Directed max-cut and some generalizations

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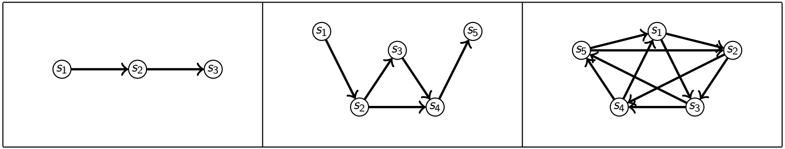

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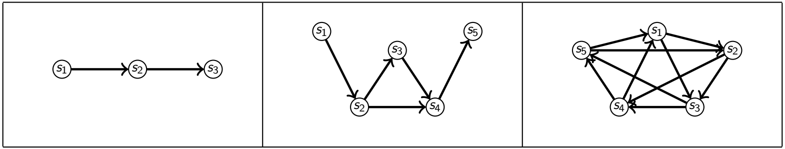

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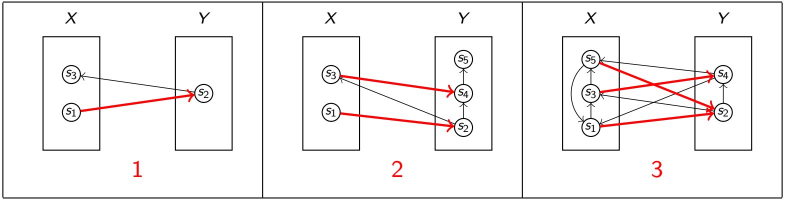

Regular digraphs

Let $mac(D)$ denote the maximum number of arcs in a (X, Y)-cut in a digraph D and let $a_D(X, Y)$ denote the number of (X, Y)-arcs in D.

Analogously, let $mac(G)$ denote the maximum number of edges in a (X, Y)-cut in a (undirected) graph G.

Question: If D is a eulerian digraph, what is $mac(D)$?

Answer: $mac(D) = \frac{mac(UG(D))}{2}$.

Let $G = UG(D)$ (the underlying graph of D) and let (X, Y) be any cut in G.

As $d^+(x) = d^-(x)$ for all $x \in V(D)$ we note that $a_D(X, Y) = a_D(Y, X)$ (any eulerian tour enters and leaves X equally many times in D), so there are exactly half as many (X, Y)-arcs in D and there are edges in G.

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Regular tournament

A tournament is an orientation of a complete graph.

Theorem 1: If T is a regular tournament of order n then $mac(T) = \frac{1}{2} \cdot \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor = \lfloor \frac{n^2}{8} \rfloor$.

Proof: As T is eulerian we note that $mac(T) = \frac{mac(K_n)}{2} = \frac{1}{2} \cdot \lceil \frac{n}{2} \rceil \cdot \lfloor \frac{n}{2} \rfloor = \frac{1}{2} \cdot \lfloor \frac{n^2}{4} \rfloor = \lfloor \frac{n^2}{8} \rfloor$. QED

For a regular tournament T of order n and size m we have $m = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2}$, so $mac(T) = \lfloor \frac{n^2}{8} \rfloor = \lfloor \frac{m}{4} + \frac{n}{8} \rfloor$.

So, the maximum cut contains slightly more than a quater of the arcs ($mac(T) \approx \frac{m}{4} + \frac{1+\sqrt{1+8m}}{16} \approx \frac{m}{4} + \frac{\sqrt{2m}}{8}$).

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We will consider the weighted version

We give a weight for each arc and we want to find a cut (X, Y) where the sum of the weights of all (X, Y)-arcs is maximum.

Let $w^+(x)$ denote the sum of the weight on the arcs leaving x and let $w^-(x)$ denote the sum of the weight on the arcs entering x.

