Open problems on digraphs

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Supereulerian digraphs

A digraph $D=(V,A)$ is supereulerian if it contains a spanning connected subdigraph $D'=(V,A')$, $A' \subseteq A$ s.t $d^+_{D'}(v) = d^-_{D'}(v) \quad \forall v \in V$

In particular every hamiltonian digraph is supereulerian

Theorem (B-J, Besoy)

It is NP-complete to decide whether a digraph is supereulerian

Problem

For which classes $\mathcal{C}$ of well-studied digraphs can we decide a digraph D from $\mathcal{C}$ is supereulerian?

True for $\mathcal{C}$ = semicomplete multipartite, in-semicomplete, quasi-transitive

An Eulerian factor of a digraph $D=(V,A)$ is a subdigraph $D'=(V,A')$ s.t each strongly connected component D'_i of D is supereulerian

Proposition (B-J, Maddaloni)

One can check whether D has an eulerian factor and if so, produce an eulering factor in pol. time

Note similarity with cycle factors:

an eulerian factor is a spanning collection of arc-disjoint cycles

So being supereulerian is a relaxation of being eulerian and of being hamiltonian

A digraph $D=(V,A)$ is semicomplete multipartite if its underlying undirected graph is complete multipartite.

<u>Theorem (B-J, Maddaloni)</u>

A semicomplete multipartite digraph D is supereulerian if and only if it is strong and has an eulerian factor

Theorem (Chvatal, Erdös)

if $\kappa(G) \geq \alpha(G)$ then G is hamiltonian

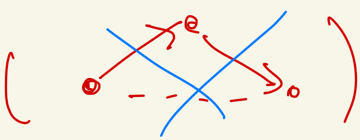
Proposition suppose $|V(G)| \geq 3$, then

$\lambda(G) \geq \alpha(G)$ implies that G is supereulerian

Conjecture (B-J, Thomassé)

$$\lambda(D) \geq \alpha(D) \Rightarrow D \text{ is supereulerian}$$

- True for symmetric digraphs
- True for semicomplete multipartite digraphs
- True for quasi-transitive digraphs



- Open even for $\alpha = 2$ whether $\exists K$ s.t. $\lambda(D) \geq K$ and $\alpha(D) = 2$ $\Rightarrow$ supereulerian

Theorem (Fraise, Thomason)

Every k-strong tournament T has a hamiltonian cycle avoiding any prescribed set of $k-1$ arcs of T

Conjecture (B-J, Havet, Yeo)

$\forall$ semicomplete $D=(V,A)$ with $\lambda(D) \geq k+1$ and $\forall A' \subseteq A$ $|A'|=k$:
$D-A'$ has a spanning eulerian subdigraph.

True if $\lambda(D) \geq \left\lceil \frac{6k+1}{5} \right\rceil$ (B-J, Déprés, Yeo)

Increasing vertex-connectivity by arc reversal/addition

$a_k(D)$ = min # new arcs to add to $D=(V,A)$ s.t $D' = (V, A \cup F)$ is k-strong

$a_k(D) < \infty$ if $|V(D)| \geq k+1$

$a_k(D)$ can be found in polynomial time (Frank, Jordan)

$r_k(D)$ = min # arcs to reverse in D s.t. result is k-strong

Clearly $a_k(D) \leq r_k(D)$ $\forall D$

Can find $r_1(D)$ in polynomial time via submodular flows

$r_k(D)$ NP-hard for $k \geq 2$

Durand de Gevigny for $k \geq 3$

B-J, Hoersch, Kriesell for $k=2$

Conjecture (B-J, 1994)

Every tournament T on $n \geq 2k+1$ vertices satisfies that $r_k(T) \leq \binom{k+1}{2}$

Easy to show that $r_k(T) \leq \binom{4k-2}{2}$

Theorem (B-J, Johansen, Yeo)

For every semicomplete digraph D on $n \geq k+1$ vertices

$a_k(D) \leq \binom{k+1}{2}$

Theorem (B-J, Jordán)

For every semicomplete digraph D on $n \geq 3k-1$ vertices we have $a_k(D) = r_k(D)$

$\Rightarrow$ Conjecture above holds if $|V(T)| \geq 3k-1$ and $r_k(T) \leq \binom{3k-2}{2}$ always

Deorienting arcs to increase (arc-) connectivity

By deorienting an arc $u \to v$ in D we mean adding the arc $v \to u$

$\mathrm{deor}_k(D)$ = min # of arcs to deorient in D s.t. result is k-strong

NP-hard to calculate $\mathrm{deor}_k(D)$ when $k \geq 3$ (B-J, Hoersch, Kriesell)

