A collocated MOR for advective-dominant PDEs: hyper-reduction of ADER scheme on evolving chimera meshes

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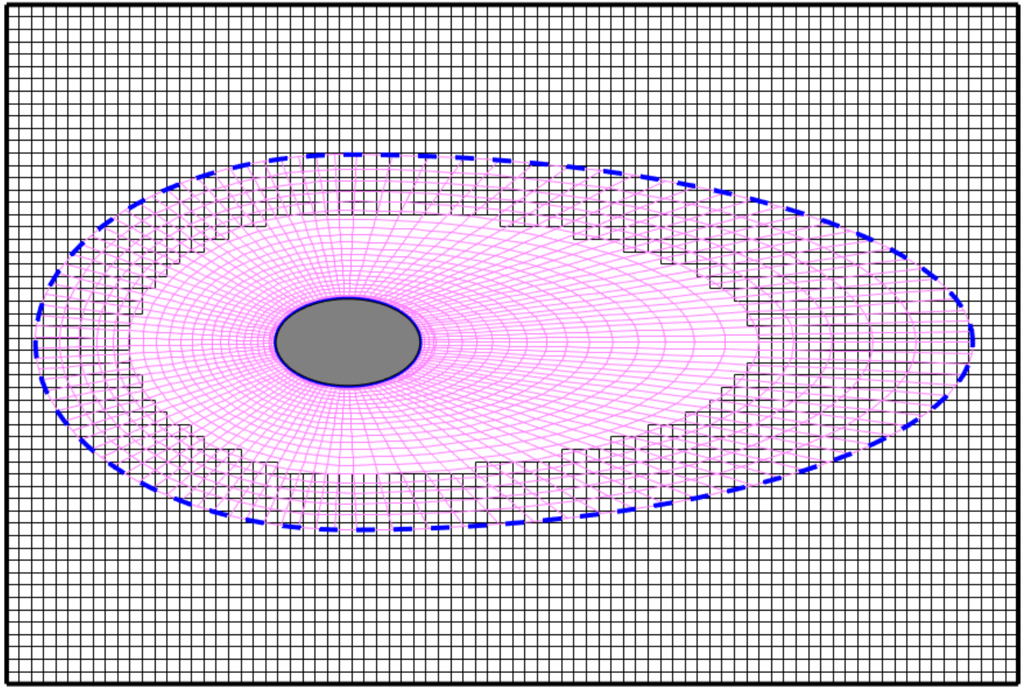
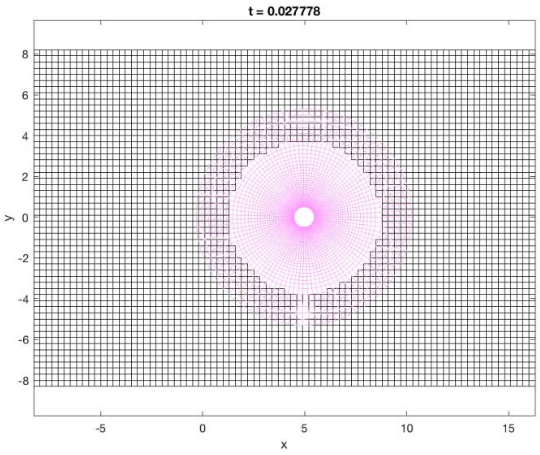
Outline

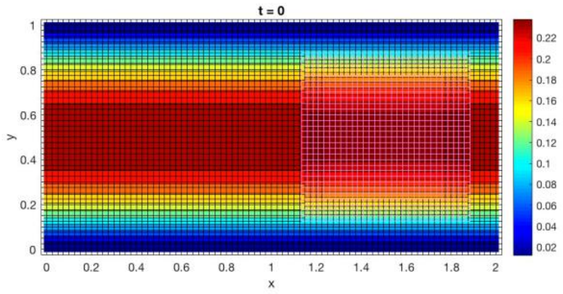
- Chimera mesh and cell classification
- The high-fidelity method
- The hyper-reduced approach
- Beyond the Kolmogorov n-width
- Numerical results
- Conclusions and Future Perspectives