If $w^+(x) = w^-(x)$ for all $x \in V(D)$ then $mac(D) = \frac{mac(UG(D))}{2}$, where $mac(D)$ ($mac(G)$, resp.) now denotes the maximum weight of a cut in D (G, resp)

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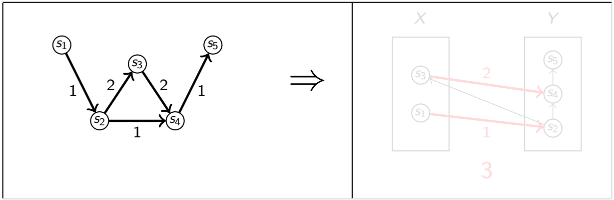

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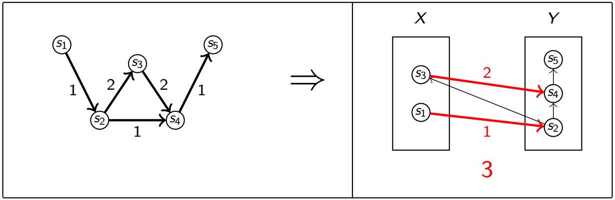

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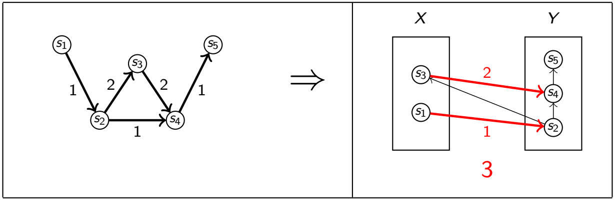

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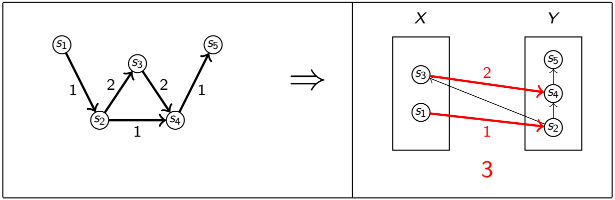

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Let D be an arc-weighted digraph and let $w(D)$ denote the sum of all weights in D.

Let $\theta(D) = \frac{\sum_{x \in V(D)} \max\{0, w^+(x) - w^-(x)\}}{w(D)}$.

If D is weighted-eulerian ($w^+(x) = w^-(x)$ for all x) then $\theta(D) = 0$.

If we multiply all arcs in D by some constant $c > 0$ then this does not change $\theta(D)$ (and does not change which cut is maximum).

Theorem 2, [1]: $mac(D) \geq I(\theta(D)) \cdot w(D)$, where

$$I(\theta) = \begin{cases} \left(\frac{1}{4} + \frac{\theta^2}{4(1-2\theta)}\right) & \text{if } \theta < 1/3; \\ \theta & \text{if } \theta \geq 1/3. \end{cases}$$

The bound is tight.

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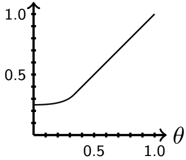

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Theorem 2, intuition of proof

If $\theta(D) \geq 1/3$ then we simply put all vertices, x, with $w^+(x) > w^-(x)$ in X and all other vertices in Y.

If $\theta(D) < 1/3$, then the proof uses a probabilistic argument

Let $\bar{p} = \frac{\theta}{2(1-2\theta)}$ and place any vertex with $w^+(x) > w^-(x)$ in X with probability $(1/2 + \bar{p})$.

Any vertex with $w^+(x) \leq w^-(x)$ we place in Y with probability $(1/2 + \bar{p})$.

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Theorem 2 is tight

To show the bound is tight we let D_k be a digraph consisting of two vertex disjoint regular tournament, A_k and B_k, of order k and arc-weights 1.

We then add all arcs from A_k to B_k with weight $Q = \frac{\theta(1-1/k)}{1-\theta}$.

$\theta(D_k) = Qk^2/(k^2 - k + Qk^2) = Q/(1 + Q - 1/k) = \theta$.

$\mathrm{mac}(D_k) = Qxy + x(k-x)/2 + y(k-y)/2$, where $x = |V(A_k) \cap X|$ and $y = |V(B_k) \cap Y|$ for optimal (X, Y).

