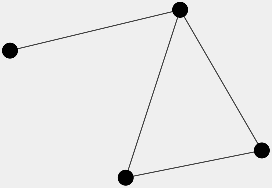
Maximum local arc-connectivity and dichromatic number

P. Aboulker[1], G. Aubian[1,2], P. Charbit[2]

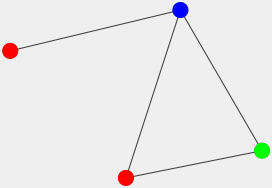
[1]TALGO, École Normale Supérieure
[2]IRIF, Université Paris-Cité

Non-oriented case

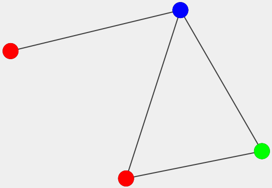
Chromatic number



Chromatic number

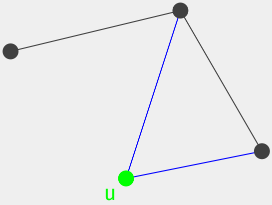


Chromatic number



$\chi(G) = 3$

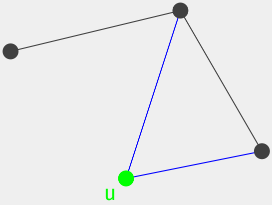
Degree



$$d(u) = |\{uv \in E(G)\}| = 2$$

$$\Delta(G) = \max_{v \in V} d(v) = 3$$

Degree



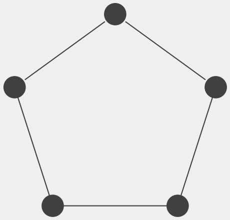
$$d(u) = |\{uv \in E(G)\}| = 2$$

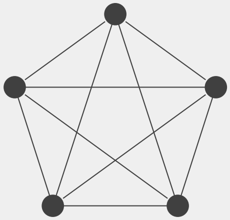
$$\Delta(G) = \max_{v \in V} d(v) = 3$$

Lemma

$$\chi(G) \leq \Delta(G) + 1$$

Brooks' theorem

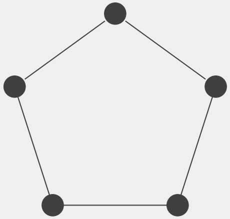


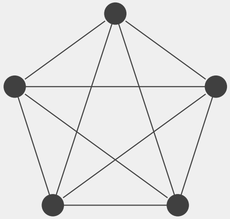


Theorem (Brooks, 1941)

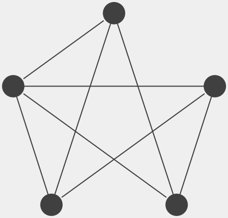
Let G be a connected graph.
$\chi(G) \leq \Delta(G) + 1$ *and equality occurs if and only if G is :*

- *a cycle on an odd number of vertices or*
- *a complete graph on $\Delta(G) + 1$ vertices.*





Maximum local edge-connectivity



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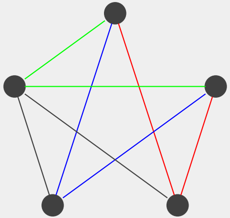
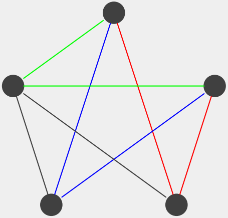


$$\lambda(u, v) = 3$$

Maximum local edge-connectivity



$$\lambda(u, v) = 3$$

$$\lambda(G) = \max_{u \neq v} \lambda(u, v) = 3$$

The goal

Lemma

$$\chi(G) \leq \lambda(G) + 1$$

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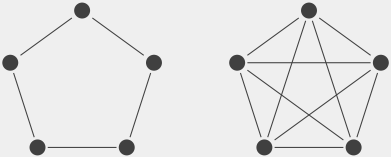
Lemma

$$\chi(G) \leq \lambda(G) + 1$$

G is k-extremal if it is biconnected and $\chi(G) = \lambda(G) + 1$

What are the k-extremal graphs ?