The Overset Grid or Chimera Mesh



The Overset Grid or Chimera Mesh

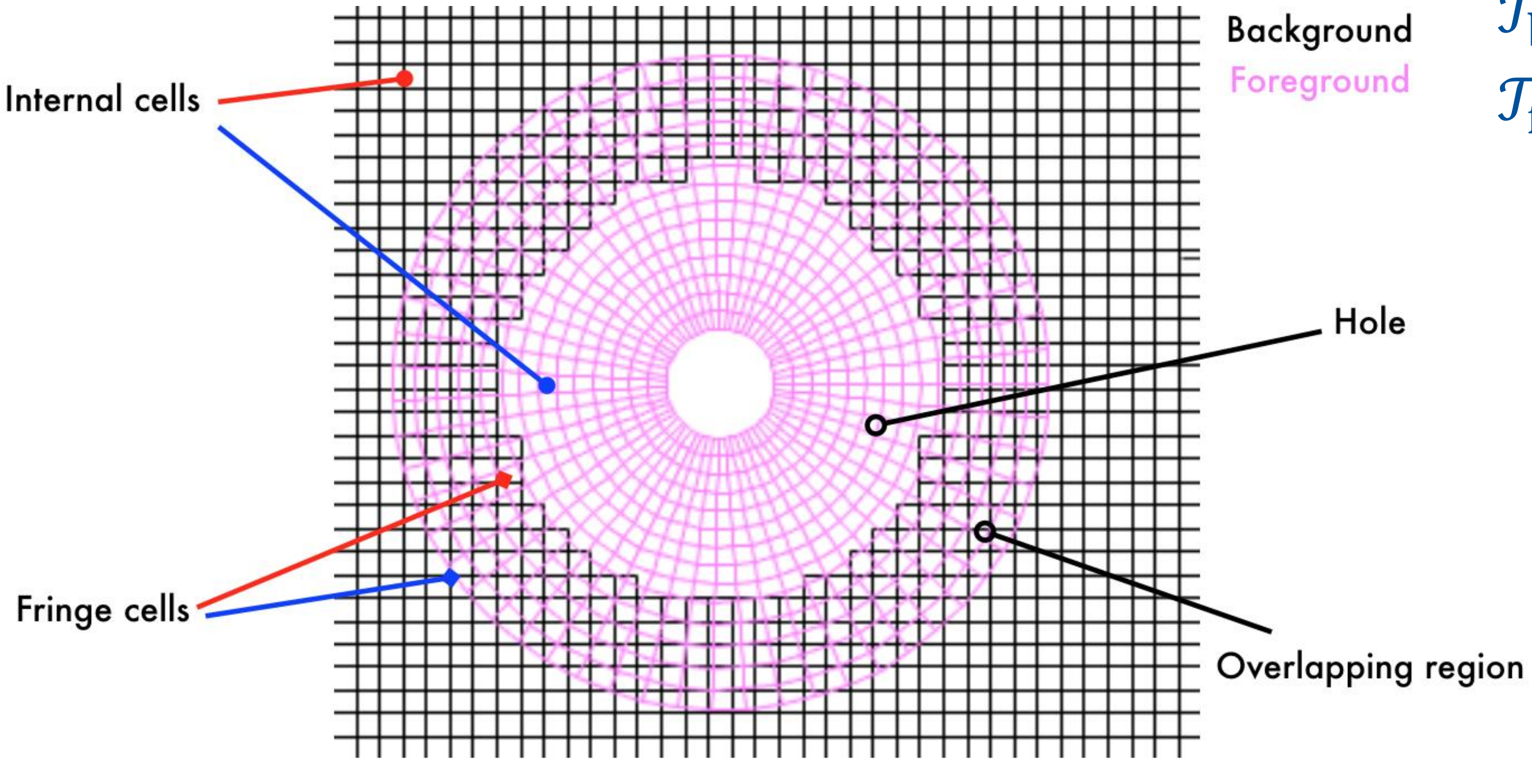




Pro: high fidelity solution wrt the real geometry of the physical phenomenon

Cons: i) Communcation among different meshes; ii) Mesh movement and deformation

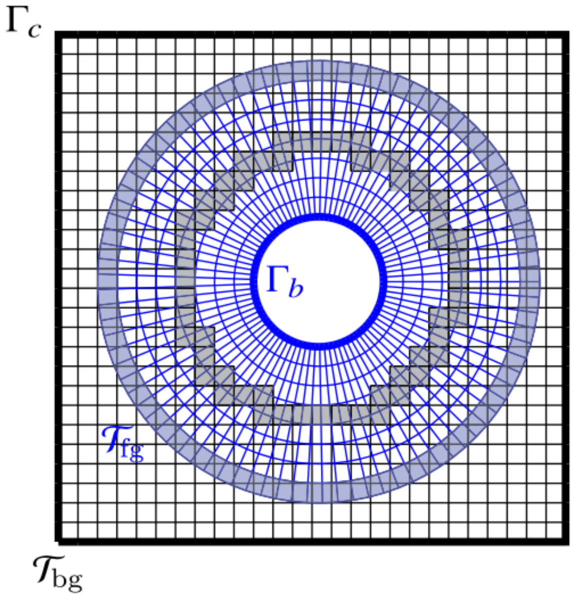
A Cell Classification



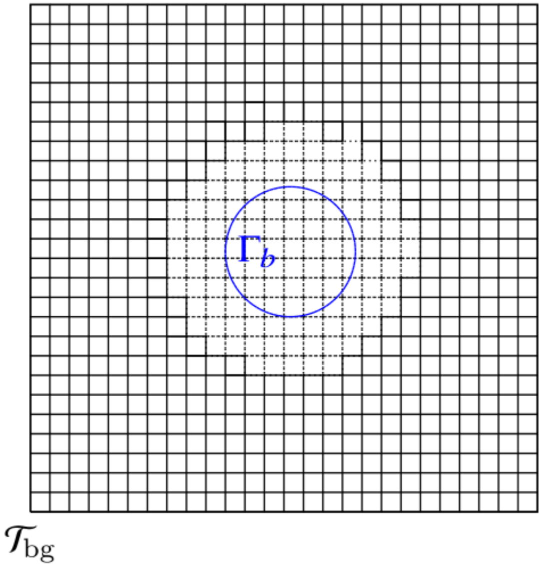
$\mathcal{T}_{\mathrm{bg}}$

$\mathcal{T}_{\mathrm{fg}}$

Another cell classification

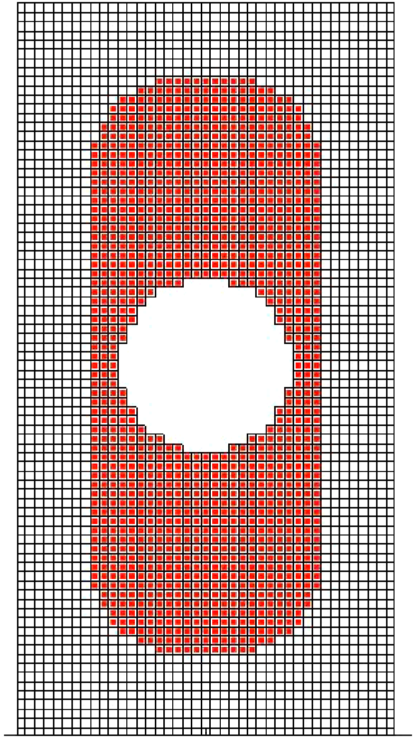


$\mathcal{T}_{\text{bg}}^{\text{hole}}(\text{t})$



$$\mathcal{T}_{\text{bg}}(\text{t}) = \mathcal{T}_{\text{bg}}^{\text{hole}}(\text{t}) \cup \mathcal{T}_{\text{bg}}^{\text{ol}}(\text{t}) \cup \mathcal{T}_{\text{bg}}^{\text{nol}}(\text{t})$$

Another cell classification



$$\bigcup_t \mathcal{T}_{\mathrm{bg}}(t) = \bigcup_t \mathcal{T}_{\mathrm{bg}}^{\mathrm{ol}}(t) \cup \bigcup_t \mathcal{T}_{\mathrm{bg}}^{\mathrm{nol}}(t) = \mathcal{T}_{\mathrm{bg}}^{\mathrm{HF}} \cup \mathcal{T}_{\mathrm{bg}}^{\mathrm{ROM}}$$

$$\mathcal{T}_{\mathrm{bg}}^{\mathrm{hole}}(t) \subset \mathcal{T}_{\mathrm{bg}}^{\mathrm{HF}} \quad \forall t$$

Static partitions: $\mathcal{T}_{\mathrm{bg}}^{\mathrm{ROM}}$ and $\mathcal{T}_{\mathrm{fg}}$.

Dynamic partition: $\mathcal{T}_{\mathrm{bg}}^{\mathrm{HF}}$.

The high-fidelity method

Find $\boldsymbol{u}: \Omega(t) \times (0, T] \rightarrow \mathbb{R}^{\delta}$ such that

$\boldsymbol{F}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{\beta}(\boldsymbol{u}) - \nu(\boldsymbol{u}) \nabla \mathbf{u}$

$$\partial_t \boldsymbol{u} + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{f}$$

Properly closed by boundary conditions an initial conditions.

Motion equation for $\boldsymbol{X}_{fg} \in \Omega_{fg}$:

$$\begin{cases} \dot{\boldsymbol{X}} = \boldsymbol{V}(\boldsymbol{x}, t; \boldsymbol{u}) \\ \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases}$$

Arbitrary high-order DERivatives: ADER (FV version)

Prediction

- From solution at $t = t^n$: (spatial) local $\mathcal{P}_2^{loc}$-interpolation over any cell $\Omega_i^n = \Omega_i(t^n)$.
- Local-space time solution over any $\mathcal{C}_i^n = \Omega_i(t) \times (t^n, t^{n+1})$.

Correction

- FV scheme by integrating the PDE over $\mathcal{C}_i^n$.
- Flux discretization by employing the local space-time predictor

Overset mesh

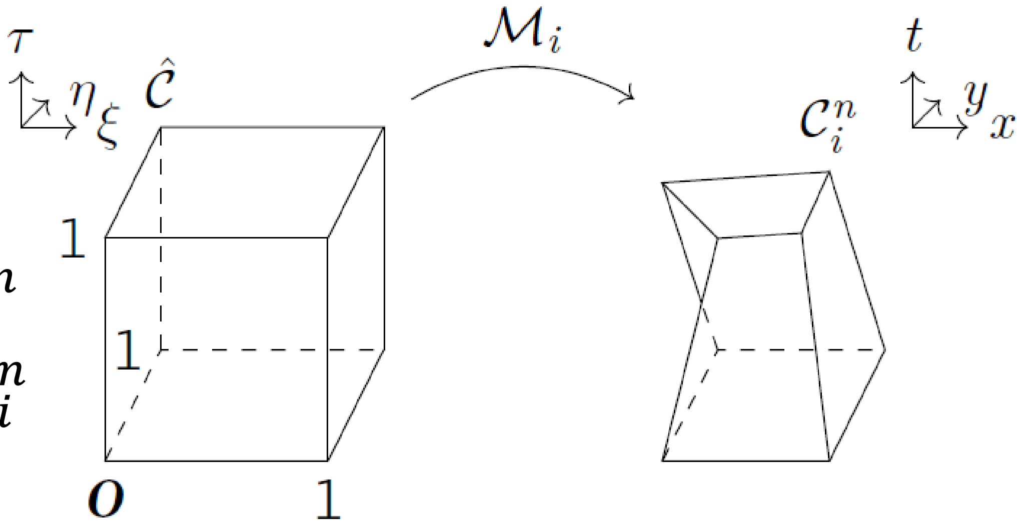
- Cell management of the overlapping zone

[1]Dumbser *et al.*, SIAM J. Sci. Comp. (2017); Titaterev & Toro, J. Comp. Phys. (2005); Bergmann *et al.*, SIAM J. Sci. Comp. (2021); Bergmann *et al.*, J. Sci. Comp. (2022).

ADER: Prediction

Space-time cell: $\mathcal{C}_i^n = \Omega_i(t) \times (t^n, t^{n+1})$.
From the mesh motion equation:

$$\mathcal{M}_i: \begin{cases} x = x(\xi,\eta,\tau) \\ y = y(\xi,\eta,\tau) \\ t = t^n + \Delta t\, \tau \end{cases}$$



Find the local space-time solution $\boldsymbol{q}: \mathcal{C}_i^n \to \mathbb{R}^\delta$ s. t.

$$\begin{cases} \partial_t \boldsymbol{q} + \nabla \cdot \boldsymbol{F}(\boldsymbol{q}, \nabla \boldsymbol{q}) = \boldsymbol{f}; \; in\ \mathcal{C}_i^n \\ \boldsymbol{q}|_{t=t^n} = \mathcal{P}_2^{loc} \boldsymbol{u}^n; \qquad on\ \Omega_i^n \end{cases}$$

The equation is solved via FE.

ADER: Correction

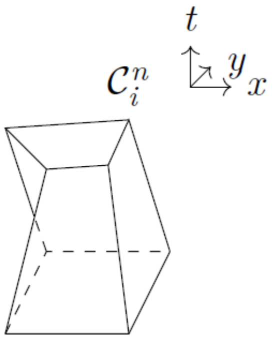
Definition: $\nabla_{x,t} := [\partial_t, \nabla]^T$; $\boldsymbol{\mathcal{U}} := [\boldsymbol{u}, \boldsymbol{F}(\boldsymbol{u}, \nabla \boldsymbol{u})]^T$

$$\partial_t \boldsymbol{u} + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}, \nabla \boldsymbol{u}) = \boldsymbol{f} \Leftrightarrow \nabla_{x,t} \cdot \boldsymbol{\mathcal{U}} = \boldsymbol{f}$$

Integration over $\mathcal{C}_i^n$:

$$-|\Omega_i^n| \boldsymbol{\mathcal{U}}_i^n + \left|\Omega_i^{n+1}\right| \boldsymbol{\mathcal{U}}_i^{n+1} + \sum_{j=1}^{4} \int_{\Gamma_{ij}^n} \boldsymbol{\mathcal{U}} \cdot \boldsymbol{n} \, d\Gamma = \int_{\mathcal{C}_i^n} \boldsymbol{f} \, d\mathcal{C}$$

with $\boldsymbol{\mathcal{U}}_i^n = \frac{1}{|\Omega_i^n|} \int_{\Omega_i^n} \boldsymbol{u}(\boldsymbol{x}, t^n) \, d\Omega$.



Rusanov approach[1]:

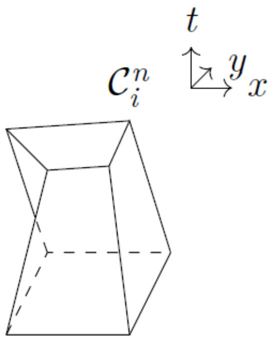
$$[\boldsymbol{\mathcal{U}} \cdot \boldsymbol{n}]_{\Gamma_{ij}^n} \approx \frac{1}{2} \left(\boldsymbol{\mathcal{U}}_j^+ + \boldsymbol{\mathcal{U}}_j^- \right) \cdot \boldsymbol{n}_{x,t} - \frac{s}{2} (\boldsymbol{q}_j^+ - \boldsymbol{q}_j^-)$$

[1]Bergmann *et al.*, SIAM J. Sci. Comp. (2021).