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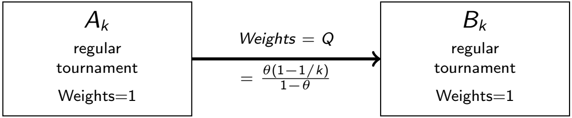

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Acyclic digraphs

For (unweighted) acyclic digraphs. Alon et. al. proved the following

Theorem 3 (Alon et al): There exists a constant k_1^s, such that for every integer $m \geq 1$ there exists an acyclic digraph D_m^s with m arcs and $mac(D_m^s) \leq \frac{m}{4} + k_1^s m^{0.8}$.

Theorem 4 (Alon et al): There exists a constant k_2^s, such that $mac(D) \geq \frac{m}{4} + k_2^s m^{0.6}$ for all acyclic digraphs D of size m.

We generalize to multi-digraphs and arc-weighted digraphs.

Theorem 5, [1]: There exists a constant k_1, such that for every integer $m \geq 1$ there exists an acyclic multi-digraph D_m with m arcs and $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.

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Theorem 6: For all acyclic arc-weighted digraphs ($w \geq 1$): $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$.

Why do we need $w \geq 1$? Otherwise Theorem 6 is not true (consider a digraph with one arc of weight q such that $q < k_2^{2.5}$ which implies that $k_2 w(D)^{0.6} = k_2 q^{0.6} > q = w(D) = mac(D)$)

We first outline the proof of Theorem 5.

Let $V(D) = \{v_1, v_2, \ldots, v_n\}$ and add an acyclic tournament on $I_i = (v_i, v_{i+1}, \ldots, v_{i+q-1})$ where all arcs go "forward" in the order of I_i and all indices are taken modulo n.

This gives us a regular multi-digraph (where n and q will be decided later).

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The result we call D_m^*, which is a regular multi-digraph.

We now delete all "backward" arcs and call the result D_m.

As $A(D_m^*)$ can be partitioned into n tournaments on q vertices we note that $mac(UG(D_m^*)) \leq n \cdot mac(K_q) = n \cdot \lfloor \frac{q^2}{4} \rfloor \leq \frac{nq^2}{4}$.

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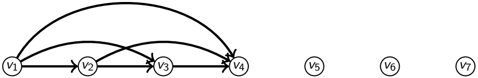

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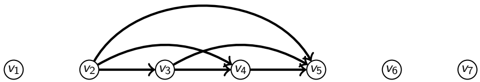

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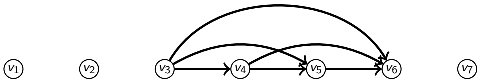

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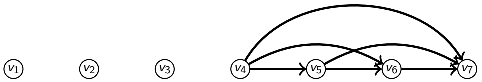

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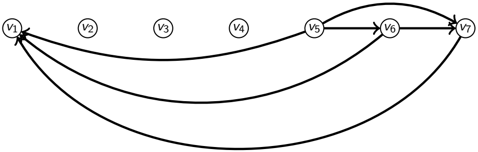

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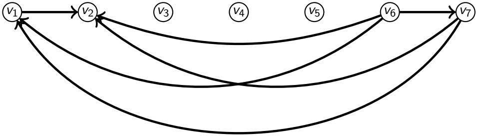

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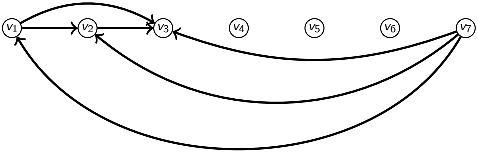

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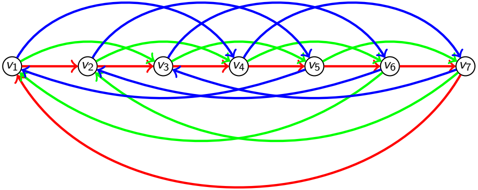

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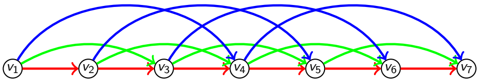

