On the minimum number of arcs in 4-dicritical oriented graphs

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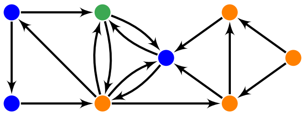
Digraph dicolouring

- *k*-**dicolouring** of D: partition of $V(D)$ in k parts inducing an acyclic subdigraph.



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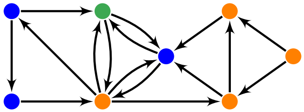

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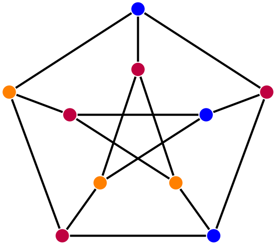

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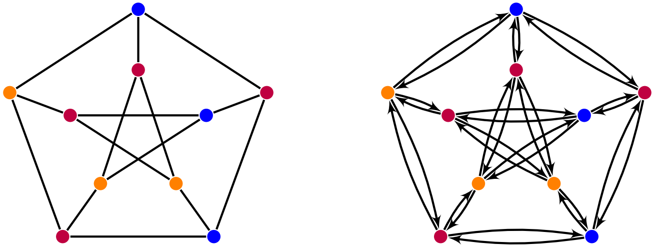

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First Easy Bound: $d_k(n) \geq (k-1)n$.

Known results

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- **Best bound:** $d_k(n) \geq (k - \frac{1}{2} + \frac{2}{k-1})n - \frac{k(k-3)}{(k-1)}$. [Aboulker and Vermande '22]

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Theorem

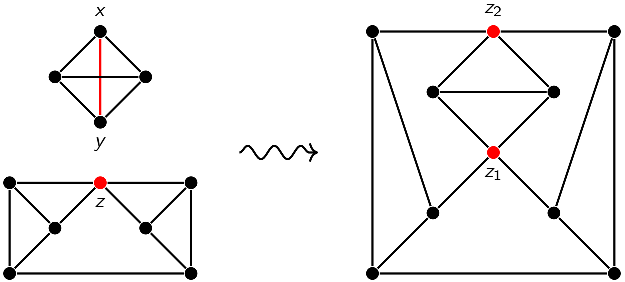
If $\vec{G}$ is a 4-dicritical oriented graph, then $m(\vec{G}) \geq \left(\frac{10}{3} + \frac{1}{51}\right) n(\vec{G}) - 1.$

Which improves $m(D) \geq \frac{10}{3}n(D) - \frac{4}{3}$ (Kostochka and Stiebitz) in general.

4-Ore digraphs

Definition

The class of 4-Ore graphs is the smallest containing K_4 closed under Ore-composition.



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4-Ore digraphs are the bidirected 4-Ore graphs.

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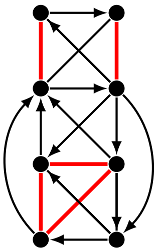
- $T(D) = \max\{d + 2t \mid \exists$ packing of d digons and t bidirected triangles$\}$



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Let $\varepsilon, \delta \geq 0$ such that $\delta \geq 6\varepsilon$ and $3\delta - \varepsilon \leq \frac{1}{3}$. If D is 4-dicritical, then

1. $\rho(D) \leq \frac{4}{3} + \varepsilon n - \delta \frac{2(n-1)}{3}$ *if D is 4-Ore, and*
2. $\rho(D) \leq 1$ *otherwise.*

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If $\vec{G}$ is 4-dicritical, then $m(\vec{G}) \geq \left(\frac{10}{3} + \frac{1}{51}\right) n(\vec{G}) - 1.$

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If D is 4-dicritical, then $m(D) \geq \frac{10}{3}n - \frac{4}{3}$ *and equality holds only if D is 4-Ore.*

Main technique of the proof

$\rho = (\frac{10}{3} + \varepsilon)n - m - \delta T$

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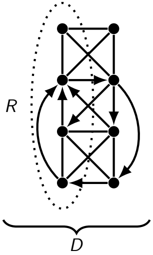

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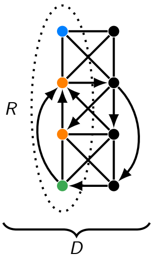

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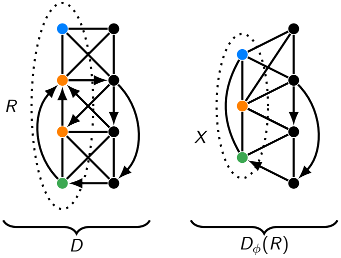

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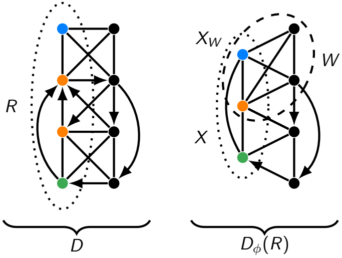

