A discretize-then-map approach for the treatment of parameterized geometries in model order reduction

Tommaso Taddei

Inria, Team MEMPHIS

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Joint work with L Zhang (Inria)

Collaborators:[1]

A Ferrero (Politecnico di Torino)

A Iollo (Inria)

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[1] TT would also like to thank Y Maday, P Mounoud, AT Patera and M Yano for fruitful discussions.

Objective

Model reduction in parametric domains

The goal of parameterized Model Order Reduction (pMOR) is to reduce the **marginal** cost associated with the solution to parameterized problems.

pMOR is motivated by *real-time* and *many-query* problems
design and optimization, UQ, control.

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The aim of this talk is to discuss the **treatment of parameterized geometries** in projection-based pMOR.

We denote by

$\mu = [\mu_1, \ldots, \mu_P] \in \mathcal{P} \subset \mathbb{R}^P$ a vector of parameters;

$\{\Omega_\mu : \mu \in \mathcal{P}\}$ a family of parametric domains;

$U_\mu : \Omega_\mu \to \mathbb{R}^D$ the solution field.

Mapped formulation

We recast the problem in a parameter-independent domain Ω through a mapping Φ such that

Φ_μ is invertible and $\Omega_\mu = \Phi_\mu(\Omega)$, for all $\mu \in \mathcal{P}$.

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Task 1: construction of Φ; RBF, FFD, ...

Task 2: solution to the mapped problem for $U_\mu \circ \Phi_\mu$.

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Agenda.

1. two paradigms to deal with geometry variations;
2. projection-based pMOR;
3. application to 2D RANS.

Treatment of parametric geometries in pMOR

Define the mesh $\mathcal{T}_{\rm hf} = \left(\{x_j^{\rm hf}\}_j, \mathtt{T} \right)$, with nodes $\{x_j^{\rm hf}\}_j$ and connectivity matrix $\mathtt{T}$, and the associated FE space $\mathcal{X}_{\rm hf}$.

Given $w \in \mathcal{X}_{\rm hf}$, denote by $\mathbf{w} \in \mathbb{R}^N$ the associated vector representation.

Given $\Phi : \Omega \to \Omega'$, define the mapped mesh $\Phi\left(\mathcal{T}_{\rm hf}\right) = \left(\{\Phi\left(x_j^{\rm hf}\right)\}_j, \mathtt{T} \right)$.

Definition. Φ is $\mathcal{T}_{\rm hf}$-*bijective* if the element mappings

$$\Psi_{k,\Phi}^{\rm hf}(x) = \sum_{i=1}^{n_{\rm lp}} \Phi(x_{i,k}^{\rm hf}) \hat{\phi}_i(x), \quad x_{i,k}^{\rm hf} = x_{\mathtt{T}_{i,k}}^{\rm hf},$$

are invertible.

Map-then-discretize (MtD) approach

Consider $-\Delta U_\mu + b_\mu \cdot \nabla U_\mu = f_\mu \text{ in } \Omega_\mu, \quad U_\mu\big|_{\partial\Omega_\mu} = 0.$

If $\Phi_\mu : \Omega \to \Omega_\mu$ is Lipschitz, $\widetilde{U}_\mu = U_\mu \circ \Phi_\mu$ solves

$$\int_\Omega \left(K_\mu \nabla \widetilde{U}_\mu \cdot \nabla v + \widetilde{b}_\mu \cdot \nabla \widetilde{U}_\mu \, v \; - \; \widetilde{f}_\mu \, v \right) \, dx \; = \; 0,$$

for all $v \in H_0^1(\Omega)$, with $K_\mu = g_\mu \nabla \Phi_\mu^{-1} \nabla \Phi_\mu^{-T}$,
$\widetilde{b}_\mu = g_\mu \nabla \Phi_\mu^{-T} b_\mu$, $\widetilde{f}_\mu = g_\mu f_\mu$, $g_\mu = \det(\nabla \Phi_\mu)$.

