Minimal number of inversion to make a digraph strong

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ANR Digraph, Sète
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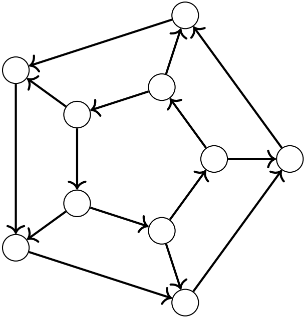
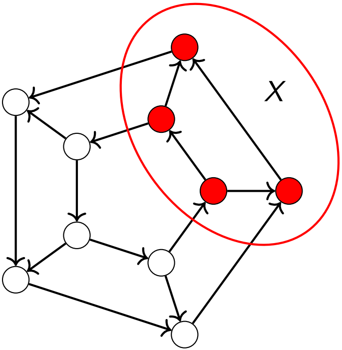
Inversion: definition



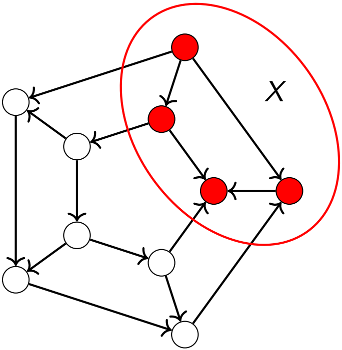
D

Inversion: definition



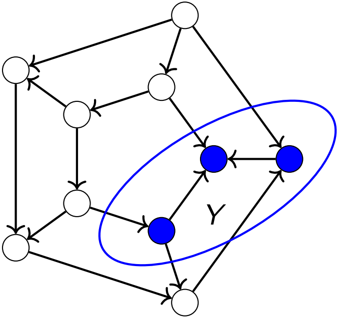
$D, X \subseteq V(D)$

Inversion: definition



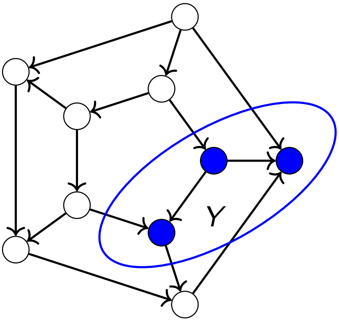
$\mathrm{Inv}(D; X)$

Inversion: definition



$\mathrm{Inv}(D; X)$, $Y \subseteq V(D)$

Inversion: definition



$$\mathrm{Inv}(D; X, Y) = \mathrm{Inv}(D; Y, X)$$

Some questions

We have a notion of distance:

$$\text{dist}(\vec{G_1}, \vec{G_2}) = \min_k \text{ s.t. } \exists X_1, \dots X_k, \vec{G_2} = \text{Inv}(\vec{G_1}, X_1, \dots, X_k).$$

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Distance to a k-strong orientation

General problem: given a digraph D, what is the minimum distance to a k-vertex-strong (resp. k-arc-strong) digraph?

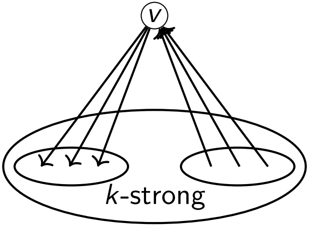
Observation: we need $n \geq 2k+1$

Notation: $\mathrm{sinv}_k(D)$ (resp. $\mathrm{sinv}'_k(D)$)

First observation

Key lemma (Folklore)

If $D - v$ is k-strong and there exists $2k$ different vertices, k of which are in $N^+(v)$ and the k others in $N^-(v)$, then D is k-strong.



Complexity

Theorem

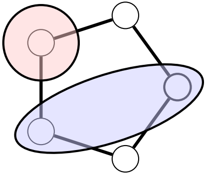
For every positive k, t the problem
Input: D a digraph.
Output: $\mathrm{sinv}_k(D) \leq t$.
Is NP-hard.

Complexity, cut cover

Idea : reduce from **CUT COVER**.

t-CUT COVER

Entry: G a graph.
Answer: $\exists X_1, \ldots, X_t$ s.t. each edge of G is contained in one of the cuts $E(X_i, X_i^c)$.