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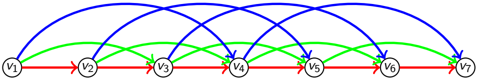

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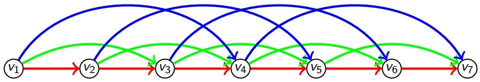

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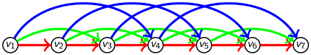
$n = 7$ and $q = 4$:



$mac(D_m) \leq \frac{nq^2}{8}$.

$$\begin{aligned} |A(D_m)| &= |A(D_m^*)| - 1\cdot(q-1) - 2\cdot(q-2) - \cdots (q-1)\cdot 1 \\ &= n\binom{q}{2} - \sum_{i=1}^{q-1} i(q-i) \\ &= \frac{nq(q-1)}{2} - q\sum_{i=1}^{q-1} i + \sum_{i=1}^{q-1} i^2 \\ &= \ldots = \frac{nq^2}{2} - \frac{nq}{2} - \frac{q^3}{6} + \frac{q}{6} \end{aligned}$$

Letting $q = \lfloor \sqrt{n} \rfloor$ and optimizing we get

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One can then extend this to all values of m....

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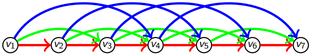

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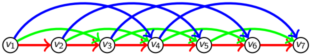

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Theorem 6

Recall Theorem 6, which we shall now give the main ideas for.

Theorem 6, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D ($w \geq 1$).

In order to prove this we need a result on arc-weighted acyclic digraphs with maximum path contiaining ν vertices.

Let c_ν be the largest number such that $mac(D) \geq c_\nu \times w(D)$ for all arc-weighted acyclic digraphs D with maximum path order at most ν.

Theorem 7, [1]: $c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

Proving Theorem 7 is the main part in proving Theorem 6.

We can show that $c_2 = 1$, $c_3 = c_4 = \frac{1}{2}$, $c_5 = c_6 = \frac{2}{5}$, $c_7 = \frac{3}{8}$, $c_8 = \frac{4}{11}$, $c_9 = \frac{13}{37}$, $c_{10} = \frac{9}{26}$ and $c_{11} = \frac{31}{92}$.

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Let D be an arc-weighted acyclic digraph with maximum path order ν.

Theorem 7: $c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

There exists independent sets $S_1, S_2, \ldots, S_\nu$ such that all arcs in D are (S_i, S_j)-arcs with $i < j$.

We contract each S_i into a vertex v_i, which gives us an acyclic digraph D' with $V(D') = \{v_1, v_2, \ldots, v_\nu\}$.

We then (for some k) partition the vertices into sets $A_{-z}, A_{-z+1}, \ldots, A_{-1}, A_0, A_1, \ldots, A_z$ ($z \approx \sqrt{k/2}$), such that $|A_i| \approx 2k - 2i^2$ for all $i \in \{-z, -z+1, \ldots, z\}$.

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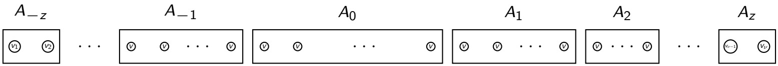

The above picture illustrates $A_{-z}, A_{-z+1}, \ldots, A_{-1}, A_0, A_1, \ldots, A_z$.

We randomly place $\frac{|A_i|-i}{2}$ vertices from A_i in X and $\frac{|A_i|+i}{2}$ vertices from A_i in Y.

Every arc in D' lies in the cut (X, Y) with probability at least $\frac{k}{4k-2}$. Why?