Complexity of finding $\mathrm{deor}_2(D)$ is open

$\mathrm{deor}_k^{\text{arc-strong}}(D)$ = min # arcs to deorient in D s.t result is k-arc-strong

$\mathrm{deor}_1^{\text{arc-strong}}(D)$ polynomial via submodular flows

$\mathrm{deor}_k^{\text{arc-strong}}(D)$ open for all $k \geq 2$

Spanning oriented subgraphs of digraphs

Conjecture (Jackson & Thomassen)

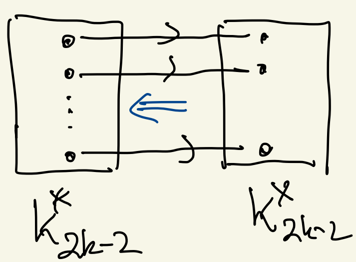
Every $2k$-strong $D=(V,A)$ contains a k-strong spanning subdigraph $D'=(V,A')$ $A'\subseteq A$
(may think of D' as obtained by deleting one arc of each 2-cycle in D)

- Open even for semicomplete digraphs
- $(3k-2)$-strong suffices for semicomplete D (Guo)

Conjecture (B-J, Jordan)

$\forall k\geq 1$ $\forall (2k-1)$-strong semicomplete D on $n\geq 2k+1$ vertices contains a spanning k-strong tournament

- would be best possible



- True for $k=2$ and 6 suffices for $k=3$

Majority Colourings

A vertex coloring c of D is a majority colouring of D if $|\{w \mid v \to w \wedge c(v) \neq c(w)\}| \geq |\{w \mid v \to w \wedge c(v) = c(w)\}|$ holds for all $v \in V(D)$

Theorem (Kreutzer et al)

Every digraph has a majority colouring with 4 colors

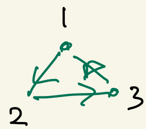
Follows from:

Every acyclic digraph has a majority col with 2 colors

Conjecture (Kreutzer et al)

Every D has a majority colouring with 3 colors

- 2 is not enough



- NPC to decide if D has a majority col with 2 colours (B-J, Bessy, Havet, Yeo)
- Open for tournaments
- True if $\delta^+(D) \geq 72 \log 3n$ $n = |V(D)|$ (Kreutzer et al)
- True if $\delta^+(D) \geq 1200$ and exponentially bounded in-degrees

2-partitions of tournaments

Theorem (Köhn et al)
$\exists f(k)$ s.t every $f(k)$-strong tournament $T=(V,A)$ has a 2-partition $V=V_1 \cup V_2$ s.t $T\langle V_1\rangle$, $T\langle V_2\rangle$ and $T\langle V_1,V_2\rangle$ are all k-strong

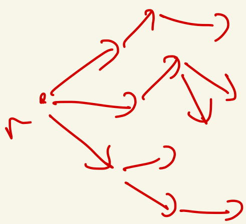
Question what is the complexity of deciding for a given semicomplete $D=(V,A)$, whether it has a 2-partition $V=V_1 \cup V_2$ such that $D\langle V_1\rangle$, $D\langle V_2\rangle$ and $D\langle V_1,V_2\rangle$ are all strong?

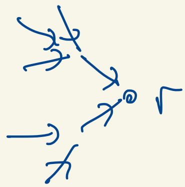
polynomial to check for 2-partition $V=V_1 \cup V_2$ s.t each of $D\langle V_1\rangle$, $D\langle V_2\rangle$ are strong (B-), Nielsen)

NPC if we want $D\langle V_1\rangle$ and $D\langle V_2\rangle$ to be strong tournaments (B-), Christiansen)

Arc-disjoint spanning subdigraphs

out-branching / in-branching





root r

Conjecture (Thomassen)

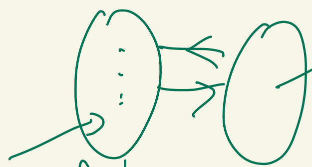
$\exists k$ s.t. $\lambda(D) \geq k \Rightarrow D$ has an out-branching which is arc-disjoint from some in-branching

- True with $k=2$ for semicomplete digraphs (B-J, Yeo)
- True with $k=3$ for D with $\alpha(D)=2$ (B-J, Bessy, Havet, Yeo)
- True with $k=3$ for D of the form $D = S[D_1, D_2, \ldots, D_s]$, $s = |V(S)|$ when S is semicomplete (B-J, Gutin, Yeo)
- open whether already $k=3$ suffices for any digraph.