Rozza, Huynh, Patera, 2007; Lassila, Rozza, 2010, ...

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Map-then-discretize.

1. Derive the mapped problem. `map`
2. Devise FE and MOR methods for the mapped problem. `discretize`.

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Discretize-then-map (MtD) approach

Consider $-\Delta U_\mu + b_\mu \cdot \nabla U_\mu = f_\mu \text{ in } \Omega_\mu, \quad U_\mu\big|_{\partial\Omega_\mu} = 0.$

Introduce the mesh $\mathcal{T}_{\rm hf}$ of Ω and the associated space $\mathcal{X}_{\rm hf}$.

`discretize`

Define the mapped mesh $\Phi_\mu(\mathcal{T}_{\rm hf})$ and approximate the problem as follows:

`map`

$$R_\mu^{\rm hf}(U_\mu^{\rm hf}, v) = \sum_{k=1}^{N_{\rm e}} r_\mu^k(U_\mu^{\rm hf}, v) = 0, \text{ for all } v \in \mathcal{X}_{{\rm hf},\Phi_\mu},$$

with $r_\mu^k(u, v) = \int_{\mathtt{D}_{k,\Phi_\mu}} \nabla u \cdot \nabla v + (b_\mu \cdot \nabla u - f_\mu)v \, dx$.

Washabaugh et al., 2016; Dal Santo, Manzoni, 2019.

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- The PDE model is left unchanged.
- MOR should be applied at algebraic level.

Washabaugh et al., 2016; Dal Santo, Manzoni, 2019.

Comments (I): equivalence; discrete bijectivity

DtM involves evaluation of Φ at mesh nodes;

$$\Psi^{\mathrm{hf}}_{k,\Phi}(x) = \sum_{i=1}^{m_{\mathrm{lp}}} \Phi(x^{\mathrm{hf}}_{i,k})\hat{\phi}_i(x)$$

MtD requires evaluation of Φ **and its derivatives** at quadrature points.

Lemma: if $\Phi \circ \Psi^{\mathrm{hf}}_k \in \mathbb{P}_p$, DtM and MtD are equivalent.

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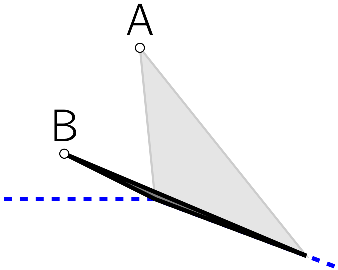
Lemma: if $\Phi \circ \Psi_k^{\mathrm{hf}} \in \mathbb{P}_p$, DtM and MtD are equivalent.

DtM might fail for large non-smooth deformations.

Enforcement of discrete bijectivity might be needed.

Taddei, Zhang, JSC, 2021

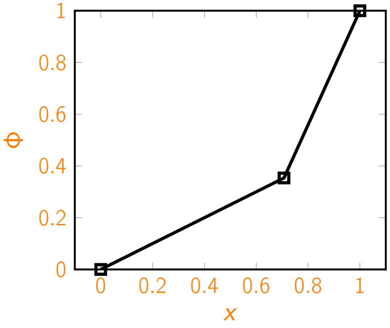


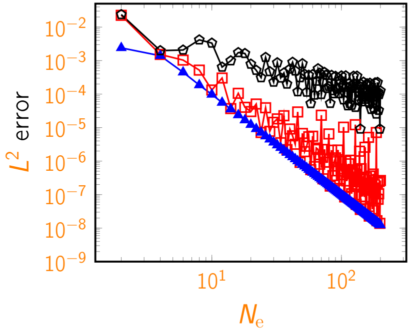
Comments (II): optimal convergence of hf solver

Consider $-\partial_{xx} u = \sin(\pi x) \quad x \in \Omega = (0,1), \quad u|_{x=0,1} = 0.$

Let $\Phi : \Omega \to \Omega$ be piecewise-linear, and let $\mathcal{T}_{\mathrm{hf}}$ be a uniform grid with N_{e} elements of degree 3.

