Colouring Kneser-type digraphs

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The dichromatic number

Let $D = (V, A)$ be a simple directed graph.

Definition. A *proper k-colouring* of D is a partition of V into k parts $V_1, ..., V_k$ such that $D[V_1], ..., D[V_k]$ are acyclic. The *dichromatic number* of D, denoted by $\vec{\chi}(D)$, is the minimum k such that D has a proper k-colouring.

The dichromatic number

Let $G = (V, E)$ be a simple undirected graph.

Definition. The *dichromatic number* of G is defined as

$$\vec{\chi}(G) = \max_{D \in \mathrm{or}(G)} \vec{\chi}(D).$$

The Erdős–Neumann Lara conjecture

Conjecture (1979)

For every integer k there exists an integer $r(k)$ such that, for any undirected graph G satisfying $\chi(G) \geq r(k)$, we have that $\vec{\chi}(G) \geq k$.

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$r(1) = 1$, $r(2) = 3$, $r(3)$?

The Erdős–Neumann Lara conjecture

Theorem (Mohar and Wu, 2016)

Let G be an undirected graph satisfying $\chi_f(G) \geq k$. Then

$$\vec{\chi}_f(G) \geq \frac{k}{4\log_2(2ek^2)}.$$

Kneser graphs

Definition. Let n, k be positive integers. The *Kneser graph* $KG(n, k)$ is the graph with vertex set $\binom{[n]}{k}$ and where two vertices u, v are adjacent iff $u \cap v = \emptyset$.

Kneser graphs

Theorem (Lovász, 1978)

$\chi(KG(n,k)) = n - 2k + 2.$

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Theorem (AH & GPS, 2023)

$\vec{\chi}(KG(n,k)) \geq \lfloor \frac{n-2k+2}{16} \rfloor.$

Some recurrent tricks: dense subgraphs

Lemma

Let G be a graph of order $n \geq 2$ and D the random orientation of G obtained by orienting each edge independently with probability $1/2$. If $\ell \geq 5 \log_2 n$, then a.a.s. every subgraph of G isomorphic to $K_{\ell,\ell}$ has a directed cycle in D.

Some recurrent tricks: tensor products

Definition. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. Their *tensor product* $G \times G'$ is the graph with vertex set $V \times V'$ and where two vertices (u, u') and (v, v') are adjacent iff both u, v and u', v' are adjacent.

Some recurrent tricks: tensor products

Lemma [MW16]

Let G be an undirected graph and k, m positive integers with $m/k \geq 2\log_2 m + 2$. There is an orientation D of $G \times K_m^*$ such that, if D is k-colourable, then G is k-colourable. (K_m^* is the complete graph on m vertices with loops.)

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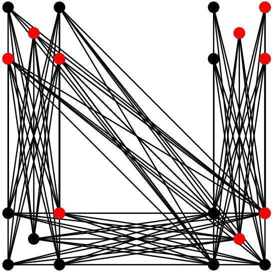
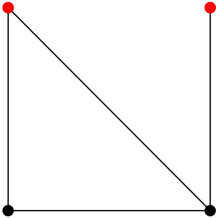


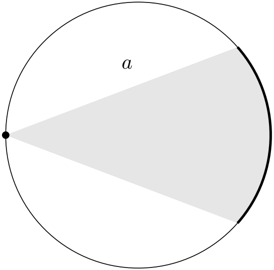
$G \times K_5^*$



G

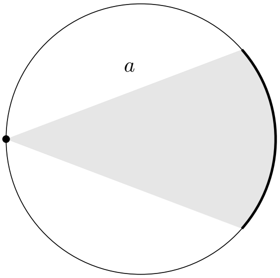
Borsuk graphs

Definition. Let n be a non-negative integer and $a \in (0, 2)$ a real number. The *Borsuk graph* $BG(n+1, a)$ is the graph with vertex set $\{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$ and where two vertices x, y are adjacent iff $\|y - x\| \geq a$.