Stabilization of the scheme

Let $\tilde{\boldsymbol{n}} = \boldsymbol{n} / \sqrt{n_x^2 + n_y^2}$.

$$s_{AD} = \max|\rho(\mathcal{A})| = \frac{1}{2}\left|\tilde{\sigma} + \sqrt{\left(a_x^2 + \frac{4\nu}{\varepsilon}\right)\tilde{n}_x^2 + 2a_x a_y \tilde{n}_x \tilde{n}_y + \left(a_y^2 + \frac{4\nu}{\varepsilon}\right)\tilde{n}_x^2}\right|$$



Optimal relaxation parameter ε

Corollary: Optimal relaxation time[1]

For a given mesh size h and a numerical scheme of order p for $u_{hyp,h}$ solving the hyperbolized problem derived by the original parabolic problem, the optimal relaxation time ε_p is

$$\varepsilon_p = \mathcal{O}(1)\frac{h^p}{C_p}$$

Chosen relaxation time: $\varepsilon = \varepsilon_2/2$.

[1]Toro & Montecinos, SIAM J. Sci. Comp. (2014); Carlino, PhD Thesis (2021); Bergmann *et al.*, SIAM J. Sci. Comp. (2021)

The advective-dominant limit

For a hyperbolic equation ($\nu = 0$), the stabilization coefficient is the maximum eigenvalue of the ALE Jacobian matrix

$$A_{\tilde{\boldsymbol{n}}}^{\boldsymbol{V}} = \sqrt{n_x^2 + n_y^2}\left(\frac{\partial \boldsymbol{F}}{\partial u}\,\widetilde{\boldsymbol{n}_{\boldsymbol{x}}} - \boldsymbol{V}\cdot\tilde{\boldsymbol{n}}\mathrm{I}\right)$$

Namely:

$$s_A = \max\left|\rho\left(A_{\tilde{\boldsymbol{n}}}^{\boldsymbol{V}}\right)\right| = \left|\boldsymbol{a}\cdot\widetilde{\boldsymbol{n}_x} + \widetilde{n_t}\right|$$

Proposition: Limit of vanishing diffusion[1]

$$\lim_{\nu\to 0} s_{AD} = \frac{1}{2}\left|\,\tilde{\sigma} + \boldsymbol{a}\cdot\widetilde{\boldsymbol{n}_{\boldsymbol{x}}} + n_t\right| = s_A$$

[1]Carlino, PhD Thesis (2021); Bergmann *et al.*, SIAM J. Sci. Comp. (2021)

The hyper-reduced approach

Find $\boldsymbol{u}: \Omega(t) \times (0, T] \times \mathcal{P} \to \mathbb{R}^{\delta}$, with $\mathcal{P} \subset \mathbb{R}^{p}$, such that

$$\partial_t \boldsymbol{u}(z) + \nabla \cdot \boldsymbol{F}(\boldsymbol{u}(z), \nabla \boldsymbol{u}(z)) = \boldsymbol{f}$$

With $\boldsymbol{z} \in \mathcal{P}$ parameter vector.

Motion equation for $\boldsymbol{X}_{fg} \in \Omega_{fg}$:

$$\begin{cases} \dot{\boldsymbol{X}} = \boldsymbol{V}(\boldsymbol{x}, t; \boldsymbol{u}; \boldsymbol{z}) \\ \boldsymbol{X}(0) = \boldsymbol{X}_0 \end{cases}$$

The reduced basis $\mathrm{span}\{\phi_\mathrm{i}\}_{i=1}^{N_\mathrm{ROM}}$

$\Phi \in \mathbb{R}^{N \times N_\mathrm{ROM}}$
$W \in \mathbb{R}^{N \times N}$
$\hat{\boldsymbol{u}}_\mathrm{HF} \in \mathbb{R}^{N}$

It holds

$$u_\mathrm{ROM}(\boldsymbol{x}, t; \boldsymbol{z}) = \sum_{i=1}^{N} a_i(t; \boldsymbol{z})\, \phi_i(\boldsymbol{x}) \Rightarrow \hat{\boldsymbol{u}}_\mathrm{ROM} = \Phi \boldsymbol{a}$$

and

$$a_i(t; \boldsymbol{z}) = \int_{\Omega(t;\boldsymbol{z})} u_\mathrm{HF}(\boldsymbol{x}, t; \boldsymbol{z}) \phi_i(\boldsymbol{x})\ \mathrm{d}\boldsymbol{x} \Rightarrow \boldsymbol{a} = \Phi^\top W \hat{\boldsymbol{u}}_\mathrm{HF}.$$

Thus:

$$\hat{\boldsymbol{u}}_\mathrm{ROM} = \Phi\Phi^\top W \hat{\boldsymbol{u}}_\mathrm{HF}$$

W is the matrix collecting the Gauss quadrature weigths.

The collocated reduced approach

The quadrature weights matrix W can be replaced by an empirical quadrature weights matrix $\widetilde{W}$. This new matrix is found by NNLS wrt the integral

$$\int_{\Omega(t;\boldsymbol{z})} u_{\mathrm{HR}}(\boldsymbol{x},t;\boldsymbol{z})\,\phi_i(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

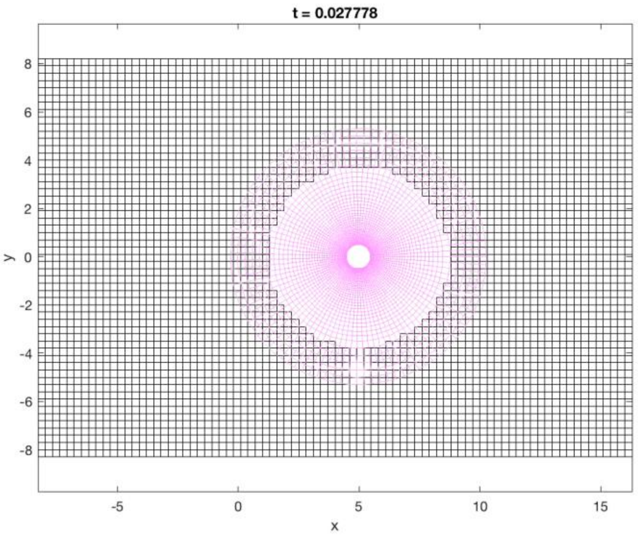
for any $i = 1,\ldots,N$, $\boldsymbol{z} \in Z$ and $(x,t) \in \Omega(t;z) \times [0,T]$.

Thus the projection reads

$$\widehat{\boldsymbol{u}}_{\mathrm{HR}} = \Phi\Phi^{\top}\widetilde{W}\widehat{\boldsymbol{u}}_{\mathrm{HF}}$$

$$\Phi \in \mathbb{R}^{N_{\mathrm{HR}}\times N_{\mathrm{ROM}}}$$
$$\widetilde{W} \in \mathbb{R}^{N_{\mathrm{HR}}\times N_{\mathrm{HR}}}$$
$$\widehat{\boldsymbol{u}}_{\mathrm{HF}} \in \mathbb{R}^{N_{\mathrm{HR}}}$$

The issue of the background mesh

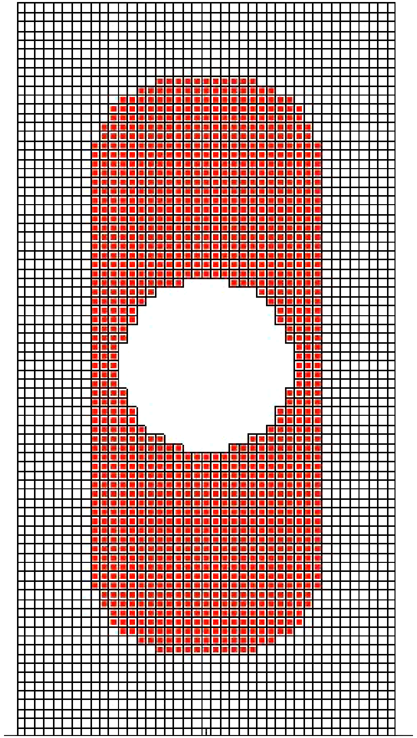


A domain decomposition approach.
Two reduced basis:

$$\text{span}\left\{\phi_i^{\text{bg}}\right\}_{i=1}^{N_{\text{ROM}}^{\text{bg}}} \text{ on } \mathcal{T}_{\text{bg}}^{\text{ROM}} = \cup_{t,\boldsymbol{z}} \mathcal{T}_{\text{bg}}^{\text{nol}}$$

$$\text{span}\left\{\phi_i^{\text{fg}}\right\}_{i=1}^{N_{\text{ROM}}^{\text{fg}}} \text{ on } \mathcal{T}_{\text{fg}}$$

The issue of the background mesh



On $\mathcal{T}_{\mathrm{bg}}^{\mathrm{HF}} = \bigcup_{t,\boldsymbol{z}} \mathcal{T}_{\mathrm{bg}}^{\mathrm{ol}}$ the recovered solution is HF.

Beyond the Kolmogorov n-width

Let $\varphi: \mathcal{P} \to \Omega$ be the mapping associating any $\boldsymbol{z} \in \mathcal{P}$ to a parametric solution $\boldsymbol{u} = \varphi(\boldsymbol{z}) \in \Omega$ under the constraint given by the PDE:

$$\mathcal{R}(u(\boldsymbol{z}), \boldsymbol{z}) = 0 \quad \forall \boldsymbol{z} \in \mathcal{P}$$

The residual implicitly define a manifold in the Hilbert space $(V, \|\cdot\|)$. Thus $\mathrm{im}\varphi \subseteq V$.

