- DIGRAPHS Workshop : Open problem sessions -

- Problem 1 - Kolja : Orientation of the hypercube -

Does There exist an integer c such that for every $d \in \mathbb{N}$, every strong orientation of Q_d, the hypercube of dimension d, has a directed cycle of lenght $\leq c$?

Remarks :
- The value $c = 6$ could work.
- There exists strong orientation of Q_4 without directed 4-cycle.
- Come from a problem of connectivity of the sets of orientations of a graph.

- Problem 2 - Raul : Disjoint branchings with few leaves in tournaments -

There exist tournaments without two arc-disjoint hamiltonian paths, and we can caracterize them : a tournament T fails to have two arc-disjoint Hamiltonian paths if and only if T contains two consecutive strong components of size one or T contains a strong component of odd size that is formed by a transitive tournament where the longest arc is reversed.
We also know that every tournament has arc-disjoint in- and out-branchings with both at most 2 leaves. However, we are concerned with this algorithmic/caracterization problem : given T a tournament, k an integer, x and y vertices of T, does T contain arc-disjoint in- and out-arborescences with roots x and y and both with at most k leaves ?

Progress report :
Joergen : Always true as soon as $k \geq 2$: If it exists arc-disjoint in- and out-branching S and T, remove S, we obtain $\alpha = 2$, find an out-branching T' with at most 2 leaves, do the same with T'.

- Problem 3 - Clément : Diameter of the inversion graph -

For a digraph D and a subset X of vertices of D, the *inversion* of X, is the digraph obtained from D by inversing all the the arcs of D with both ends in X.
From a graph G, link two orientations of G if you can pass from one to the other by the inversion of a set. Build like that the *inversion graph* $\mathscr{L}(G)$ of G on all the orientations of G.
What is the diameter of $\mathscr{L}(G)$? Do we have $diam(\mathscr{L}(G)) \leq \Delta(G)$?

Remarks :
- We have $diam(\mathscr{L}(G)) \leq 2\Delta(G) - 1$.

- Problem 4 - Colin : Logic and tournaments -

There exists c, and ϕ logical formula, such that for all tournament T there exists a total order on the vertices, a coloring λ such that $x < y$ iff T, λ satifies $\phi(x, y)$.

Does it work for a class of tournaments with bounded clique-width and CMSO logic ? In general, with FOL + modulo counting ?

Remarks :
- First Order Logic does not work because of a lexicographical product of $\overrightarrow{C_3}$.
- First Order Logic works with random tournaments.

- Problem 5 - Stéphan : realizing every pair as forward paths -

We have shown that there exists c such that for all strong digraphs D, there exists an enumeration $v_1, \dots, v_n$ such that the number of forward path is $\geq c.n^2$.
Question : What is the minimal k such that for every digraph D there exists enumerations $L_1, \dots, L_k$ with for every $v_i \neq v_j \in V(D)$ either there is a forward path from v_i to v_j in some L_l or there is a forward path from v_j to v_i in some L_l.

Remarks :
- True for $k = n$, open for $k = \log n$

Progress report :
Colin : False, there is an example where you need $k = \Omega(n)$: take a cycle, double every edge and subdive them. To go from one added point to its twin forces $n/3$ orders.
If you allow cyclic orders, then $\log n$ is enough... (Ask Colin...)

- Problem 6 - Hugo : parameterized complexity of directed pathwidth -

There exists an XP algorithm to compute the directed pathwidth for the digraphs in general. It is open to know if an FPT algorithm exists or not (but probably not). It is conjectured that the problem is XNLP-complete, and in particular W[1]-hard.

Remarks :
- Some example of XNLP-complete problems : bandwidth$\leq k$ (for undirected graphs), binary CSP parameterized by cutwidth (for undirected graphs)

- Problem 7 - Colin : dichromatic number of a class of digraphs -

Build $\mathscr{C}$ a class of digraphs as follow : oriented K_2 is in $\mathscr{C}$, adding a twin to a vertex of a graph of $\mathscr{C}$ results in a graph of $\mathscr{C}$ and the graph obtained from two members of $\mathscr{C}$ glued on at most 2 non-adjacent vertices is a member of $\mathscr{C}$. Does the defined class have bounded dichromatic number ?

Remarks :
- The digraphs of the class are triangle-free (in the undirected sense)
- In the case of undirected, the (usual) chromatic umber is unbounded (see Nicolas's question...)