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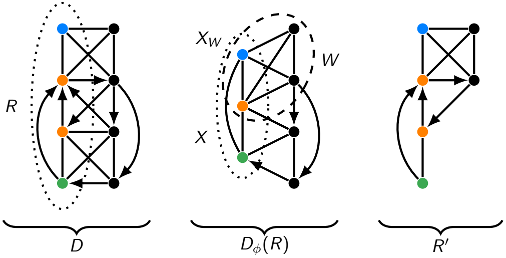

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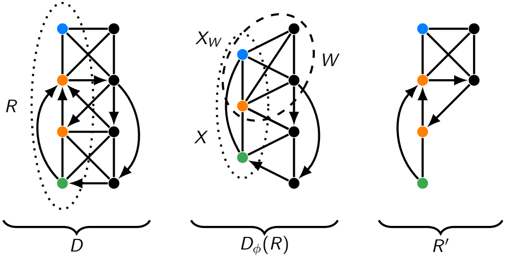

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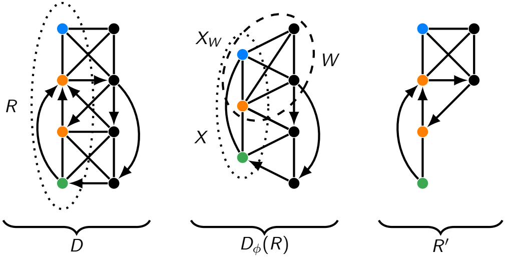

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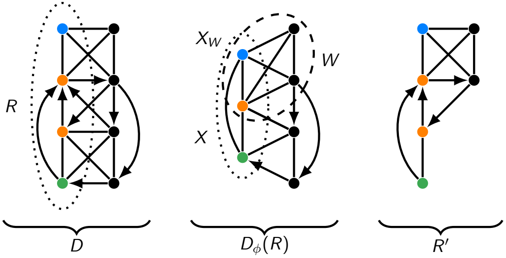

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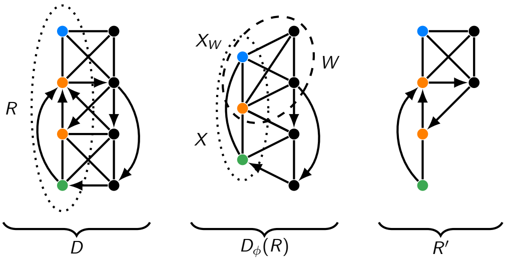

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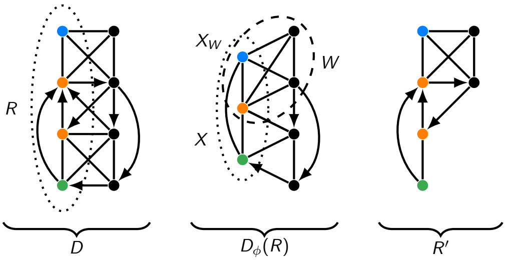

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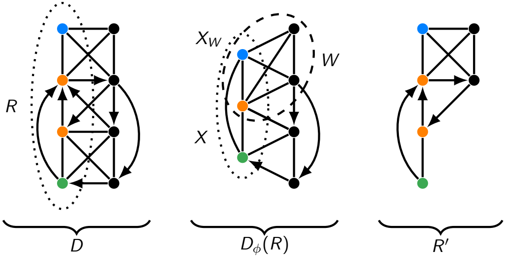

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- With more work: $\rho(R) \geq \rho(D) + \frac{8}{3} - \varepsilon - \delta$.

A useful tool

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Claim

$\forall R \subsetneq_{ind} D$, $R \cup \{uv, u'v'\}$ is 3-dicolourable.

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$\forall R \subsetneq_{ind} D$, $R \cup \{uv, u'v'\}$ is 3-dicolourable.

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$$\Rightarrow \rho(D) \leq 1$$

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$\rho = (\frac{10}{3} + \varepsilon)n - m - \delta T$

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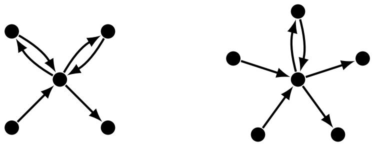

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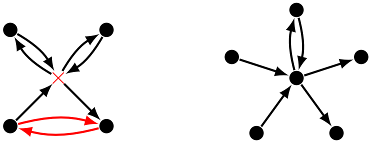

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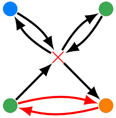

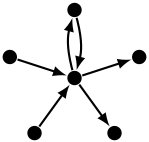

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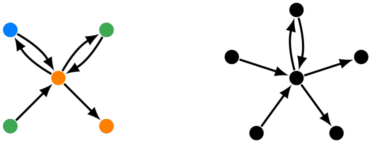