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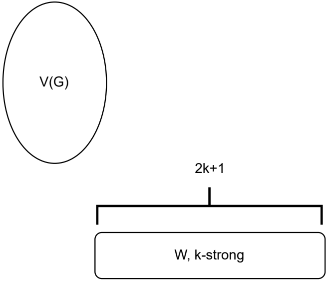
Remark

For any graph G, the cut cover number of G is $\lfloor \log \chi(G) \rfloor$.

Corollary

There is no $(2 - \varepsilon)$-approximation of sinv_k.

Sketch proof



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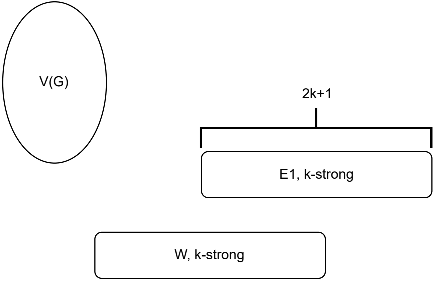

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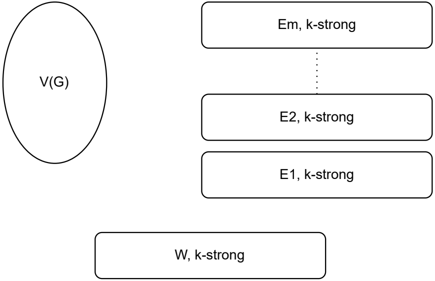

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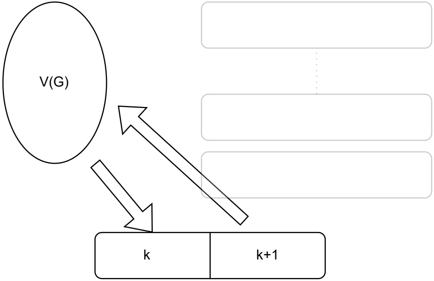

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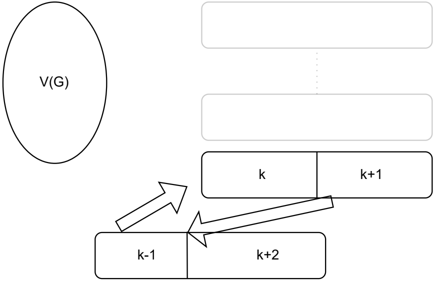

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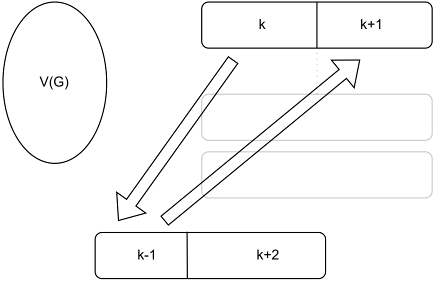

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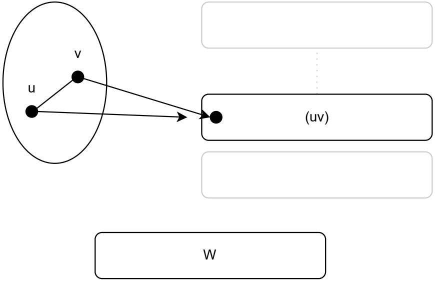

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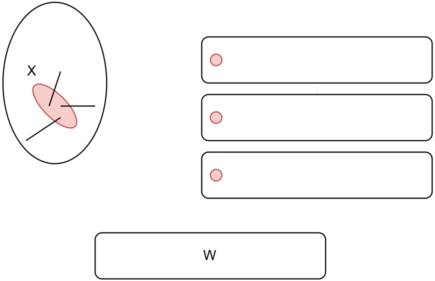