$BG(2, a)$

Borsuk graphs



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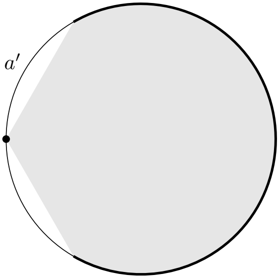

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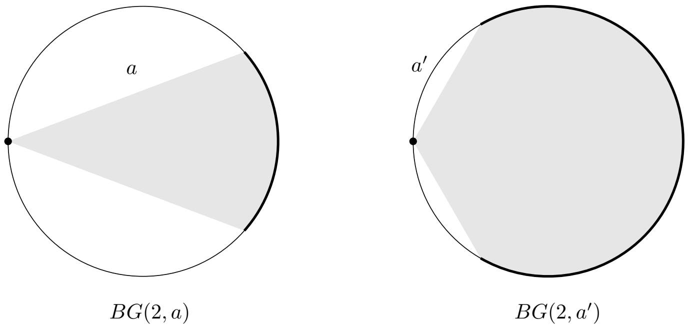

Theorem

$\chi(BG(n+1, a)) \geq n+2.$

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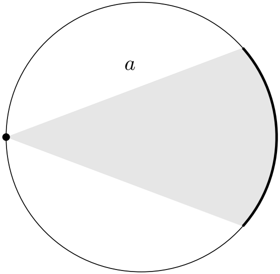

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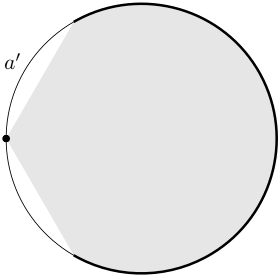

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Theorem (AH & GPS, 2023)

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The list (di)chromatic number

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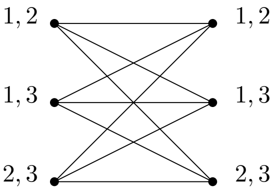
Let G be a (di)graph.

Definition. G is *k-list colourable* if for every assignment of k-lists to its vertices there is a proper colouring of G assigning to each vertex a colour from its list. The *list (di)chromatic number* of G, denoted by $\chi_\ell(G)$ (resp. by $\vec{\chi}_\ell(G)$), is the minimum k such that G is k-list colourable.



The list dichromatic number

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Definition. The *list dichromatic number* of G is defined as

$$\vec{\chi}_\ell(G) = \max_{D \in \mathrm{or}(G)} \vec{\chi}_\ell(D).$$

Kneser graphs (II)

Theorem (Bulankina and Kupavskii, 2022)

For any $\varepsilon \in \mathbb{R}^+$ and $2 \leq k < n^{1/2-\varepsilon}$ there exist positive constants c_1, c_2 such that $c_1 n \ln n \leq \chi_\ell(KG(n,k)) \leq c_2 n \ln n$.

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Q: What about larger k?

Some recurrent tricks (II): covering partitions

Let $G = (V, E)$ be an undirected graph of order n.

Definition. An (s, t)-*collection* of V is a collection $\mathcal{C}$ of at most s subsets of V each of which has size at most t. We say that $\mathcal{C}$ *covers* a class of subsets of V if for every subset P in that class there is some $C \in \mathcal{C}$ with $P \subseteq C$.

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Lemma [BK22]

Assume that the independent sets of G are covered by an (s,t)-collection. Let L be the random ℓ-list assignment for G where each list is chosen uniformly and independently from a palette of u colours. If $4tu \leq (u-\ell)n$, then the probability that G can be properly coloured assigning to each vertex v a colour from $L(v)$ is at most

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Some recurrent tricks (II): semicovering tensor products

Definition. Let D be an orientation of $K_2 \times G$, $\mathcal{C}$ an (s,t)-*collection* of $V(G)$, and $\lambda \in \mathbb{R}^+$. We say that the pair $(\mathcal{C}, \lambda)$ *semicovers* the acyclic sets of D if for every acyclic set $S = (\{1\} \times S_1) \cup (\{2\} \times S_2)$

either $S_1 \subseteq C_1$ and $S_2 \subseteq C_2$ for some $C_1, C_2 \in \mathcal{C}$,

or $S_i \subseteq C$ and $|S_i| < \lambda$ for some $i \in \{1,2\}$ and some $C \in \mathcal{C}$.