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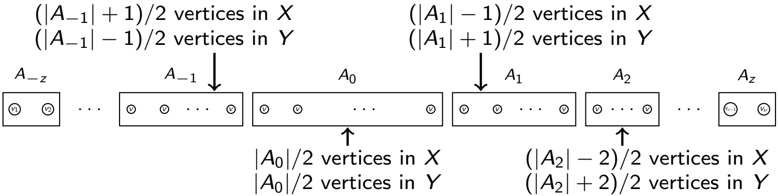

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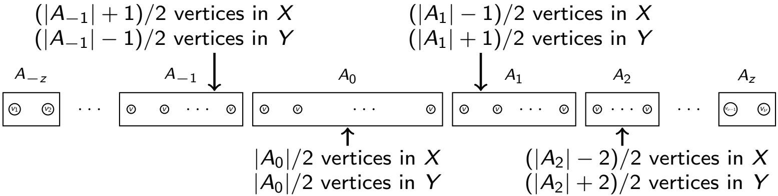

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To show: $P(a \in (X,Y)) \geq \frac{k}{4k-2}$

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$$P(a \in (X,Y)) \geq \frac{|A_i|-i}{2|A_i|} \times \frac{|A_i|+i}{2(|A_i|-1)} = \frac{|A_i|^2-i^2}{4|A_i|(|A_i|-1)}$$

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Analogously, one can show that all arcs within sets A_i satisfy the condition. One can then show that it also holds for arcs between sets.

Theorem 7

To show: $P(a \in (X,Y)) \geq \frac{k}{4k-2}$

If a is an arc within A_i, then

$$P(a \in (X,Y)) \geq \frac{|A_i|-i}{2|A_i|} \times \frac{|A_i|+i}{2(|A_i|-1)} = \frac{|A_i|^2-i^2}{4|A_i|(|A_i|-1)}$$

E.g. $|A_0| = 2k$ implies that $P(a \in (X,Y)) \geq \frac{(2k)^2}{4 \cdot 2k(2k-1)} = \frac{k}{4k-2}$.

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Analogously, one can show that all arcs within sets A_i satisfy the condition. One can then show that it also holds for arcs between sets.

Theorem 7

One can also show that $\nu \geq k^{3/2}$. So,

$$\begin{aligned} mac(D) &\geq \frac{k}{4k-2} \times w(D) \\ &= \left(\frac{1}{4} + \frac{1}{8k-4}\right) w(D) \\ &\geq \left(\frac{1}{4} + \frac{1}{8\nu^{2/3}-4}\right) w(D) \end{aligned}$$

In Theorem 7 we show $mac(D) \geq \left(\frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}\right) w(D)$, which is because we need the bound to hold for all ν, not just the ones we consider above.

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Theorem 6

Theorem 6: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D ($w \geq 1$).

Theorem 7:

$c_\nu \geq \frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times \nu^{2/3}}$.

Proof: Let D be a arc-weighted acyclic digraphs D.

Let $P = p_1 p_2 p_3 \ldots p_n$ be a longest path in D.

We consider the cases when $w(P) \leq w(D)^{0.6}$ and $w(P) \geq w(D)^{0.6}$ seperately.

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Theorem 6, Case 1 proof

Case 1: $w(P) \leq w(D)^{0.6}$.

As all weights are at least one, we have $|A(P)| \leq w(P) \leq w(D)^{0.6}$. So Theorem 7 implies,

$$\begin{aligned} mac(D) &\geq \left(\frac{1}{4} + \frac{1}{8 \times 3^{2/3} \times |A(P)|^{2/3}}\right) w(D) \\ &\geq \frac{w(D)}{4} + \frac{w(D)}{8 \times 3^{2/3} \times w(D)^{0.4}} \\ &\geq \frac{w(D)}{4} + k_2 \cdot w(D)^{0.6} \end{aligned}$$

Theorem 6, Case 2 proof

Case 2: $w(P) \geq w(D)^{0.6}$.

Let M_1 and M_2 be two matchings in $A(P)$ such that $A(M_1) \cup A(M_2) = A(P)$.

W.l.o.g assume that $w(M_1) \geq w(M_2)$, and for each arc, uv, in M_1 assign u to X and v to Y with probability $1/2$ and assign u to Y and v to X with probability $1/2$. Any vertex not in $V(M_1)$ gets assigned to X or Y with probability $1/2$.

