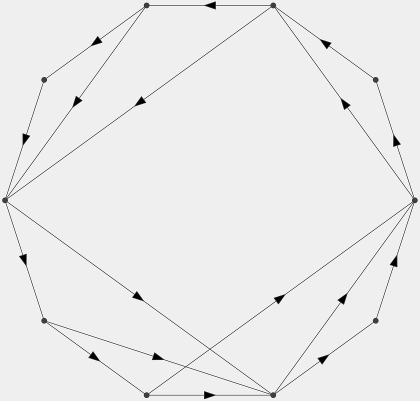
Proof :



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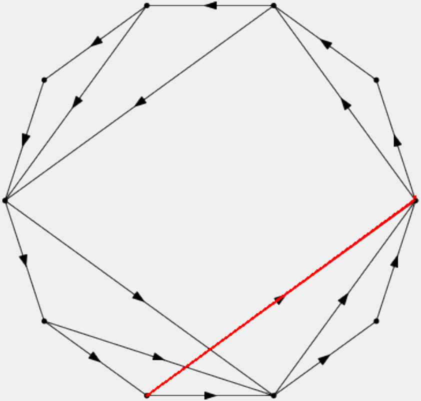

Theorem

In-round oriented graphs are 2-dicolourable.

Theorem

For any in-round oriented graph G and vertex v, G admits a 2-dicolouring where $v^+ \cup \{v\}$ is monochromatic.

Theorem (Aboulker, Aubian, Charbit / Steiner, 2021)

For any locally out-transitive oriented graph D, D admits a 2-dicolouring.

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For any locally out-transitive oriented graph D **and transitive subtournament** T, D admits a 2-dicolouring **in which T is monochromatic**.

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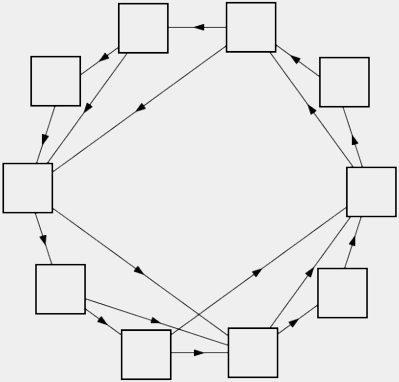

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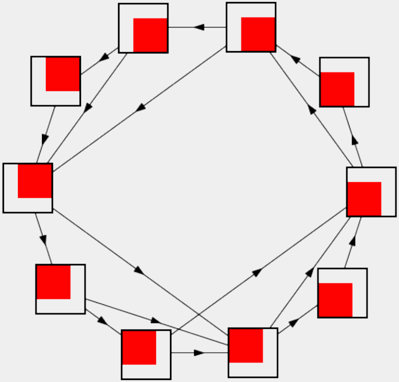

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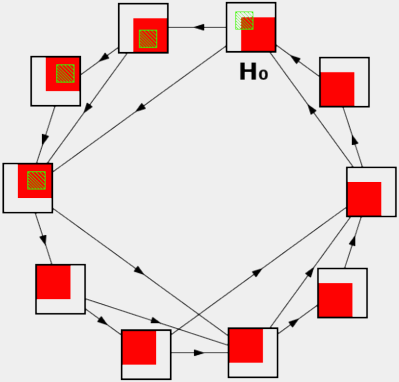
For any locally out-transitive oriented graph *D* and transitive subtournament *T*, *D* admits a 2-dicolouring in which T is monochromatic.



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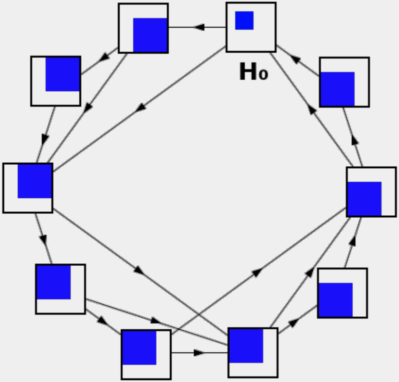

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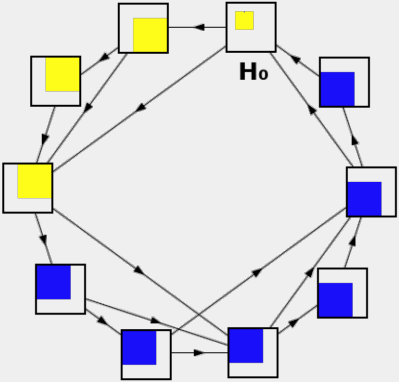


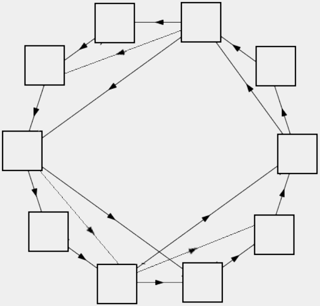
Caccetta-Häggkvist

Conjecture, (Caccetta-Häggkvist)

Let $k \geq 2$ be an integer. Every digraph D on n vertices with no directed circuit of length at most k contains a vertex of out-degree less than n/k.