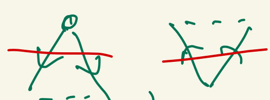
Conjecture (B-J, Yeo)

$\exists\, k$ s.t. every k-arc-strong $D=(V,A)$ has an arc-partition $A=A_1 \cup A_2$ s.t. $D_i=(V,A_i)$ is strong for $i=1,2$

- True with $k=3$ semicomplete D (B-J, Yeo) one exception for $k=2$
- True with $k=3$ for digraphs of the form

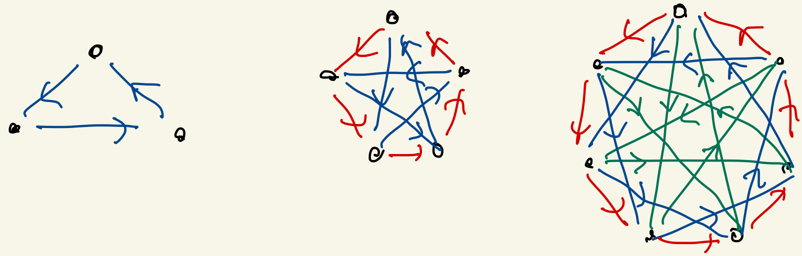


The underlying graphs of such digraphs are exactly the class of split graphs.

- True with $k=3$ for locally semicomplete digraphs () B-J, Huang

Conjecture (Kelly)

Every k-regular tournament decomposes into k arc-disjoint hamiltonian cycles



- True for very large k (Kühn et al)
- would follow for all k from the following (Every k-regular T has $\lambda(T) \geq k$)

Conjecture (B-J, Yeo)

Every k-arc-strong tournament $T=(V,A)$ has an arc-decomposition $A = A_1 \cup A_2 \cup \dots \cup A_k$ s.t. $D=(V,A_i)$ is strong for $i=1,2,\dots,k$

This is true if $\delta^+(T), \delta^-(T) \geq 37k$ (B-J, Yeo)

Weaknesses of the conjecture:

Conjecture (B-J, Yeo)

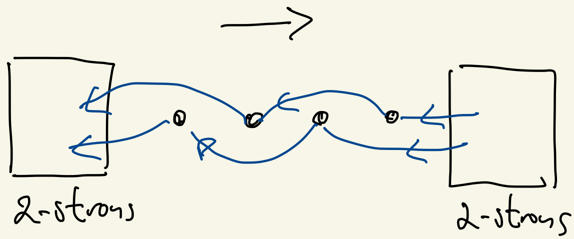
$\forall k \geq 2$ and s_1, s_2 s.t $k = s_1 + s_2$ every tournament $T = (V, A)$ with $\lambda(T) = k$ has arc-decomp $A = A_1 \cup A_2$ s.t. $\lambda(D_i) \geq s_i$ for $i = 1, 2$ when $D_i = (V, A_i)$

Conjecture (B-J, Yeo)

$\forall k \geq 1$ every k-arc-strong tournament $T = (V, A)$ has a strong spanning subdigraph $H = (V, A^*)$ s.t $\lambda(T - A^*) \geq k - 1$

Thomassen proved that every 2-arc-strong tournament has a hamiltonian path P s.t $T - A(P)$ is strong

This does not hold if we replace path by cycle:



Problem (B-J, Yeo)

Is there a polynomial algorithm for deciding whether a strong tournament T has a hamiltonian cycle C s.t. $T - A(C)$ is strong?

Conjecture (Thomassen)

Every 3-strong tournament has a pair of arc-disjoint hamiltonian cycles.

would follow from

Conjecture (B-J, Yeo)

If a tournament has no pair of arc-disjoint hamiltonian cycles, then there are 2 arcs $a, a' \in A(T)$ s.t $T - \{a, a'\}$ has no hamiltonian cycle

By the Fraisse-Thomassen theorem, every 3-strong tournament contains a hamiltonian cycle avoiding any two prescribed arcs

So Thomassens conjecture follows if the above conjecture holds.

Mixed linkings in digraphs

Question (Thomassen) What is the complexity of deciding whether the underlying graph $G(D)$ of a digraph D has edge-disj. spanning trees T_1, T_2 s.t. T_1 is oriented as an out-branching in D?

Answer: the problem is NP-complete (B-J, Yeo)

Theorem (B-J, Kriesell) The following is NPC

Given a digraph $D=(V,A)$ and distinct $s_1,s_2,t_1,t_2 \in V$
Does $G(D)$ contain disjoint paths P_1,P_2 s.t. P_i is an (s_i,t_i)-path $i=1,2$ and in D the path P_1 is a directed (s_1,t_1)-path?