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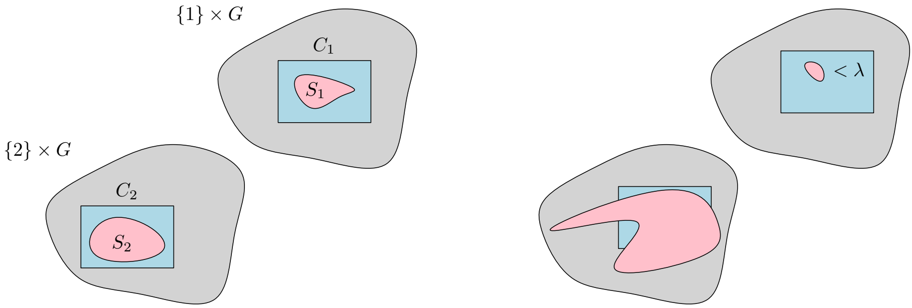

Some recurrent tricks (II): semicovering tensor products

Lemma

Let G, H be graphs. Let m_G be the size of G and n_H the order of H. Let D be an orientation of $K_2 \times H$, and $(\mathcal{C}, \lambda)$ an (s, t)-semicover of all acyclic sets of D. Let ℓ_1, ℓ_2 be positive integers such that

1. $8t\ell_1 \leq (\ell_1 - \ell_2)n_H$,
2. $m_G s^{4\ell_1} \exp\left\{-n_H 2^{-\frac{8\ell_1\ell_2 t}{(\ell_1-\ell_2)n_H}}\right\} < 1$,
3. $\lambda\ell_1 \leq n_H$.

If $\vec{\chi}_\ell(G \times H) \leq \ell_2$, then $\chi_\ell(G) \leq \ell_1$.

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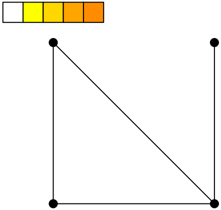

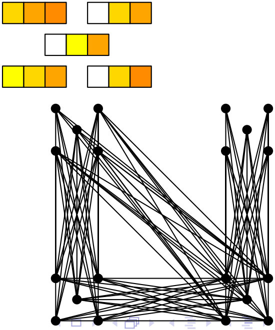

Idea of the proof

Theorem

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Idea of the proof

1) $KG(n/2, k-2) \times KG(n/2, 2)$ is a subgraph of $KG(n, k)$.

By looking at the vertices of the form $A \cup B$ with $A \in \binom{[n/2]}{k-2}$ and $B \in \binom{[n/2]+n/2}{2}$.

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1) $KG(n/2, k-2) \times KG(n/2, 2)$ is a subgraph of $KG(n, k)$.

2) $K_{n/4} \times K_{n/4}$ is a subgraph of $KG(n/2, 2)$.

By looking at the vertices of the form $\{i, j\}$ with $1 \leq i \leq n/4 < j \leq n/2$.

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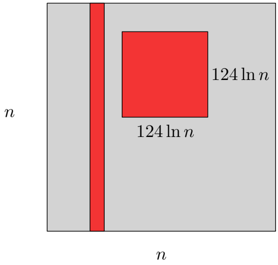
3) There is an orientation D of $K_2 \times (K_n \times K_n)$ such that all acyclic sets of D are semicovered by $(\mathcal{C}_n, 2^{13} \ln^2 n)$, where $\mathcal{C}_n$ are the sets of the form



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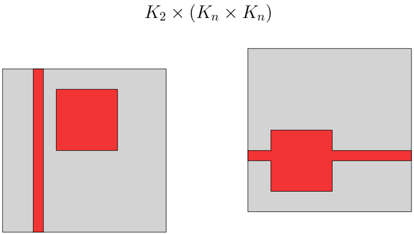

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4) The lemma about semicovering tensor products is applied to $KG(n/2, k-2) \times (K_{n/4} \times K_{n/4})$.

Complete multipartite graphs

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Theorem (Alon, 1992)

There exist $c_1, c_2 \in \mathbb{R}^+$ such that $c_1 r \ln m \leq \chi_\ell(K_{m*r}) \leq c_2 r \ln m$ for every $m, r \geq 2$.

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Q: What about smaller m? We know that these asymptotics are not true in general: if $m \leq \ln r$ then $\vec{\chi}_\ell(K_{m*r}) \leq \vec{\chi}_\ell(K_{mr}) = O(r)$.

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- Random graphs on n vertices
- Large spectral gap: $\lambda \leq d/k - 2(\log_2 k + 4)^2 \implies \vec{\chi}(G) > k$ (G d-reg.)