- Problem 8 - Florian : partitionning into two linear forests -

A linear forest is the union of directed paths.
Algorithmic question : can we partition the arcs of a diraph D into two linear forests ? Same question with bounded length for the components of one of the linear forest ?

Remarks :
- Its NPC for undirected graph, but could be tractable for digraphs
- What about if D is the orientation of a cubic graph

Progress report :
Florian : Polynomial for $(1,2)$ (ie : a matching and a set of vertex disjoint paths of length ≤ 2). Still open for $(1,3),(2,2)$...

- Problem 9 - Lucas : reducing the mad by removing an acyclic set of arcs -

It is well known that for every digraph $D=(V,A)$ we can find $A' \subseteq A$ such that A' is acyclic and in $A \setminus A'$ the total degree is at most $1/2$ of the initial one (take for A' a maximal acyclic set of arcs). Can we ask the same for the mad of the digraph ?

Remarks :
- Even reducing the mad by 1% could be nice.

- Problem 10 - Amadeus : In-dominating set with small chromatic number -

An *in-dominating* set X of a digraph D is a subset of vertices of D such that every vertex not in X has at least one out-neigbour in X.
Does every digraph D with maximal out-degree k admit an in-dominating set X with $\chi(D[X]) \leq \log k$ (where χ is the usual chromatic number of the underlying graph) ?

Remarks :
- It is tight for tournaments.
- k instead of $\log k$ is 'easy', we konw that it is true for $k-1$ but we do not have better bound.

- Problem 11 - Pierre A. : hypergraph of the induced directed cycles -

From a digraph $D=(V,A)$, build the hypergraph $\mathscr{H}_D$ on V whose hyperedges are the induced directed cycles of D. We have $\overrightarrow{\chi}(D) = \chi(\mathscr{H}_D)$.
What are the digraphs D such that $\mathscr{H}_D$ is critical (in sens of the hyperedges) ?
Given a (3-uniform) hypergraph $\mathscr{H}$, decide if there exist D such that $\mathscr{H}_D = \mathscr{H}$.

Remarks :
- There is example where D is $\overrightarrow{\chi}$-critical, but not $\mathscr{H}_D$

Progress report :
Anders : NP-hard, even for 3-uniform hypergraphs. Reduction from NAE-3-SAT (ask Anders for the gadgets...)

- Problem 12 - Nicolas : chromatic number of a class of graphs -

Build $\mathscr{C}$ a class of (undirected) graph as follow : K_1 and K_2 are in $\mathscr{C}$, and a graph obtained from two members of $\mathscr{C}$ glued on at most 2 non-adjacent vertices is a member of $\mathscr{C}$. Can you build a graph of $\mathscr{C}$ with chromatic number 4 ?

Remarks :
- 3 is not hard, and you can show that the graphs of $\mathscr{C}$ are 4-degenerate, so you cannot build graph with chromatic number 5.

Addendum :
Is it true that every graph that contains no triangle and no 3-connected subgraph is 3-colorable ?

Note that C_5 is a such a graph with chromatic number 3, so 3 is best possible. Also it is easy to prove that every such graph is 4-colorable, by showing that every such graph has a vertex of degree at most 3 by counting edges. More precisely, an easy induction shows that every triangle-free graph with no 3-connected subgraph on at least 3 vertices satisfies m<=2n-4.
Note that containing no triangle is also an important hypothesis because there exists graphs with no 3-connected subgraph and chromatic number 4. Here is one : consider a cycle 1-2-3-4-5-1. Add a vertex adjacent to 1,2,3 and a vertex adjacent to 1, 5, 4. This has chromatic number 4 and no 3-connected subgraph. However, I am not aware of such a graph with chromatic number 5 (but I did not try to find one, while for the first question I did try).

In the open problem session, the problem was presented in a slightly different form.

- Problem 13 - Pierre A. : substructures in highly colourable tournament -

Conjecture from Nguyen, Seymour and Scott. There exists a function f such that for every tournament T with $\overrightarrow{\chi}(T) \geq f(k)$ contains two disjoint sets of vertices A and B such that A dominates B and $\overrightarrow{\chi}(A) \geq k$ and $\overrightarrow{\chi}(B) \geq k$.
What about if we ask for $\overrightarrow{\chi}(A) \geq k$ and $\min\{\delta^+(B), \delta^-(B)\} \geq k$ or the converse ?