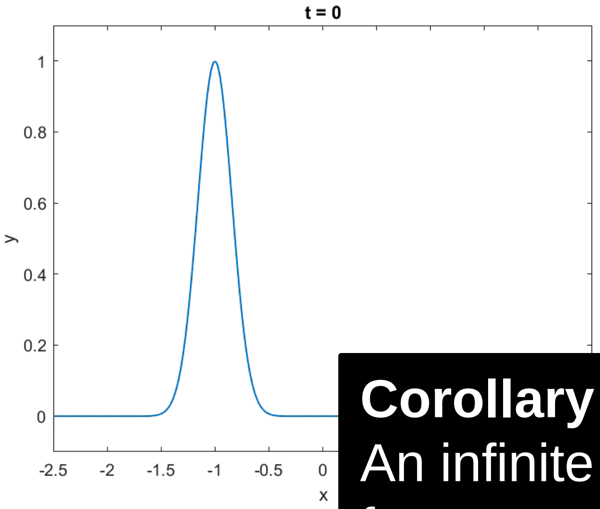
Let V_n be a n-dimensional ($n < +\infty$) subspace of V. It holds

$$d_{V_n} = \inf_{v \in \mathrm{im}\varphi} \sup_{v_n \in V_n} \|v - v_n\| \geq d_n(\mathrm{im}\varphi) = \sup_{\substack{W_n \subset V \\ n = \dim W_n < +\infty}} \inf_{w \in \mathrm{im}\varphi} \sup_{w_n \in W_n} \|w - w_n\|$$

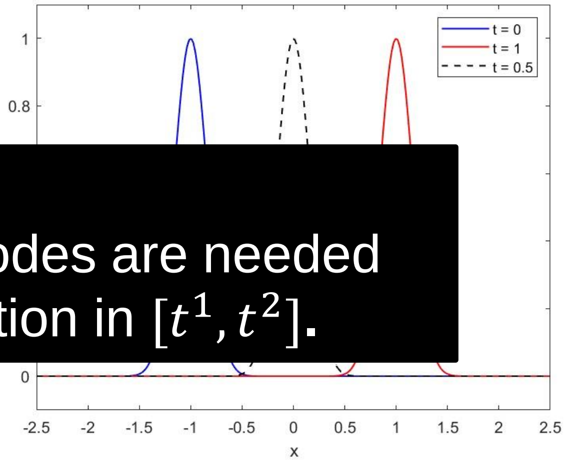
With W_n a generic n-dimensional subspace of V.

An example: the advection of a compact support function

Fourier ≅ SVD at fixed time



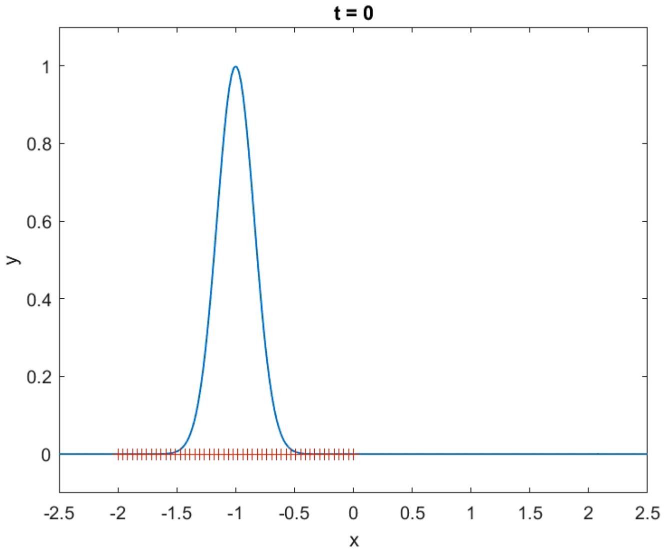
Proposition
There not exists a linear combination representing the evolution of the solution from t^1 to t^2.



Corollary
An infinite number of modes are needed for represeing the evolution in $[t^1, t^2]$.

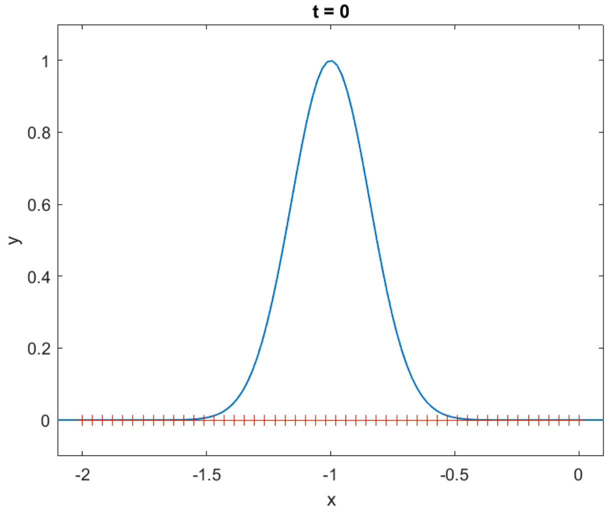
$$f(t = 0.5) \neq a_0 f(t = 0) + a_1(t = 1)$$

Reference Mapping vs Relativity of the frame



$$\varphi_{\Omega}$$

POV: the foreground mesh



$$\varphi_{\mathcal{T}_{\mathrm{bg}}^{\mathrm{HR}}};\ \varphi_{\mathcal{T}_{\mathrm{bg}}^{\mathrm{ROM}}};\ \varphi_{\mathcal{T}_{\mathrm{fg}}}$$

$$d_n(\mathrm{im}\varphi_{\Omega}) \geq d_n\left(\mathrm{im}\varphi_{\mathcal{T}_{\mathrm{bg}}^{\mathrm{HR}}}\right) + d_n\left(\mathrm{im}\varphi_{\mathcal{T}_{\mathrm{bg}}^{\mathrm{ROM}}}\right) + d_n\left(\mathrm{im}\varphi_{\mathcal{T}_{\mathrm{fg}}}\right)$$

Numerical results

Performance analysis: Hyper-Reduced (HR) vs High-Fidelity (HF).

$$\mathrm{Err} = \frac{\sqrt{\int_0^T \int_\Omega (u_{\mathrm{HR}} - u_{\mathrm{HF}})^2 \, d\boldsymbol{x} \, \mathrm{dt}}}{\sqrt{\int_0^T \int_\Omega u_{\mathrm{HF}}^2 \, d\boldsymbol{x} \, \mathrm{dt}}}$$

$$u_{\mathrm{HR}} \propto \varepsilon_{\mathrm{NNLS}}$$

$$\varepsilon_{\mathrm{NNLS}} = 0 \;\Rightarrow\; u_{\mathrm{HR}} = u_{\mathrm{ROM}}$$

Hyper-reduction analysis:

Total number of cells: N

Number of cells in $\mathcal{T}_{\mathrm{fg}}$: N_{fg}

Number of cells in $\mathcal{T}_{\mathrm{bg}}^{\mathrm{HF}}$: $N_{\mathrm{bg}}^{\mathrm{HF}}$

Number of cells in $\mathcal{T}_{\mathrm{bg}}^{\mathrm{ROM}}$: $N_{\mathrm{bg}}^{\mathrm{ROM}}$

Number of cells in $\mathcal{T}_{\mathrm{bg}}$: $N_{\mathrm{bg}} = N_{\mathrm{bg}}^{\mathrm{HF}} + N_{\mathrm{bg}}^{\mathrm{ROM}}$

Number of HR cells in $\mathcal{T}_{\mathrm{bg}}^{\mathrm{ROM}}$: $N_{\mathrm{bg}}^{\mathrm{HR}}$

Number of HR cells in $\mathcal{T}_{\mathrm{fg}}$: N_{fg}^{HR}

The best approximation: full projection on ROM basis

Let $V_n = \mathrm{span}\left\{\phi_i^{\mathrm{bg}}\right\}_{i=1}^{n} \cup \mathrm{span}\left\{\phi_i^{\mathrm{fg}}\right\}_{i=1}^{n} \cup \tilde{V}_n\Big|_{\mathcal{T}_{\mathrm{bg}}^{\mathrm{HF}}}$ be the employed ROM subspace.

$$\mathrm{Err}\Big|_{\varepsilon_{\mathrm{NLLS}}\neq 0} \geq \mathrm{Err}\Big|_{\varepsilon_{NNLS}=0} \Rightarrow d_{V_n}\Big|_{\varepsilon_{\mathrm{NNLS}}\neq 0} \geq d_{V_n}\Big|_{\varepsilon_{\mathrm{NNLS}}=0}$$

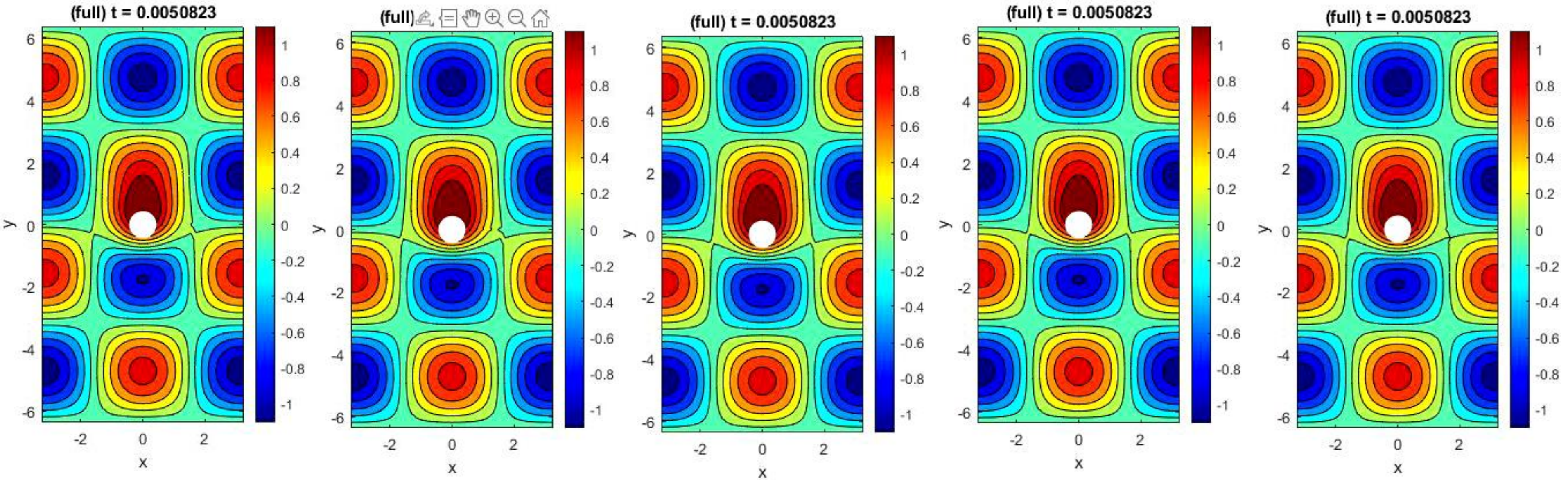
Since ϕ_i^* $(i = 1, \dots, n,\ * = \mathrm{bg}, \mathrm{fg})$ is defined all over $\mathcal{T}_*$, we expect

$$\mathrm{Err}\Big|_{\varepsilon_{\mathrm{NLLS}}^1} \leq \mathrm{Err}\Big|_{\varepsilon_{NNLS}^2}$$

For $\varepsilon_{\mathrm{NNLS}}^1 \leq \varepsilon_{\mathrm{NNLS}}^2$.