The average weight of the cut (X, Y) is the following.

$$\frac{w(D)}{4} + \frac{w(M_1)}{4} \geq \frac{w(D)}{4} + \frac{w(P)/2}{4} \geq \frac{w(D)}{4} + \frac{w(D)^{0.6}}{8}$$

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Open problem

Theorem 5, [1]: There exists a constant k_1, such that for every integer $m \geq 1$ there exists an acyclic multi-digraph D_m with m arcs and $mac(D_m) \leq \frac{m}{4} + k_1 m^{0.75}$.

Theorem 6, [1]: There exists a constant k_2, such that $mac(D) \geq \frac{w(D)}{4} + k_2 w(D)^{0.6}$ for all arc-weighted acyclic digraphs D ($w \geq 1$).

Open Problem: Close the gap between 0.6 and 0.75 for arc-weighted acyclic digraphs D.

Small cycles

Let *circ*(D) denote the *circumference* of a digraph D (i.e. the length of a longest cycle in D). Assume all weights are at least 1.

Theorem 8, [1]: Assume that there exist constants $k > 0$ and $0 < \alpha < 1$ such that $\mathrm{mac}(H) \geq \frac{w(H)}{4} + kw(H)^\alpha$ for all acyclic digraphs H. If D is an arbitrary arc-weighted digraph then,

$$\mathrm{mac}(D) \geq \frac{w(D)}{4} + \frac{k}{(4k+1) \cdot circ(D) + 1} \times w(D)^\alpha$$

The proof of Theorem 8 uses the following theorem.

Theorem 9 (Bondy, 1976): For all strong digraphs D we have $\chi(D) \leq circ(D)$.

So, any result holding for acyclic digraphs also holds for digraphs where the circumference is bounded by a constant.

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Final open problem

For simple digraphs the following holds.

Theorem 3 (Alon et al): There exists a constant k_1^s, such that for every integer $m \geq 1$ there exists an acyclic digraph D_m^s with m arcs and $mac(D_m^s) \leq \frac{m}{4} + k_1^s m^{0.8}$.

Theorem 4 (Alon et al): There exists a constant k_2^s, such that $mac(D) \geq \frac{m}{4} + k_2^s m^{0.6}$ for all acyclic digraphs D of size m.

Open Problem: Close the gap between 0.6 and 0.8 for simple acyclic digraphs D.

End of first part of the talk

This completes the first part of the talk, which was based on the paper

[1] Jiangdong Ai, Stefanie Gerke, Gregory Gutin, Anders Yeo and Yacong Zhou. *Bounds on Maximum Weight Directed Cut.* Submitted.

The second part of the talk will be based on the paper

[2] Argyrios Deligkas, Eduard Eiben, Gregory Gutin, Philip R. Neary and Anders Yeo *Complexity of Efficient Outcomes in Binary-Action Polymatrix Games with Implications for Coordination Problems.* Accepted at IJCAI 2023.

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Generalization of max-cut in digraphs

Let D be a digraph, such that for each arc $a \in A(D)$ we are given values $xx(a)$, $xy(a)$, $yx(a)$ and $yy(a)$. We want to find a partition (X, Y) of $V(D)$ that maximizes $\sum_{a \in A(D)} val(a)$, where

$$val(uv) = \begin{cases} xx(uv) & \text{if } u, v \in X \\ xy(uv) & \text{if } u \in X \text{ and } v \in X \\ yx(uv) & \text{if } u \in Y \text{ and } v \in X \\ yy(uv) & \text{if } u, v \in Y \end{cases}$$

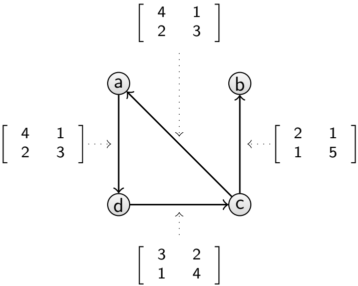
We denote the values $(xx(a), xy(a), yx(a), yy(a))$ by

$$M(a) = \begin{bmatrix} xx(uv) & xy(uv) \\ xy(uv) & yy(uv) \end{bmatrix}$$

Example

Consider the following example

$$M(a) = \begin{bmatrix} xx(uv) & xy(uv) \\ xy(uv) & yy(uv) \end{bmatrix}$$



What is an optimal partition?