Conjecture (B-J, Kriesell)
$\exists K$ s.t. $\forall D$ which is K-strong and $s_1,s_2,t_1,t_2 \in V(D)$: $G(D)$ contains P_1,P_2 as above

$K \geq 6$ must hold as there are planar 5-connected non 2-linked graphs

Certificates for k(-arc)-strong connectivity

Every digraph $D=(V,A)$ on n vertices with $\lambda(D)=k$ has a spanning k-arc-strong subdigraph $D'=(V,A')$ s.t. $\lambda(D')=k$ and $nk \le |A'| \le 2k(n-1)$. (*)

D' is called a certificate for D being k-arc-strong

(*) follows from Edmonds branchings theorem

as D has k arc-disjoint out-branchings $B^+_{s,1},\dots,B^+_{s,k}$

and k arc-disj. in-branchings $B^-_{s,1}, B^-_{s,2},\dots,B^-_{s,k}$

and taking $A'=\left(\bigcup_i A(B^+_{s,i})\right)\cup\left(\bigcup_j A(B^-_{s,j})\right)$ works

Let $f(n,k)$ be the smallest integer s.t. every k-arc-strong tournament T on n vertices has a sp. k-arc-strong $D'=(V,A')$ with $|A'|\le f(n,k)$

Theorem (B-J, Huang, Yeo)

$$f(n,k)\le nk+136k^2$$

Let $\delta_{\ge k}(D)$ denote the minimum number of arcs in a spanning subdigraph D' of D in which

$\delta^+_{D'},\delta^-_{D'}\ge k$

Conjecture (B-J, Huang, Yeo)

For every k-arc-strong tournament T the number of arcs in a smallest certificate for $\lambda(T)=k$ is exactly $\delta_{\geq k}(T)$

Theorem (B-J, Huang, Yeo)

Every tournament $T=(V,A)$ with $\lambda(T)=k$ has a spanning subdigraph $D'=(V,A')$ with $\delta^+_{D'}, \delta^-_{D'} \geq k$ and $|A'| \leq nk + \binom{k+1}{2}$. If T is also k-arc-strong then we can find D' s.t. $|A'| \leq nk + \binom{k}{2}$
A' can be found in polynomial time (via flows)

Conjecture (B-J, Huang, Yeo)

$\exists$ polynomial algorithm for finding an optimal (minimum size) certificate for k-arc-strong connectivity for a given tournament T with $\lambda(T)=k$.

Theorem (Kang et al)

Every k-strong tournament on n vertices contains a spanning k-strong subdigraph on at most $nk + 750k^2\log(k+1)$ arcs

Orienting a graph to avoid even directed cycles

Theorem (Chung et al)

Every strong digraph on n vertices and at least $\left(\frac{n+1}{2}\right)^2 + 1$ arcs has an even cycle.

Best possible!

Every strong orientation of the Petersen graph has an even cycle

Problem (B-J)

What is the complexity of deciding whether a given input graph G has a strong orientation with no even cycle?

Problem (B-J)

Does there exist a polynomial algorithm which given a graph G decides whether G has a strong orientation D with no two cycles C, C' s.t. $\left| |C| - |C'| \right| = 1$?

Linkages in 'almost acyclic' digraphs

Theorem (Fortune et al)

The k-linkage and the weak-k-linkage (arc-disj paths) is polynomial for every fixed k in acyclic digraphs

The **circumference** of a digraph D is the length of a longest cycle in D

Theorem (B-J, Christiansen)

If the circumference of D is at most p then D has DAG width at most p and hence directed-tree width at most 3p+1

Corollary For every integer p

The k-linkage problem is polynomial

Problem (B-J, Christiansen)

Determine the complexity of the weak-k-linkage problem for digraphs of circumference p.

- polynomial for $p=2$
- open already for $p=3$

Miscellaneous problems

Problem (B-J)

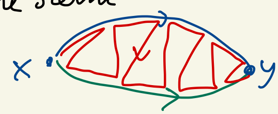
What is the complexity of finding a hamiltonian cycle C in a strong semicomplete digraph $D=(V,A)$ which minimizes $|A(C) \cap A_2|$ when $A_2 = \{uv \in A \mid vu \in A\}$

Problem (B-J)

Can we decide if there is a hamiltonian cycle with no arc from A_2 in polynomial time?

A digraph D is path-mergeable if the following holds

$\forall P_1, P_2$ internally disjoint paths with the same initial vertex x and terminal vertex y



$\exists$ an (x,y)-path P in D s.t $V(P) = V(P_1) \cup V(P_2)$

path-mergeable digraphs can be recognized in polynomial time.

Theorem (B-J)

A path mergeable digraph D has a hamiltonian cycle if and only if $G(D)$ is 2-connected and D is strong. One can find a hamiltonian cycle in polynomial time when one exists

Problem (B-J)

Determine the complexity of the hamiltonian path problem for path-mergeable digraphs.