Apply MtD, iso-parametric DtM and sub-parametric DtM.





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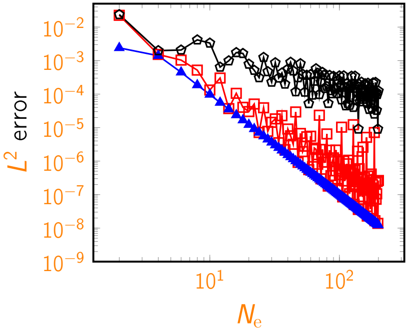
MtD fails to recover optimal rate.

DtM might have inverted elements.

DtM recovers optimal rate.



Comments (III): simplicity of implementation

Consider SUPG stabilization of the advection term:

$$\int_{\mathtt{D}_{k,\Phi}} (-\Delta u + b \cdot \nabla u - f) \frac{b}{\|b\|_2} \cdot \nabla v \, dx = 0.$$

Note that $\displaystyle\int_{\mathtt{D}_{k,\Phi}} \Delta u \left(\frac{b}{\|b\|_2} \cdot \nabla v \right) dx =$

$$\int_{\mathtt{D}_k} \left(\left(\nabla \Phi^{-T} \nabla \right) \cdot \left(\nabla \Phi^{-T} \nabla u \right) \right) \left(\frac{\nabla \Phi^{-1} b}{\|b\|_2} \cdot \nabla v \right) g \, dx.$$

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Implementation of MtD requires new assembly routines

- high implementation costs;
- second-order derivatives of Φ might not be available.

Comments (IV): hyper-reduction

$$\mathcal{G}_\mu(U_\mu, v) = \sum_{k=1}^{N_e} \int_{\mathtt{D}_k} \eta_\mu(x; U_\mu) \cdot [v(x), \nabla v(x)]\, dx = 0.$$

Two strategies for hyper-reduction:

1. Affine approximation of integrand η_μ: EIM,...

$$\eta_\mu(x; U_\mu) \approx \sum_{q=1}^{Q} \eta_\mu\left(x_q^\star; U_\mu\right)\ \xi_q(x) \text{ for } \{x_q^\star\}_q \subset \Omega,$$

$$\widehat{\mathcal{G}}_\mu(U_\mu, v) = \sum_{k=1}^{N_e} \int_{\mathtt{D}_k} \widehat{\eta}_{\mu,Q}(x; U_\mu) \cdot [v(x), \nabla v(x)]\, dx = 0.$$

2. Reduced integration domain: ECSW, EQP,...

$$\widehat{\mathcal{G}}_\mu(U_\mu, v) = \sum_{k \in \mathtt{I}_{\mathrm{eq}}} \rho_k^{\mathrm{eq}} \int_{\mathtt{D}_k} \eta_\mu(x; U_\mu) \cdot [v(x), \nabla v(x)]\, dx = 0.$$

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MtD copes with both hyper-reduction paradigms.

DtM requires the introduction of a reduced integration domain to avoid integration over Ω_μ.

[2] For DG discretizations, evaluation of Φ_μ at nodes of neighboring elements is also required.

Comments (IV): hyper-reduction

MtD copes with both hyper-reduction paradigms.

DtM requires the introduction of a reduced integration domain to avoid integration over Ω_μ.

- We rely on **elementwise empirical quadrature** for hyper-reduction. Farhat et al., 2015; Yano 2019.

- Application of EQ within the DtM framework is straightforward: given $\mu \in \mathcal{P}$, DtM requires evaluation of Φ_μ in all nodes of the sampled elements[2].

[2] For DG discretizations, evaluation of Φ_μ at nodes of neighboring elements is also required.

Projection-based pMOR

Projection scheme: LSPG+EQ

Introduce reduced-order bases $\mathbf{Z} \in \mathbb{R}^{N,n}$, $\mathbf{Y} \in \mathbb{R}^{N,j_{\mathrm{es}}}$.