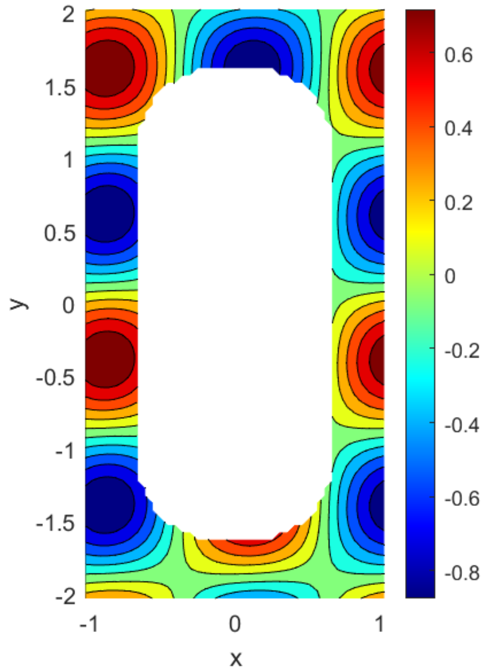
Linear test case: $\nu = 0.05$, $\boldsymbol{\beta} = [1,1]^\top$, $z \in [-2,2]$

$$u(\boldsymbol{x}, t; z) = \exp[-x^2 - (y + zt)^2 + 0.5^2] + \cos(x - 0.5t)\sin(y - 0.5t$$

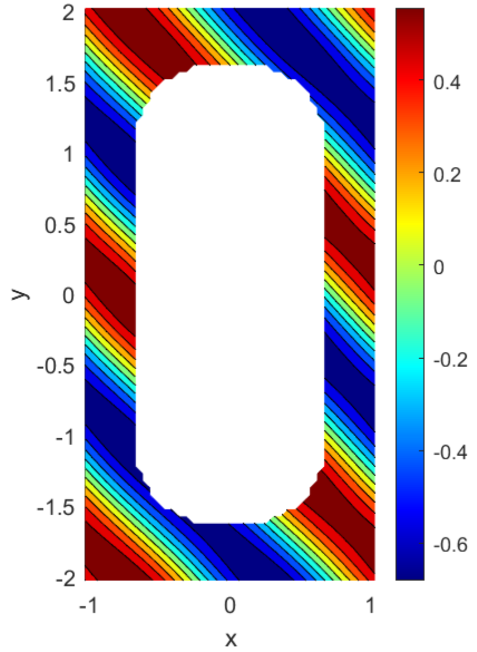


$z = -2$ $z = -1$ $z = 0$ $z = +1$ $z = +2$

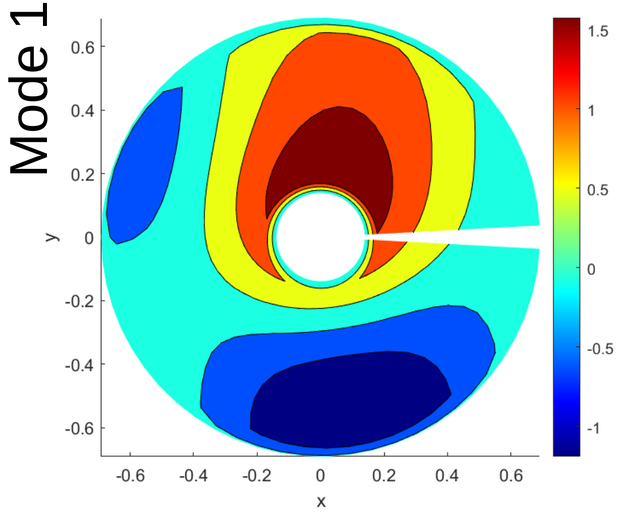
Linear test case

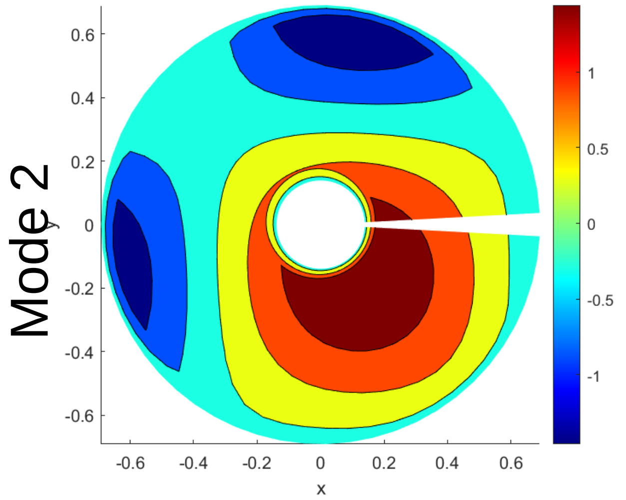


Mode 1

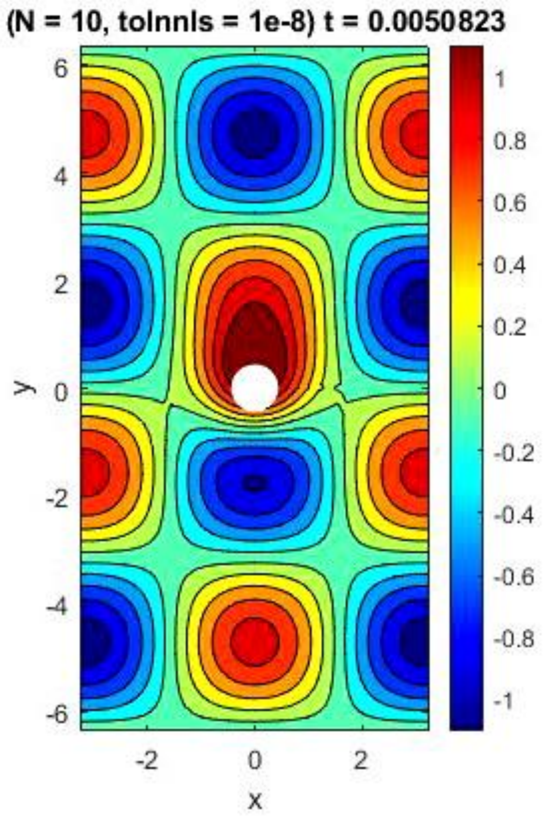


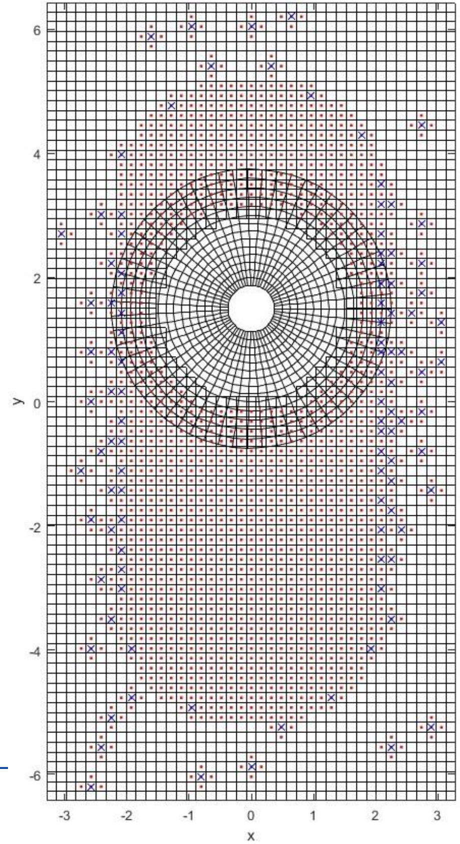
Mode 2

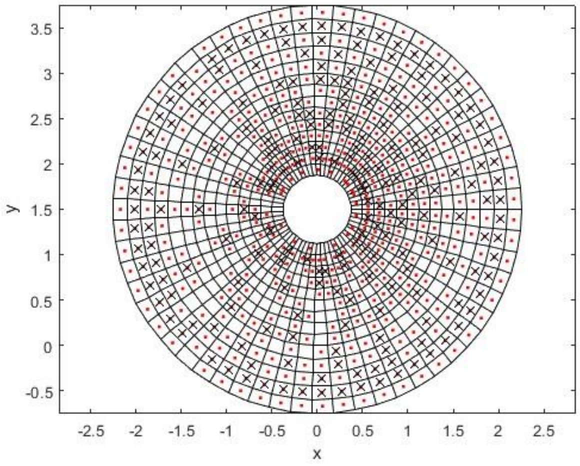




In-sample simulation ($z = +1$)







$N = 4141; N_{\mathrm{bg}} = 3321; N_{\mathrm{fg}} = 826;$
$N_{fg}^{HF} = 1350$ (40%)