For each cut X in the cut cover, we consider the inversion

$$X' = X \bigcup_{e_i \in \mathrm{cut}(X)} E_i^1.$$

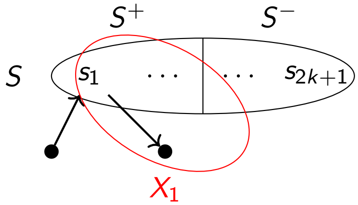
Distance to a k-strong orientation: general bound

Theorem

$$\mathrm{sinv}_k(T) \leq 2k$$

Proof sketch:

- pick $S = \{s_1, \ldots, s_{2k+1}\} \subseteq V(T)$, partition $\{s_1, \ldots, s_{2k}\}$ into S^+, S^- of size k
- iteratively make $T\langle S\rangle$ k-strong with $2k$ inversions and s.t. $S^+ \subseteq N^+(v) \cap S, S^- \subseteq N^-(v) \cap S$ for all $v \notin S$



$X_i = \{s_i\} \cup \{s_j \in S$ badly oriented with $s_i, j > i\} \cup \{v \notin S$ badly oriented with $s_i\}$

Distance to a k-strong orientation: single inversion

Theorem

If $n \geq 2^{4k-1}$ then $\mathrm{sinv}_k(T) \leq 1$.

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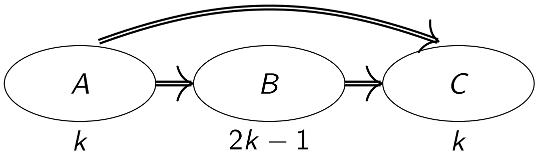

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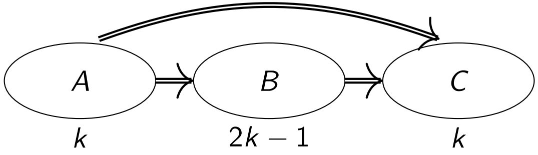

Proof of the lemma: let $S = A \cup B \cup C$.

$$X = A \cup C \cup \{v \notin S \mid |N^+(v) \cap S| < k \text{ or } |N^-(v) \cap S| < k\}$$

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Our tool: median orders

Orders that minimize the number of backward arcs.

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Our tool: median orders

Orders that minimize the number of backward arcs.

Facts on median orders.

For any tournament T, and $(v_1, \ldots, v_n)$ a median order on T, then:

- for all $i < j$, $v_i, \ldots, v_j$ is a median order of $T[v_i, v_j]$.
- for all $i < j$, v_i is adjacent to at least half of $v_i, \ldots, v_j$.

Connectivity in median orders

For a tournament T and $v \in V(T)$ let $R^+_T(v)$ the set of vertices reachable from v.

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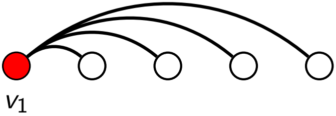

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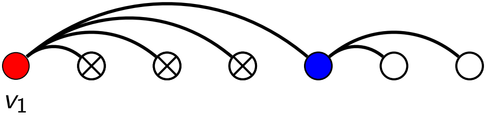

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Corollary

If $|T| \geq 4k+2$ and $|F| = k$, there is a path $v_1 \to v_n$ in $T - F$.

How to connect two subtournaments

Lemma

If A and B are two disjoint subtournaments of T of size $6k$, then in a **single** *inversion, one can ensure for every $|F| \leq k-1$:*

- *For every $a \in A \setminus F$, there is a path in $T \setminus F$ from a to $B \setminus F$.*
- *For every $b \in B \setminus F$, there is a path in $T \setminus F$ from $A \setminus F$ to b.*