Some k-extremal graphs



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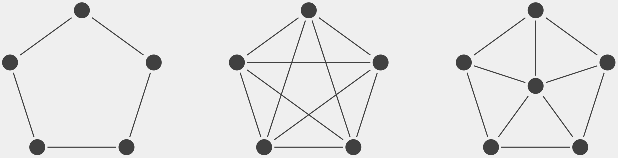
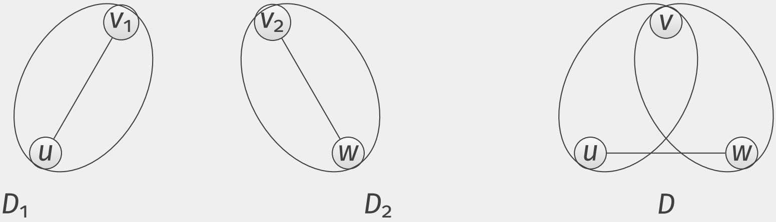
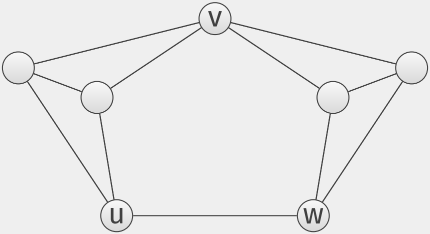


Hajós Join



HAJÓS JOIN



Lemma

Let G be the Hajós join of G_1 and G_2.
G is k-extremal if and only if so are G_1 and G_2.

Results

Theorem (Aboulker, Brettell, Havet, Marx, Trotignon, 2017)

2-extremal graphs = odd cycles.

Theorem (Aboulker, Brettell, Havet, Marx, Trotignon, 2017)

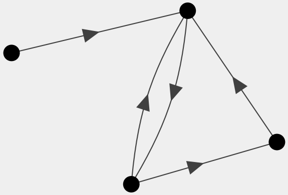
3-extremal graphs = odd wheels + Hajós joins

Theorem (Stiebitz, Toft, 2016)

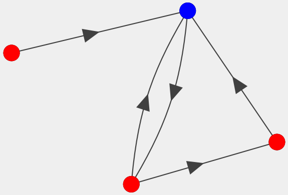
If $k \geq 4$, k-extremal graphs = K_k + Hajós joins

Directed case

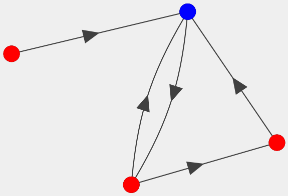
Dichromatic number



Dichromatic number

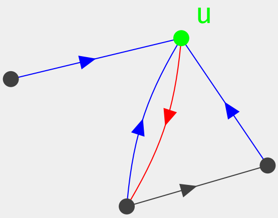


Dichromatic number



$$\vec{\chi}(D) = 2$$

Maxdegree and mindegree



$$d^{MIN}(u) = \min(d^-(u), d^+(u)) = 1$$

$$d^{MAX}(u) = \max(d^-(u), d^+(u)) = 3$$

$$\Delta_{MAX}(G) = max_{v \in V} d_{MAX}(v) = 3$$

$$\Delta_{MIN}(G) = max_{v \in V} d_{MIN}(v) = 1$$

Brooks' theorem for Δ_{MIN} ?

Lemma

$$\vec{\chi}(D) \leq \Delta_{MIN}(D) + 1$$

Lemma

$$\overrightarrow{\chi}(D) \leq \Delta_{MIN}(D) + 1$$

Theorem (Aboulker, Aubian, 2021)

Let $k \geq 2$. The problem :
Input: *a digraph D with $\Delta_{MIN}(D) = k$.*
Output: *Does there exist a k-dicoloring of D ?*
is NP-complete.

Brooks' theorem for Δ_{MIN} ?

Lemma

$$\vec{\chi}(D) \leq \Delta_{MIN}(D) + 1 \leq \Delta_{MAX}(D) + 1$$

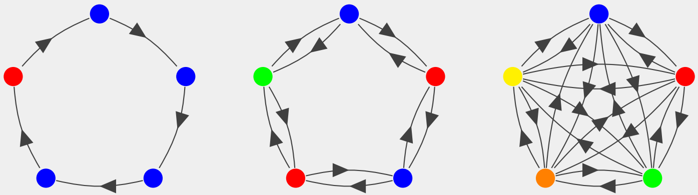
Theorem (Aboulker, Aubian, 2021)

Let $k \geq 2$. The problem :
Input: *a digraph D with $\Delta_{MIN}(D) = k$.*
Output: *Does there exist a k-dicoloring of D ?*
is NP-complete.