Define the weighted residual $R_\mu^{\mathrm{eq}}(u, v) = \sum_{k=1}^{N_\mathrm{e}} \rho_k^{\mathrm{eq}} r_\mu^k(u, v)$.

EQ LSPG ROM: find $\widehat{\mathbf{U}}_\mu \in \arg \min_{\boldsymbol{\zeta} \in \mathrm{col}(\mathbf{Z})} \sup_{\boldsymbol{\eta} \in \mathrm{col}(\mathbf{Y})} \frac{R_\mu^{\mathrm{eq}}(\boldsymbol{\zeta}, \boldsymbol{\eta})}{\|\boldsymbol{\eta}\|_{\mathcal{Y}}}$.

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Implementation requires to address several points.

- Choice of trial ROB $\mathbf{Z}$.
- Choice of test ROB $\mathbf{Y}$ and the norm $\|\cdot\|_{\mathcal{Y}}$.
- Choice of the EQ weights $\boldsymbol{\rho}^{\rm eq}$.

Taddei, Zhang, *A discretize-then-map approach for the treatment of parameterized geometries in model order reduction*, CMAME, 2021.

Residual assembly: FE routine

Notation: $\phi_{i,k}^{\mathrm{fe}}$ $i = 1, \ldots, n_{\mathrm{lp}}$ FE basis in the k-th element; $e_1, \ldots, e_D$ canonical basis (D number of equations).

For $k = 1, \ldots, N_{\mathrm{e}}$

Compute $R_{i,k,d}^{\mathrm{un}} = r^k \left(U|_{\mathtt{D}_{k,\Phi}}, \phi_{i,k}^{\mathrm{fe}} e_d \right)$,

$i = 1 \ldots, n_{\mathrm{lp}}, d = 1, \ldots, D$.

EndFor

$\mathbf{R}^{\mathrm{hf}} \leftarrow \{R_{i,k,d}^{\mathrm{un}}\}_{i,k,d}$, $\mathbf{R}^{\mathrm{hf}} \in \mathbb{R}^N$ FE vector assembly.

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- Computation of $\{R_{i,k,d}^{\mathrm{un}}\}_{i,d}$ relies on element-wise assembly routines that take as input $\{\Phi(x_{i,k})\}_i$.
- Assembly of the FE vector is independent of the PDE.

Residual assembly: MOR routine

Notation: $\mathbf{Y} = [\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{j_{\rm es}}] \in \mathbb{R}^{N, j_{\rm es}}$ test space; $\mathtt{T}_{i,k}$ index of the i-th node of the k-th element.

$\mathbf{Y}^{\rm un}_{i,k,d,j} = \mathbf{Y}_{\mathtt{T}_{i,k}+(d-1)N,j}$ unassembled test space, $\widehat{\mathbf{R}} \in \mathbb{R}^{j_{\rm es}}$ reduced residual, $\left(\widehat{\mathbf{R}}\right)_j = R^{\rm eq}_{\mu}(U, \psi_j)$.

For $k \in \mathtt{I}_{\rm eq}$

Compute $R^{\rm un}_{i,k,d} = r^k \left(U|_{\mathtt{D}_{k,\Phi}}, \phi^{\rm fe}_{i,k} e_d \right)$, $i = 1 \ldots, n_{\rm lp}, d = 1, \ldots, D$.

EndFor

$$\left(\widehat{\mathbf{R}}\right)_j = \sum_{k \in \mathtt{I}_{\rm eq}} \rho^{\rm eq}_k \left(\sum_{i=1}^{n_{\rm lp}} \sum_{d=1}^{D} \mathbf{Y}^{\rm un}_{i,k,d,j} R^{\rm un}_{i,k,d} \right), j = 1, \ldots, j_{\rm es}.$$