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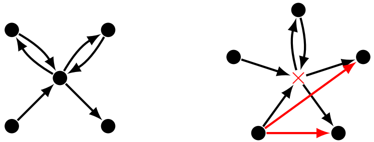

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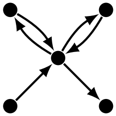

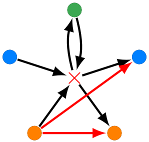

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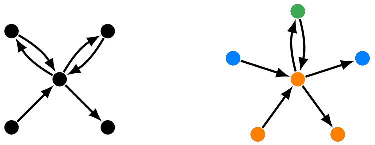

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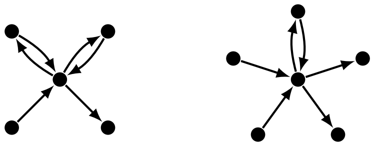

Claim

A vertex of degree 7 has 7 neighbours.

Chelou arcs

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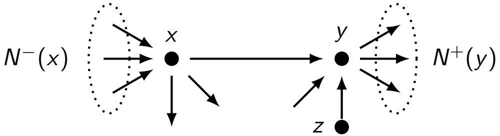
xy is chelou if $d^+(x) = 3$, $d^-(y) = 3$ and y has a simple in-neighbour $z \neq x$.



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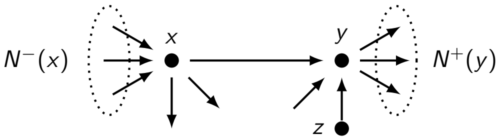

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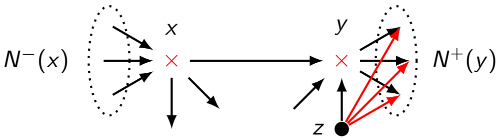

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Structure of $D[V_6]$

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- valency $val(v)$: number of arcs between v and a vertex of degree ≥ 8.
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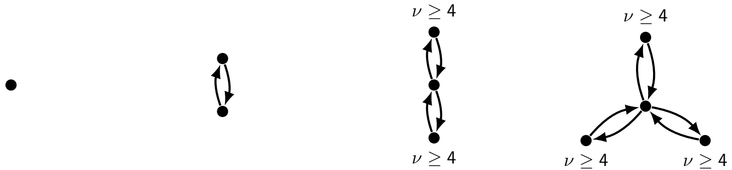
- V_6: vertices of degree 6.

Claim

Every connected component of $D[V_6]$ is one of the following.



Discharging

$\rho = (\frac{10}{3} + \varepsilon)n - m - \delta T$

- Initial charge:

$$w(v) = \frac{10}{3} + \varepsilon - \frac{1}{2}d(v) - \delta\sigma(v)$$

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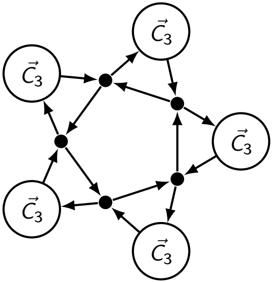
- Apply some discharging rules to obtain $w^*(v) \leq 0$, and contradict $\rho(D) > 1$.

Upper bounds

Constructions of [Aboulker, Bellitto, Havet and Rambaud '22].

$\vec{G}_3$



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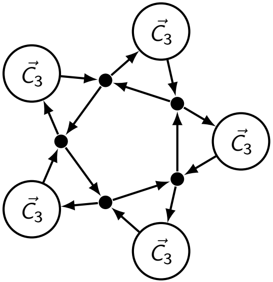

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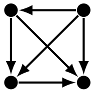

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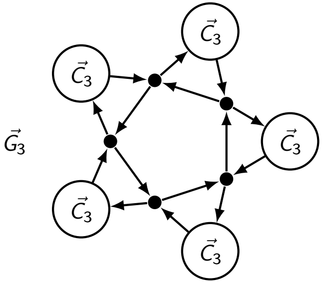

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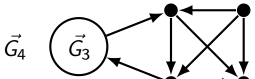

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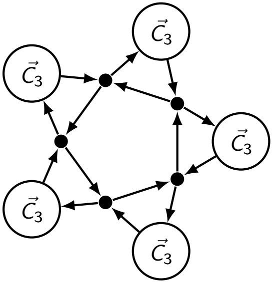

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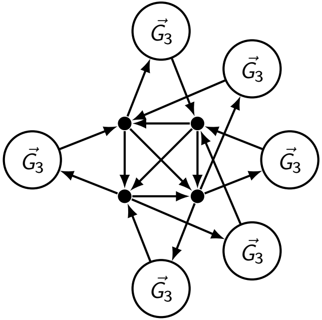

$m \leq \frac{9}{2} n$

Open questions

Conjecture (Kostochka and Stiebitz)

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Thank You !