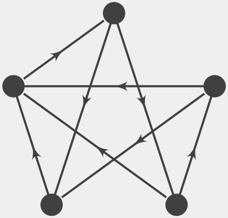
Theorem (Mohar, 2010)

Let G be a connected digraph.
$\vec{\chi}(G) \leq \Delta_{MAX}(G) + 1$ *and equality occurs if and only if G is :*

- *a directed cycle or*
- *a symmetric cycle of odd length or*
- *a complete digraph on* $\Delta_{MAX}(G) + 1$ *vertices.*



Maximum local arc-connectivity



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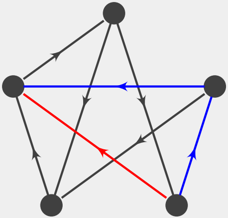

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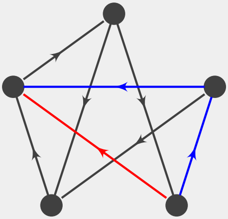

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The goal

Lemma (Neumann-Lara, 1982)

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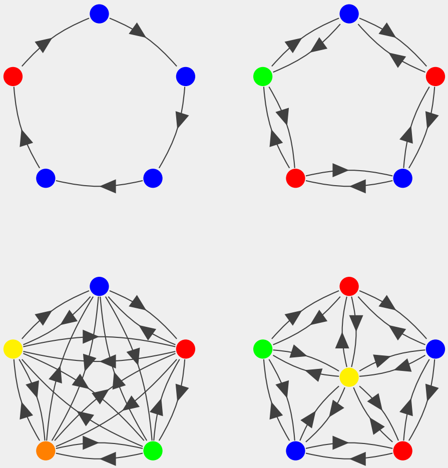
Lemma (Neumann-Lara, 1982)

$$\vec{\chi}(D) \leq \lambda(D) + 1$$

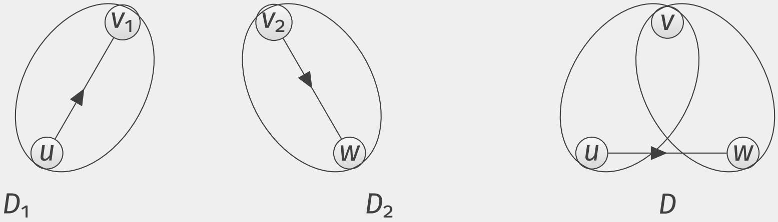
D is k-extremal if it is biconnected, strongly connected and $\vec{\chi}(D) = \lambda(D) + 1$

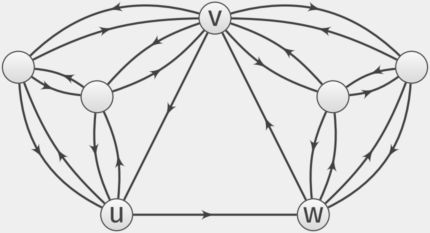
What are the k-extremal digraphs ?

Some extremal digraphs

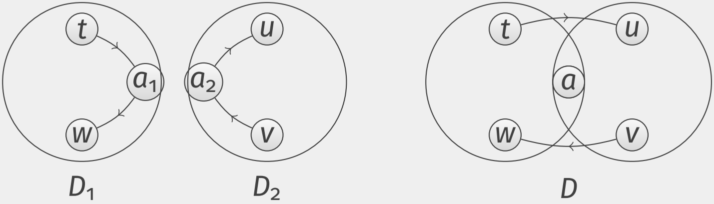


Directed Hajós Join




v
u
w

Hajós Bijoin



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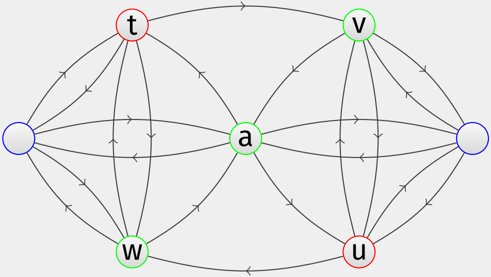

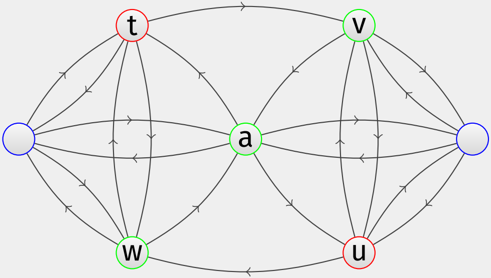

Lemma

Let D be the Hajós Bijoin of D_1 and D_2. If D is k-extremal, so are D_1 and D_2.