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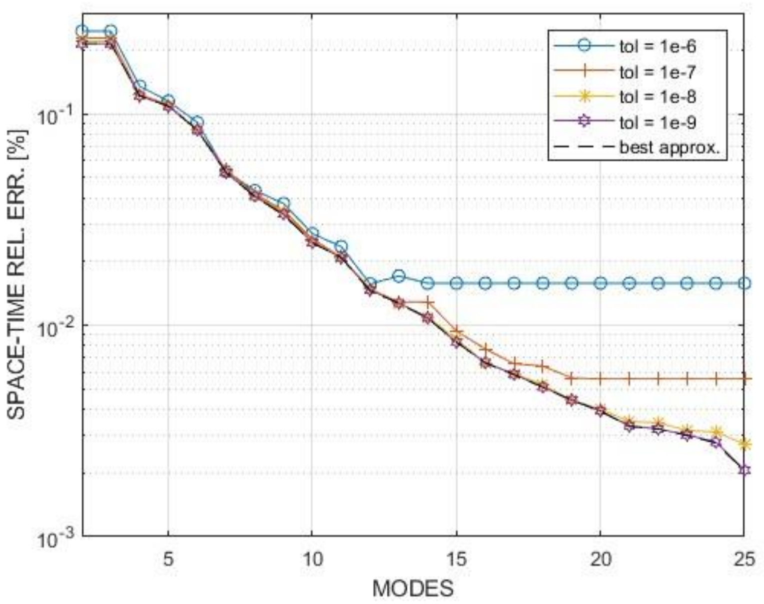
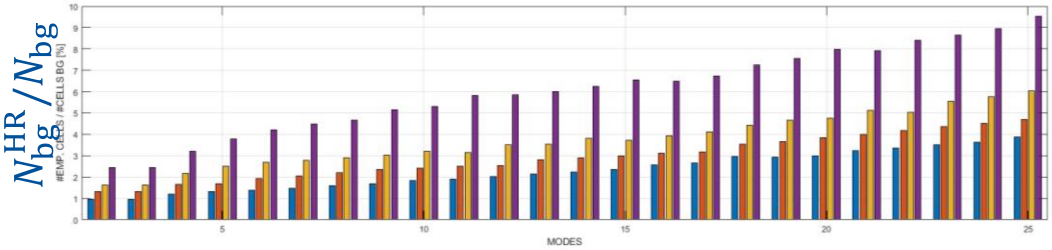
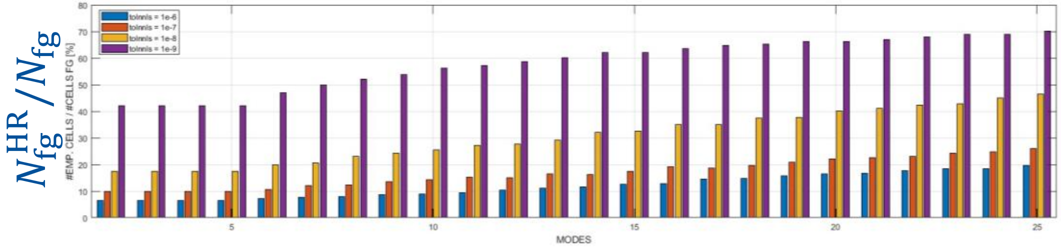
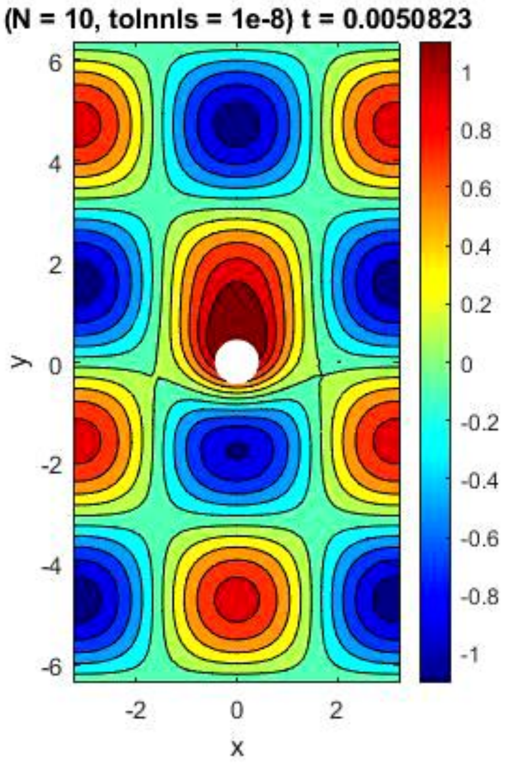
In-sample simulation

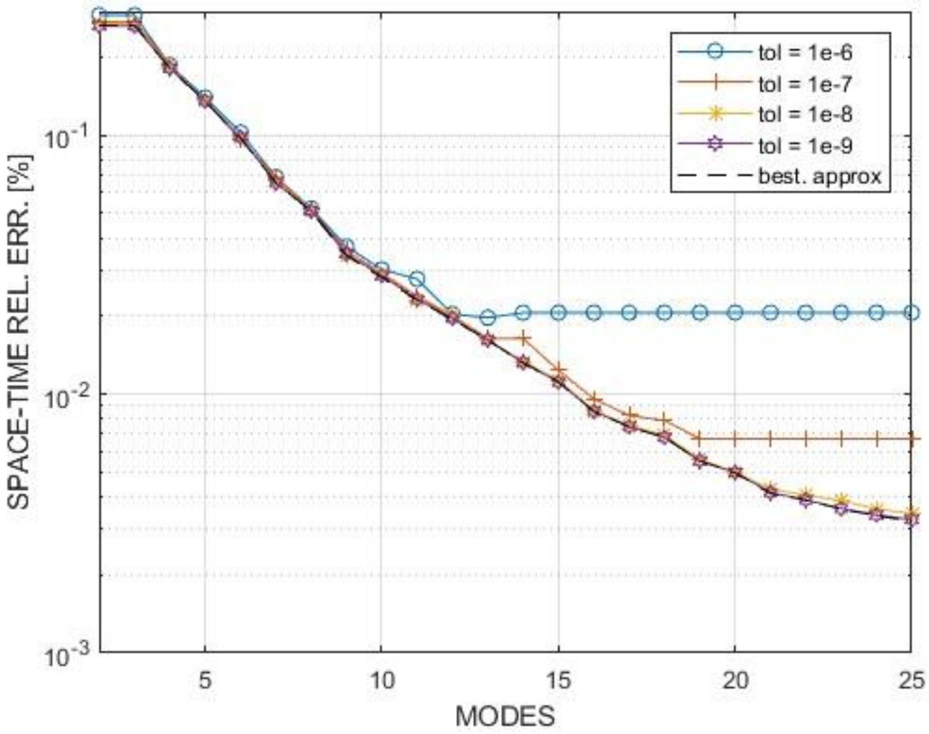




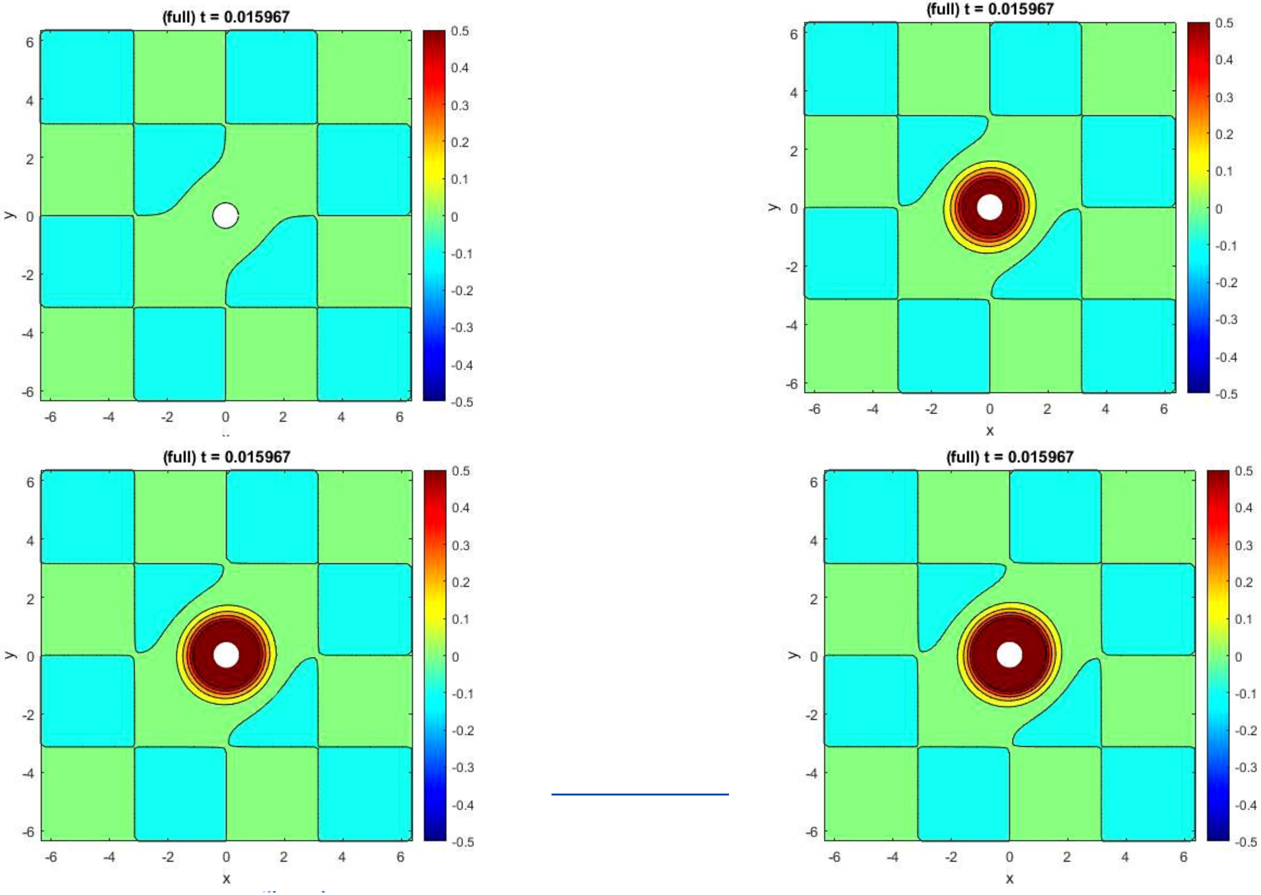


Out-of-sample simulation ($z = +1.7$)