The optimal partition is $(\{a, d\}, \{b, c\})$ with value 13.

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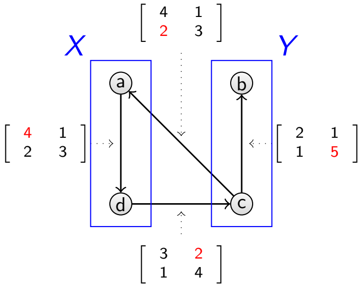

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Generalization of max-cut in dgraphs

In order to obtain a dichotomy, we will let $\mathcal{F}$ denote the list of matrices that are allowed.

We assume that if a matrix $M \in \mathcal{F}$ is allowed to be used then every multiple of M is also allowed to be used.

Example: The directed max-cut problem (we count the number of (X, Y)-arcs) can be reduced to the case when $\mathcal{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

So, if $\mathcal{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then the problem is NP-hard.

We give a dichotomy for this problem.

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Dichotomy

We are looking at the problem $MWDP(\mathcal{F})$ (Maximum Weighted Digraph Partition).

We are given a digraph, D, and functions $f : A(D) \to \mathcal{F}$ and $c : A(D) \to \mathbb{R}^+$, such that the matrix $c(a) \cdot f(a)$ is used on arc a.

Given $\mathcal{F}$ we define the following 3 properties.

(a): $m_{11} + m_{22} \geq m_{12} + m_{21}$ for all matrices $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \in \mathcal{F}$.

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We will not go through the proof of Theorem 10, but instead give some applications.

However, note that if Property (b) or Property (c) hold then the problem is trivially polynomial (by letting $X = V(D)$ or $Y = V(D)$).

If Property (a) holds then we can reduce the problem to finding a (s, t)-minimum cut in an auxilary digraph.

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Application 1, arc-weighted directed max-cut

Let (D, f, c) be an instance of $MWDP(\mathcal{F})$, where $\mathcal{F} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

By our dichotomy, $MWDP(\mathcal{F})$ is NP-hard.

Let the weight of any arc in D be $c(a)$.

Now the solution to $MWDP(\mathcal{F})$ is exactly a directed max-cut in D.

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Application 2, Poly-matrix games, from economics

We are given a number of players, which we think of as vertices in a graph, G. Each player has to chose Strategy 1 or Stratergy 2.

An edge $uv \in A(D)$ indicates that there is a pay-off depending on the stratergies players u and v have chosen.

Let $M_u(uv) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ be the matrix associated with edge uv, such that u gets pay-off m_{ij} if and only if player u choses Stratergy i and player v choses Stratergy j.

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Let $M_u(uv) = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$ be the matrix associated with edge uv, such that u gets pay-off m_{ij} if and only if player u choses Stratergy i and player v choses Stratergy j.

Analogously, we define $M_v(uv) = \begin{bmatrix} m'_{11} & m'_{12} \\ m'_{21} & m'_{22} \end{bmatrix}$ to indicate player v's pay-off.

We want to know which stratergies should be played to maximize the overall pay-out.

Application 2, Poly-matrix games, from economics

We now let D be any orientation of G.

For every edge $uv \in E(G)$ we can compute the pay-out for u and the pay-out for v, given all 4 permutations of stratergies.

We can then build a matrix M that gives us the over-all payout as seen in the following example.

$M_u(uv) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $M_v(uv) = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$ implies that $M = \begin{bmatrix} 1+5 & 2+7 \\ 3+6 & 4+8 \end{bmatrix}$, if $uv \in A(D)$.

Letting $\mathcal{F}$ contain all the obtained matrices we have transformed the problem into $MWDP(\mathcal{F})$.