Decomposition theorem

Theorem

Let $k \geq 3$. If D is k-extremal, then either :

- *D is a symmetric odd wheel (if $k = 3$)*
- $D = \overleftrightarrow{K}_k$
- *D is a Directed Hajós Join*
- *D is a Hajós Bijoin*

SKETCH OF PROOF FOR $k \geq 4$

Lemma

Let D a k-extremal digraph, and $(X, \overline{X})$ a dicut of D of size k. Then either D/X or $D/\overline{X}$ is k-extremal.

- Consider a minimum counterexample *D*

Sketch of proof for $k \geq 4$

Lemma

Let D a k-extremal digraph, and $(X, \overline{X})$ a dicut of D of size k. Then either D/X or $D/\overline{X}$ is k-extremal.

- Consider a minimum counterexample D
- If all dicuts of size k isolate a vertex, then D has at most one vertex of indegree and outdegree $> k$, conclude

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- If D/X has a Directed Hajós Join or a Hajós Bijoin, so does D

Sketch of proof for $k \geq 4$

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- Otherwise, suppose D/X is k-extremal
- If D/X has a Directed Hajós Join or a Hajós Bijoin, so does D
- Thus $D/X = \overleftrightarrow{K}_k$ and every $v \in \overline{X}$ has an {in,out}neighbour in X

Sketch of proof for $k \geq 4$

Lemma

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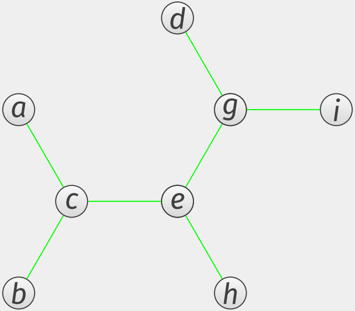
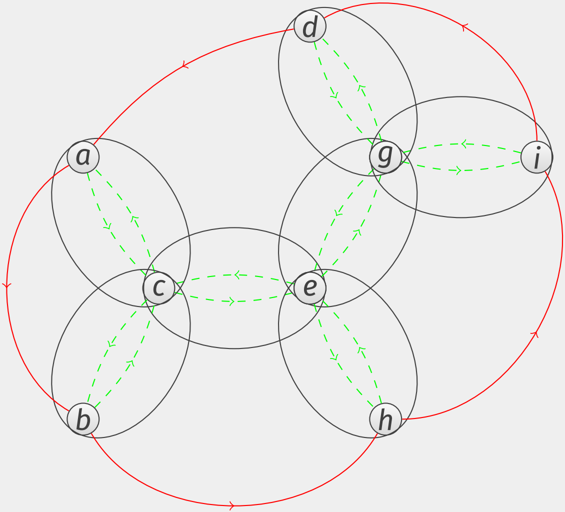
- If all dicuts of size k isolate a vertex, then D has at most one vertex of indegree and outdegree $> k$, conclude
- Otherwise, suppose D/X is k-extremal
- If D/X has a Directed Hajós Join or a Hajós Bijoin, so does D
- Thus $D/X = \overleftrightarrow{K}_k$ and every $v \in \overline{X}$ has an {in,out}neighbour in X
- Then, technical proof, sorry :'(

Hajós Tree join




d
a
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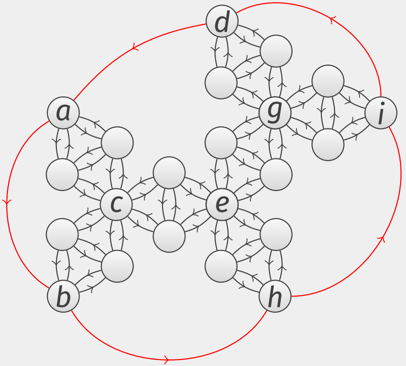
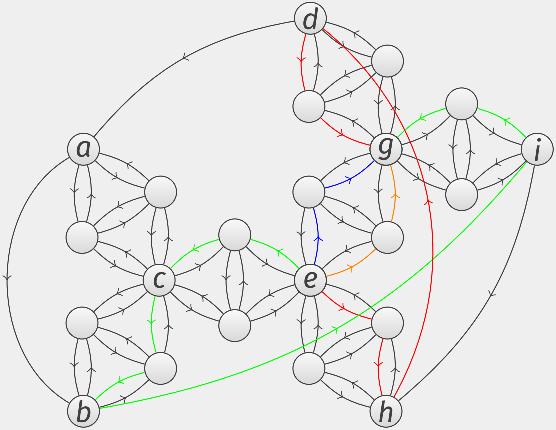
Hajós Tree join