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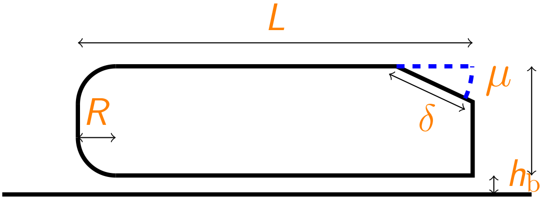
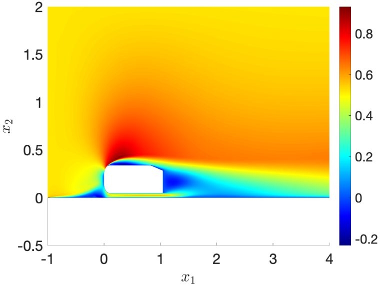
- MOR assembly exploits available FE routines.
- Geometry variations don't influence MOR assembly.

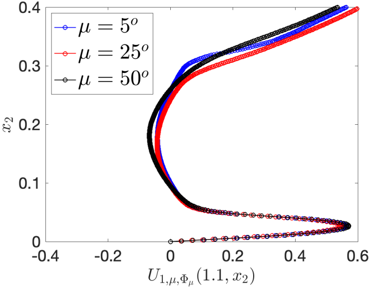
Application to 2D RANS

Ahmed's body problem

Closure model: SA. **Discretization:** stab P2-P2 FEM. $\mathrm{Re} = \frac{u_{\mathrm{in}} H_{\mathrm{c}}}{\nu} = 3 \cdot 10^3$, $\mu \in [5^o, 50^o]$.

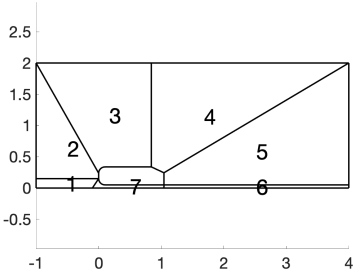


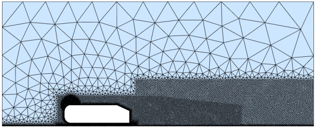




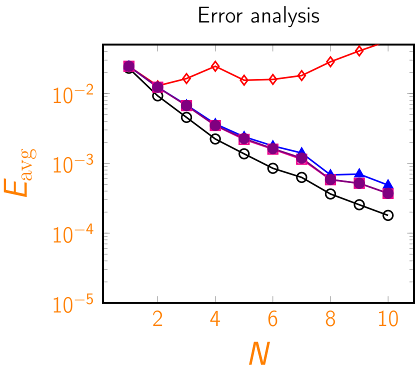
We consider a piecewise-bilinear mapping (Gordon-Hall).

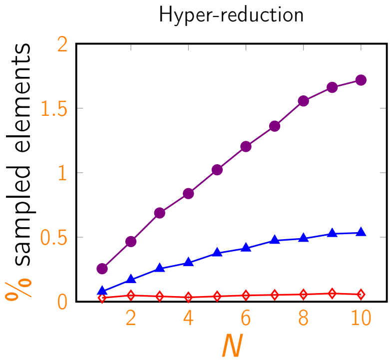
FE mesh is not conforming with the coarse-grained partition.





Relative error averaged over 10 out-of-sample configurations.





Legend: projection, ROM with HF quadrature, $tol_{\text{eq}} = 10^{-6}$, $tol_{\text{eq}} = 10^{-10}$, $tol_{\text{eq}} = 10^{-14}$.

Conclusions and perspectives

Summary

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- We consider two approaches:
  1. Discretize-then-Map;
  2. Map-then-Discretize.

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Take-aways

- DtM is simpler to implement, and can be used in combination with non-smooth mappings.
- MtD copes with a broader class of hyper-reduction methods ($\rightarrow$ possibly larger speedups), and does not require discrete bijectivity wrt the mesh.

Thank you for your attention!

For more information, visit the website:

math.u-bordeaux.fr/~ttaddei/ .

1. Taddei, Zhang; *A discretize-then-map approach for the treatment of parameterized geometries in model order reduction*, CMAME, 2021.

2. Ferrero, Taddei, Zhang; *Registration-based model reduction of parameterized two-dimensional conservation laws*, Arxiv preprint 2021.