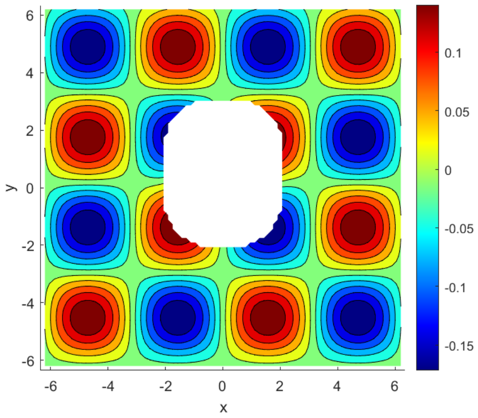




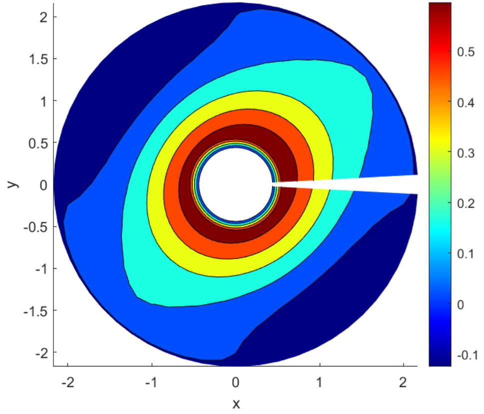
Nonlinear case: $\nu = 0.05$, $\beta = \tanh(u)\,[0,1]^\top$, $z \in [0.1,2]$, $f = \sin(x)\sin(y)$

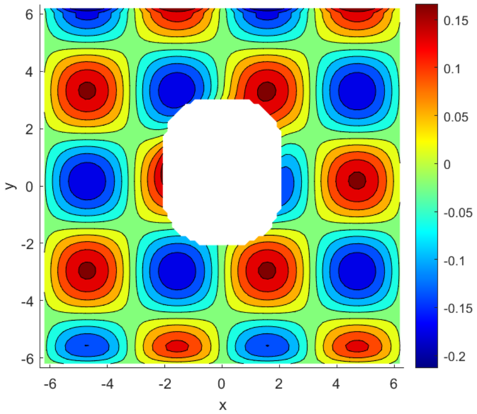


Nonlinear test case

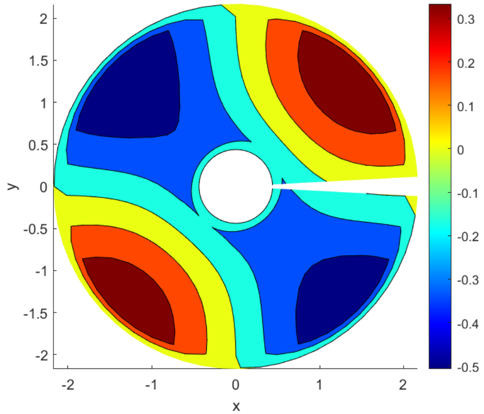


Mode 1

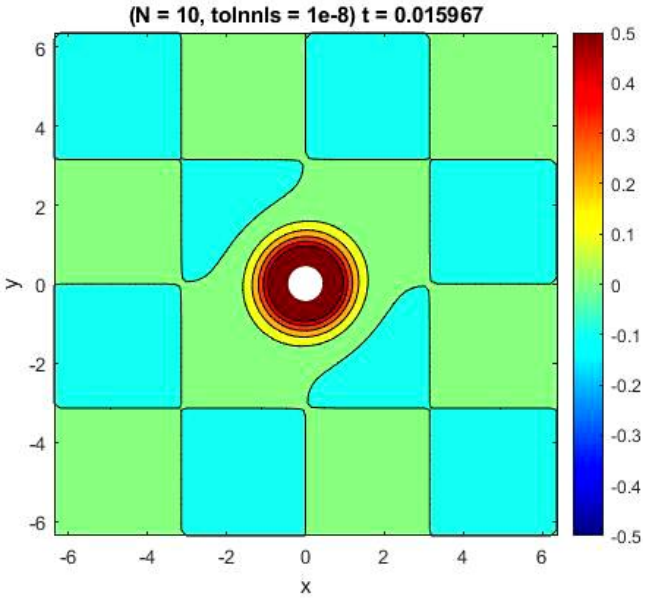


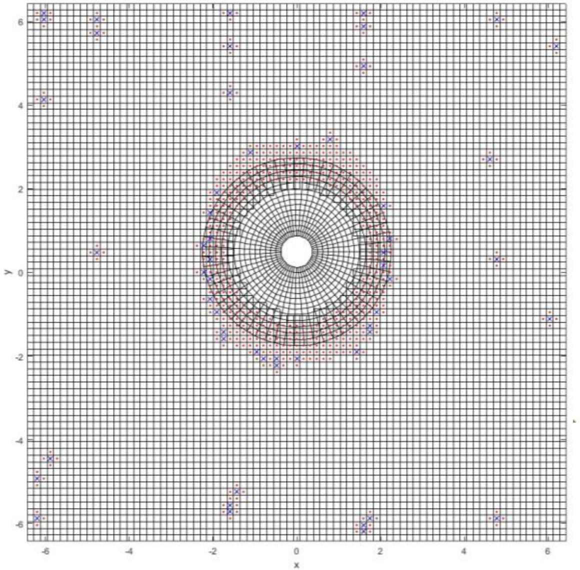


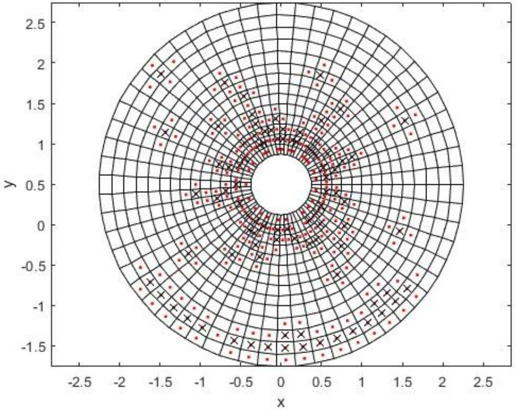
Mode 2



In-sample simulation ($z = +1$)

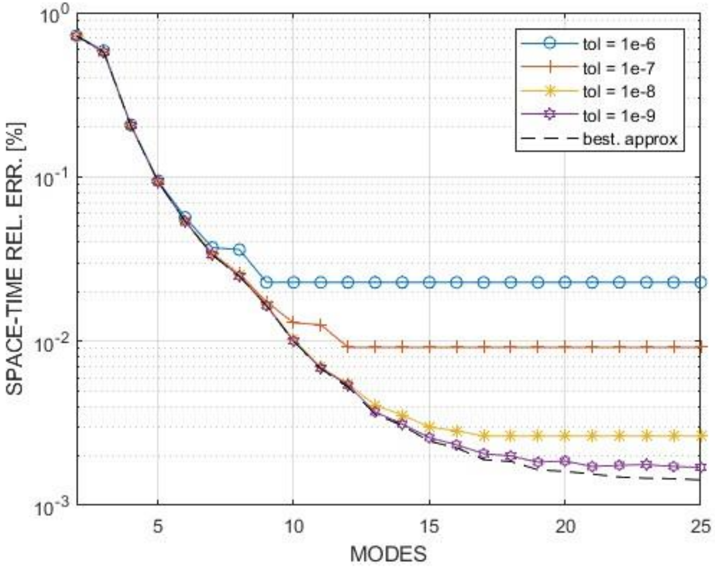




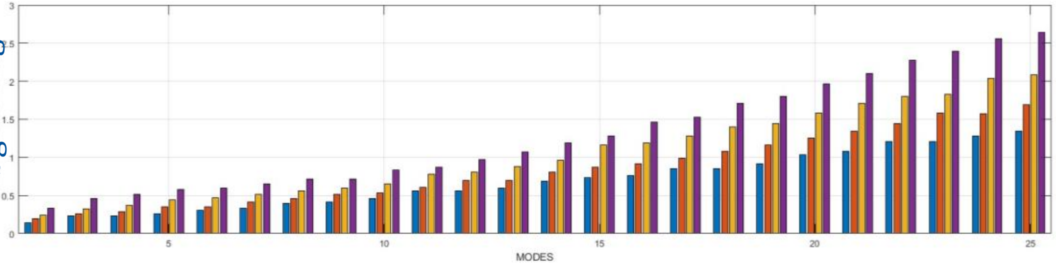


$N = 7387; N_{\mathrm{bg}} = 6561; N_{\mathrm{fg}} = 826;$
$N_{fg}^{HF} = 491$ (7.5%)