Lemma

The Hajós Tree Join of $D_1 \dots D_n$ is k-extremal if and only if $D_1 \dots D_n$ are k-extremal.

Hajós Tree join



Theorem

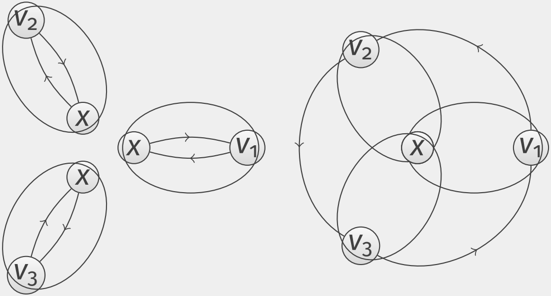
The set of 3-extremal digraphs is the smallest set containing symmetric odd wheels and stable by Directed Hajós Joins and Hajós Tree Joins.

Theorem

Let $k \geq 4$. The set of k-extremal digraphs is the smallest set containing $\overleftrightarrow{K}_k$ and stable by Directed Hajós Joins and Hajós Tree Joins.

Algorithmic results

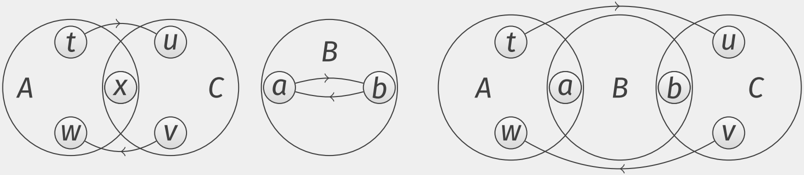
Flower Join



Lemma

A parallel Hajós join of digraphs $D_1, \ldots, D_n$ is k-extremal if and only if so is each D_i.

Parallel Join



Lemma

D is k-extremal if and only if so are both D_B and D_{AC}.

Lemma

The set of 3-extremal digraphs is the smallest set that contains symmetric odd wheels and is stable by Parallel Joins, Flower Joins and Directed Hajós Joins.

Lemma

Let $k \geq 4$. The set of k-extremal digraphs is the smallest set that contains $\overleftrightarrow{K}_k$ and is stable by Parallel Joins, Flower Joins and Directed Hajós Joins.

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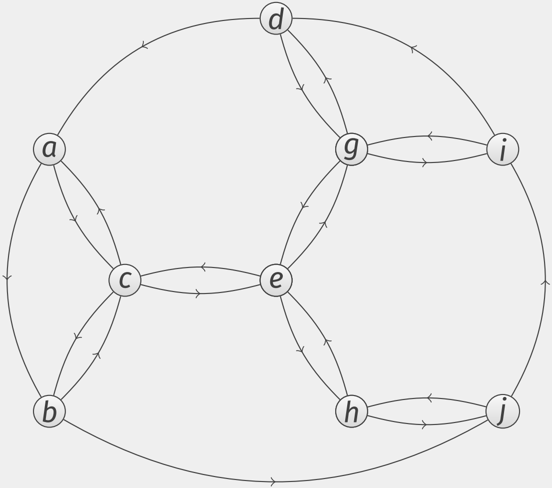
Let $k \geq 4$. The set of k-extremal digraphs is the smallest set that contains $\overleftrightarrow{K}_k$ and is stable by Parallel Joins, Flower Joins and Directed Hajós Joins.

Theorem

The problem :
Input: *a digraph D.*
Output: *Does $\overrightarrow{\chi}(D) = \lambda(D) + 1$?*
is NP-complete.

And then ?

$k = 2$



And then ?

If you are in Paris on 20/06, feel free to come to my PhD defence
More details: phd.gaubian.xyz