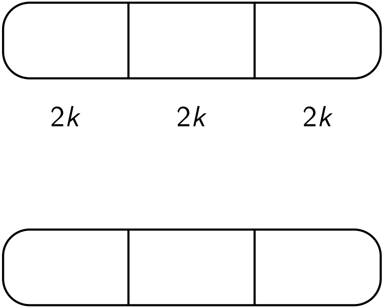
Proof sketch

$$|A| = 6k$$

median orders

$$|B| = 6k$$

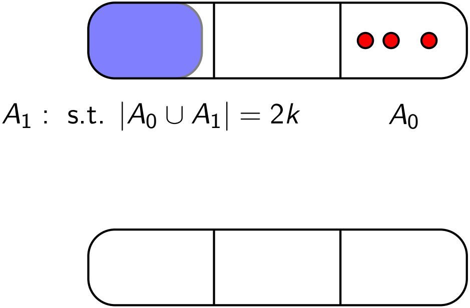
Proof sketch



Proof sketch

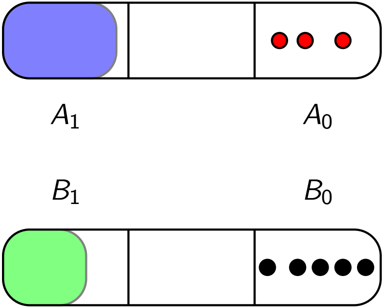
$A_0 : \{v, |N^+(v) \cap B| \le k\}$

Proof sketch



Proof sketch

$$\mathsf{Inv}(A_0 \cup A_1 \cup B_0, \cup B_1)$$



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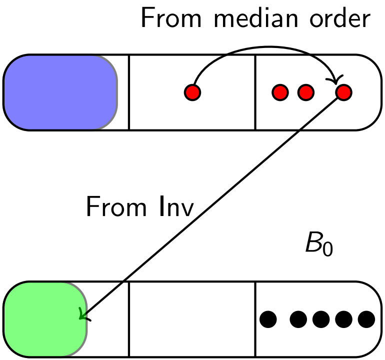
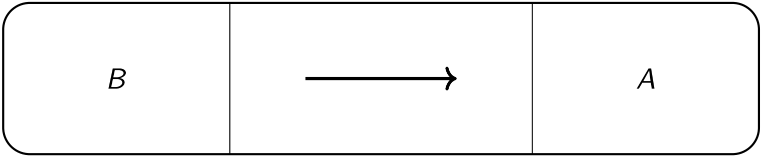


Final argument

T with median order

Final argument



Apply lemma on A and B!

Distance to a k-strong orientation: asymptotic

By drawing inversions at random:

Theorem

There is a function $f : \mathbb{R}_{>0} \to \mathbb{N}$ s.t. for every $\varepsilon > 0$ and every integer k, if T is a n-vertex tournament with $n \geq 2k + 1 + \varepsilon k$, then $\mathrm{sinv}_k(T) \leq f(\varepsilon)$.

Key lemma

If T is not k-strong then one of the following happens:

E_1 : there is a vector $z \in \mathbb{F}_2^t \setminus \{\vec{0}\}$ such that $|\{v \in V(T) \mid \vec{v} \neq z\}| \leq k$,

E_2 : there are $u, v \in V(T)$ with $\vec{u} \neq \vec{v}$ such that $\min\{|N^+_{T'}(u) \cap N^-_{T'}(v)|, |N^+_{T'}(u) \cap N^+_{T'}(v)|, |N^-_{T'}(u) \cap N^-_{T'}(v)|\} \leq (1+\varepsilon/4)\frac{k}{2}$,

E_3 : there are disjoint sets $A, B \subseteq V(T')$ with $|A|, |B| \geq (1+\varepsilon/4)\frac{k}{2}$ with no directed (A, B)-matching of size $\frac{k}{2}$.

Distance to a k-strong orientation: lower bounds

By a counting argument

Theorem

There is a tournament T on $2k+1$ vertices s.t. $\mathrm{sinv}_k(T) \geq \frac{1}{3}\log(2k+1)$.

Indeed, (McKay, 90) proved that the number of labeled Eulerian tournaments on n vertices is

$$\left(\frac{2^{n+1}}{\pi n}\right)^{(n-1)/2}\sqrt{\frac{n}{e}}(1+o(1)).$$

Open questions

Problem

What is the maximum value of $\mathrm{sinv}_k(T)$ over every tournaments on at least $2k + 1$ vertices?

Problem

Can we show that sinv_k is non-increasing?