Theorem

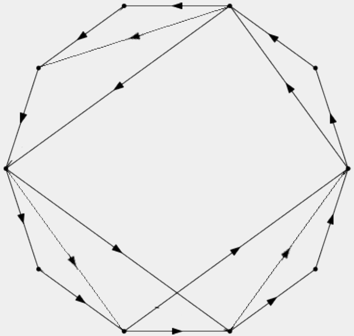
Let $D = (V, A)$ a strong locally in-tournament oriented graph without C_3, there exists a partition of V into strong subdigraphs whose contraction yields a strong out-round oriented graph D', i.e. it admits a cyclic order such that :
$\forall x, y, z \in V(D')\ (xy \in A \wedge z \in]x, y[) \Rightarrow xz \in A$



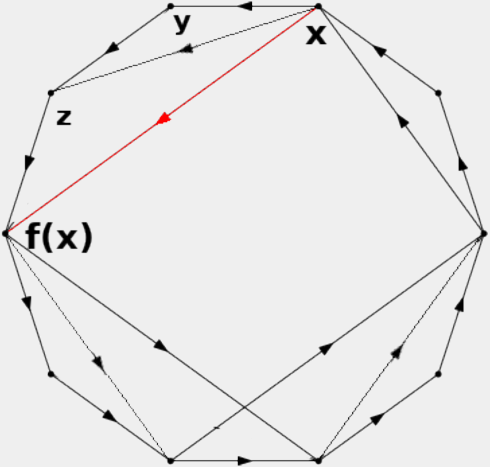
Lemma

Let $k > 2$, D a strong out-round oriented graph with no C_k and $w : V(D) \to \mathbb{R}^+$ and $W = \sum_{u \in V(D)} u$.

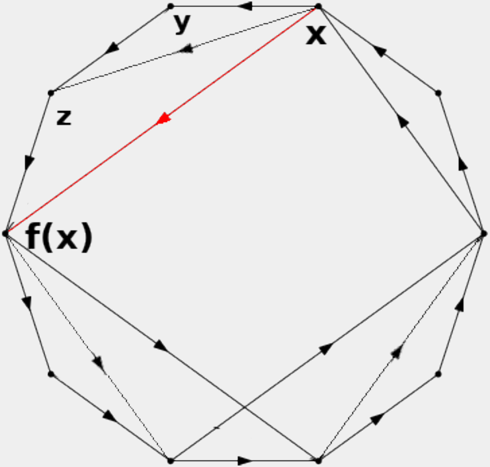
There exists u such that $\sum_{v \in u^+} w(v) \leq \frac{W - w(v)}{k}$



Lemma proof : two definitions



LEMMA PROOF : TWO DEFINITIONS



$$\phi(x) = \sum_{v \in x^+} w(v)$$

Thus $\forall x, \sum_{i=0}^{k} \phi(f^{(i)}(x)) \leq W - w(x)$

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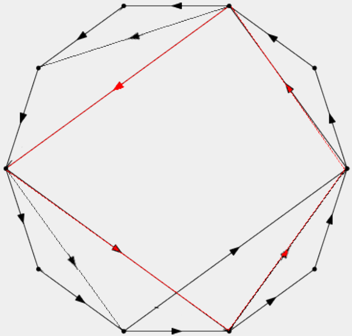

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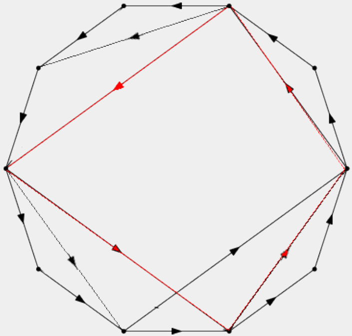

$k \sum_{x \in V(C)} \phi(x) \leq W|C| - \sum_{x \in V(C)} w(x)$

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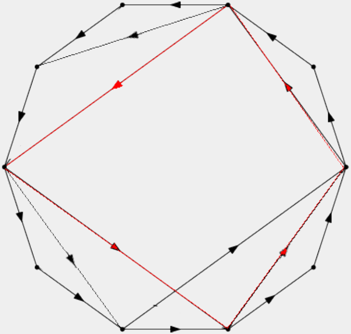


$k \sum_{x \in V(C)} \phi(x) \leq W|C| - \sum_{x \in V(C)} w(x)$

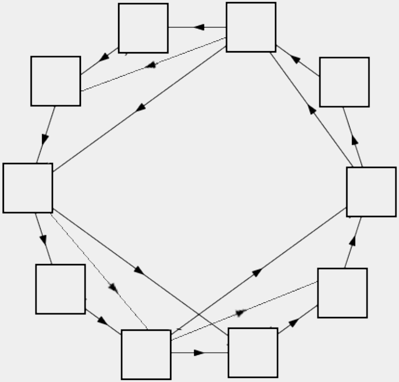
Which implies $k\phi(x) + w(x) \leq W$

Concluding the proof of our theorem

Lemma

Let $k > 2$, D a strong out-round oriented graph with no C_k and $w : V(D) \to \mathbb{R}^+$ and $W = \sum_{u \in V(D)} u$.

There exists u such that $\sum_{v \in u^+} w(v) \leq \frac{W - w(v)}{k}$



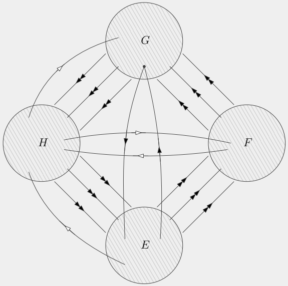
What comes next ?

A (COMPLICATED) DECOMPOSITION THEOREM

Theorem

Let D a connected locally semicomplete digraph, then either :

- *D is semicomplete with a universal vertex.*
- *There exists a partition of $V(D)$ into $k \geq 2$ subsets each inducing strong connected semicomplete digraphs such that the digraph obtained by contracting every member of the partition is a round oriented graph.*
- *there exists a partition of $V(D)$ into four sets E, F, G and H such that :*
    - *F and H are non empty, and one of E and G is non empty.*
    - *$D[E]$, $D[F]$, $D[G]$ and $D[H]$ are semicomplete;*
    - *E strictly out-dominates F, F strictly out-dominates G, G out-dominates H and H out-dominates E.*
    - *$\forall x \in G,\ x^+ \cap E \neq \emptyset$ and $x^- \cap E \neq \emptyset$*


G
H
F
E