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Our results also indicate why "coordination-games" are easy and "anti-coordination-games" are difficult (in general).

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Application 3, Directed Min (s, t)-cut

Given a digraph, D, with $s, t \in V(D)$, find a (s, t)-partition (X_1, X_2) with the fewest number of arcs from X_1 to X_2.

This is equivalent to finding the largest number of arc-disjoint paths from s to t (by Menger's Theorem).

Let $M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $S = \begin{bmatrix} |A(D)| & 0 \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} 0 & 0 \\ 0 & |A(D)| \end{bmatrix}$.

All arcs of D get associated with matrix M. We then add a new vertex s' and the arc $s's$ which we associate with matrix S. We also add a new vertex t' and the arc tt' which we associate with matrix T.

Now the maximum value we can obtain is $3|A(D)|$ minus the size of a minimum (s, t)-cut. So by our dichotomy result this is polynomial.

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Given a graph, G, and an integer k, find a vertex set $X \subseteq V(G)$ such that the induced subgraph $G[X]$ has average degree strictly greater than k.

Let $M_1 = \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ and $\mathcal{F} = \{M_1, M_2\}$.

Let D be any orientation of G after adding a pendent edge to each vertex ($|V(D)| = 2|V(G)|$).

Associate M_1 to each pendent arc and M_2 to all other arcs of D.

This gives us an instance of $MWDP(\mathcal{F})$ and let (X, Y) be an optimal solution. The value of this is the following ($x = |X \cap V(G)|$ and $y = |Y \cap V(G)|$).

$s = k \cdot x + 2e(Y, Y) = k|V(D)| - k \cdot y + 2e(Y, Y)$.

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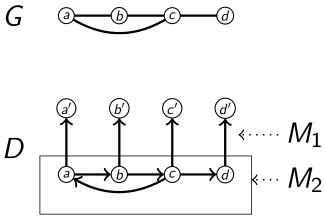

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So, $s > k|V(D)|$ if and only if $2e(Y,Y) > k|Y|$.

$s = k|V(D)| - k|Y| + 2e(Y,Y).$

This is equivalent with $k < \frac{2e(Y,Y)}{|Y|} = \frac{\sum_{y \in Y} d_Y(y)}{|Y|} = \text{Avg-deg}(Y)$.

So, there exists a subgraph with average degree greater than k if and only if the solution to $MWDP(\mathcal{F})$ is greater than $k|V(D)|$.

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Application 5, Max Density

Given a graph, G, find a vertex set $X \subseteq V(G)$ such that the number of edges divided by the number of vertices in the induced subgraph $G[X]$ is maximum possible.

This is polynomial by the above result on the Max-average-degree problem as $e(X,X)/|X|$ is maximum if and only if $2e(X,X)/|X|$ is maximum.

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Application 6, 2-color partition

Given a 2-edge-colored graph, G, find a partition (X_1, X_2) which maximizes the sum of the number of edges in X_1 of color one and the number of edges in X_2 of color two.

Let $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $M_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathcal{F} = \{M_1, M_2\}$.

By associating M_1 to any orientation of each edge of color one and associating M_2 to any orientation of each edge of color two we note that our dichotomy implies that this problem is polynomial.

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Application 7 (if time), Closeness to Eulerian

Given a digraph, D, find a partition (X_1, X_2) of $V(D)$ where the difference between the number of arcs from X_1 to X_2 and the number of arcs from X_2 to X_1 is maximized.

Note that this value is zero for Eulerian digraphs.

This value can also be shown to be equal to the minimum number of paths that need to be added to D in order to make it Eulerian.

Let $\mathcal{F}$ contain $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (or alternatively $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$).

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Open problems

One could maybe try to generalize the results to 3-partitions (using 3×3 matrices), but this is maybe difficult and I do not have any immediate applications.

But it would be interesting to see if there are any other problems that can be solved using the above dichotomy.

Or one could try to prove the same dichotomy, where we do not require that if a matrix belongs to $\mathcal{F}$ then all multiples of that matrix is also allowed to be used in the digraph.

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