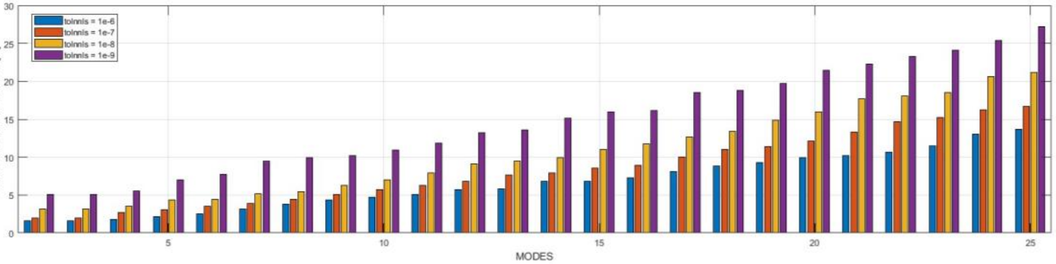
In-sample simulation



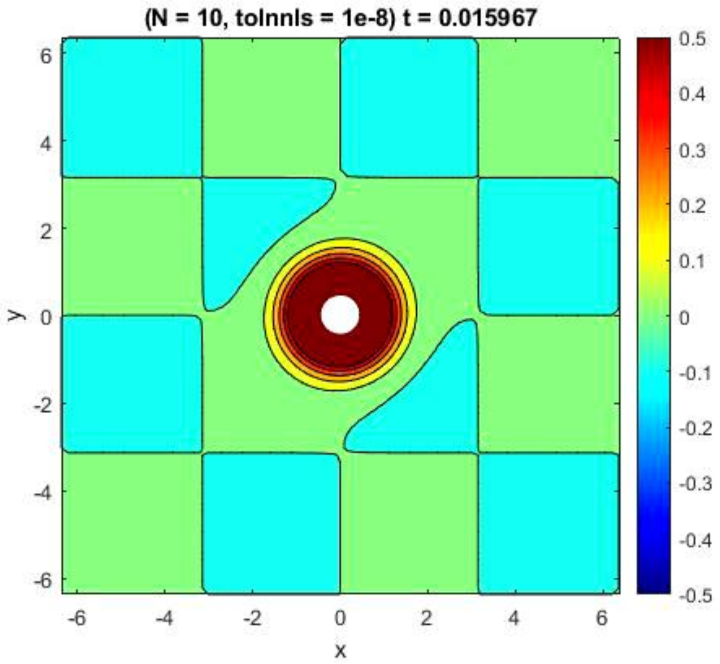
N_{bg}^{HR}/N_{bg}

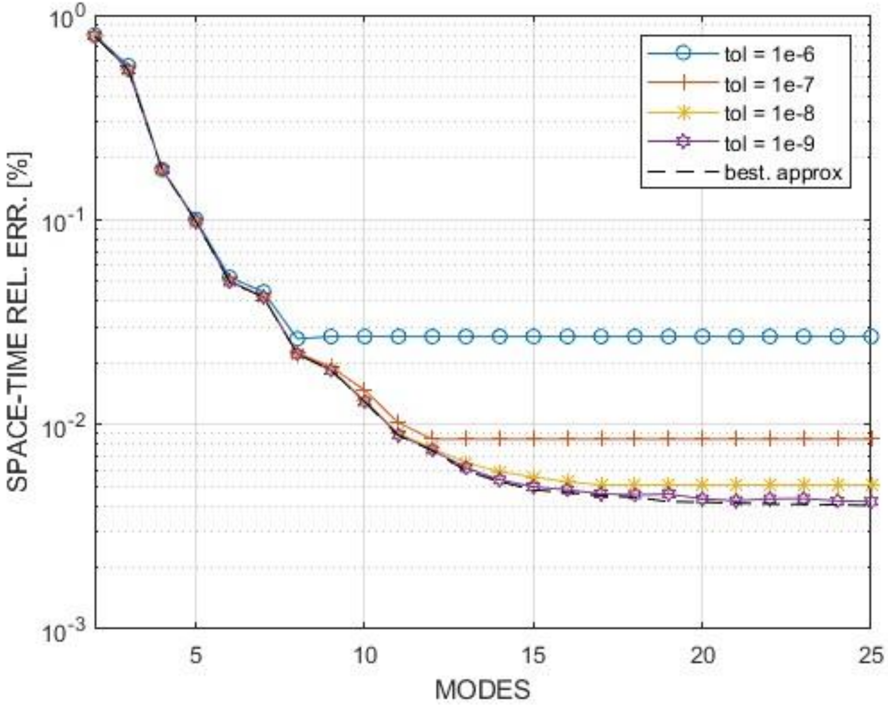


N_{fg}^{HR}/N_{fg}



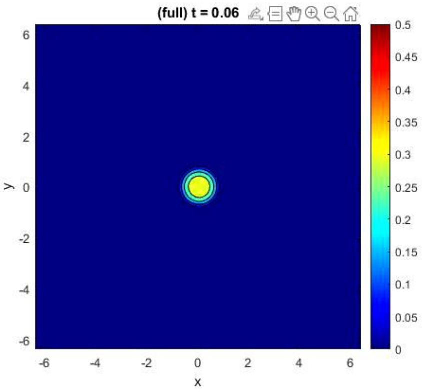
Out-of-sample simulation ($z = +1.7$)

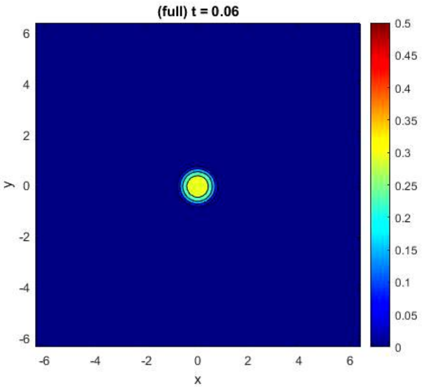


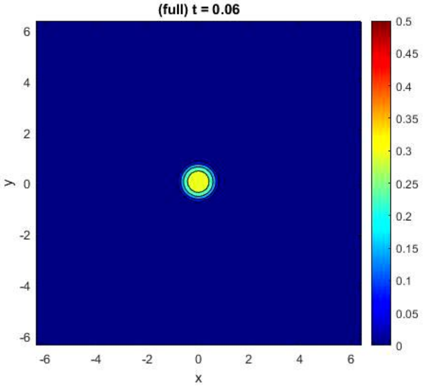


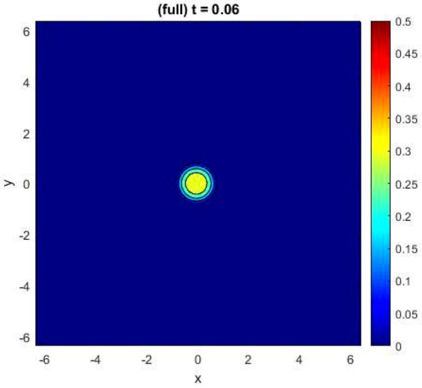
Pure advective problem for a compact support function

$$u(\boldsymbol{x}, t; z) = \exp\frac{-1}{1 - |\boldsymbol{x} - \boldsymbol{z}t|^2}\,\chi_{\{|\boldsymbol{x}-\boldsymbol{z}t|\leq 1\}}(\boldsymbol{x})$$



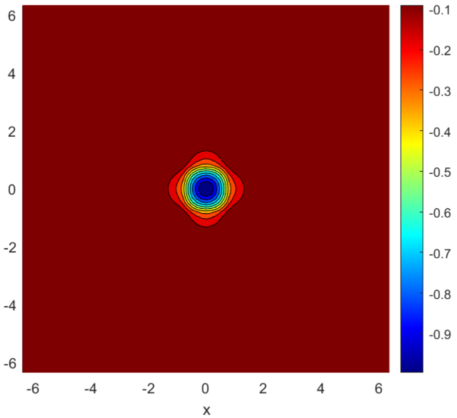




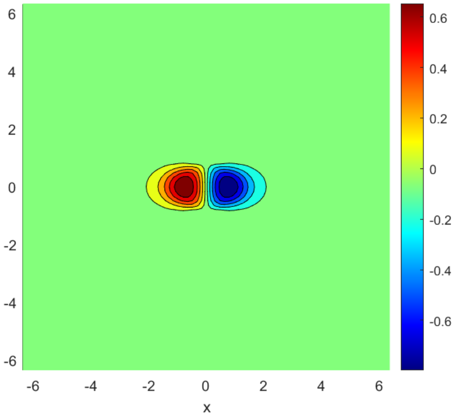


Pure advective problem (one-block approach)

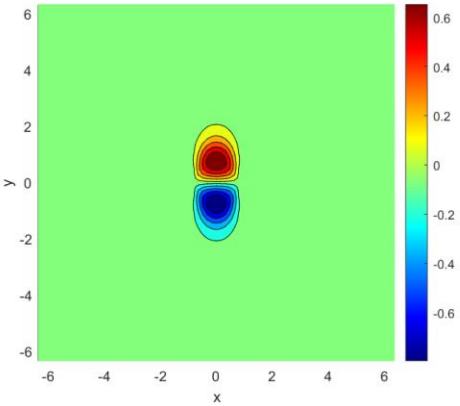
Mode 1



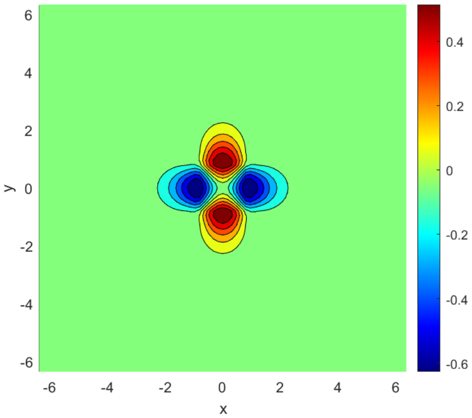
Mode 2



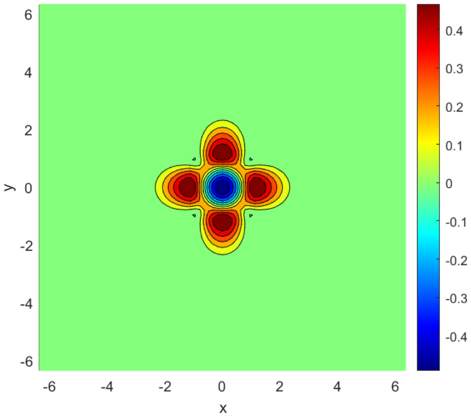
Mode 3



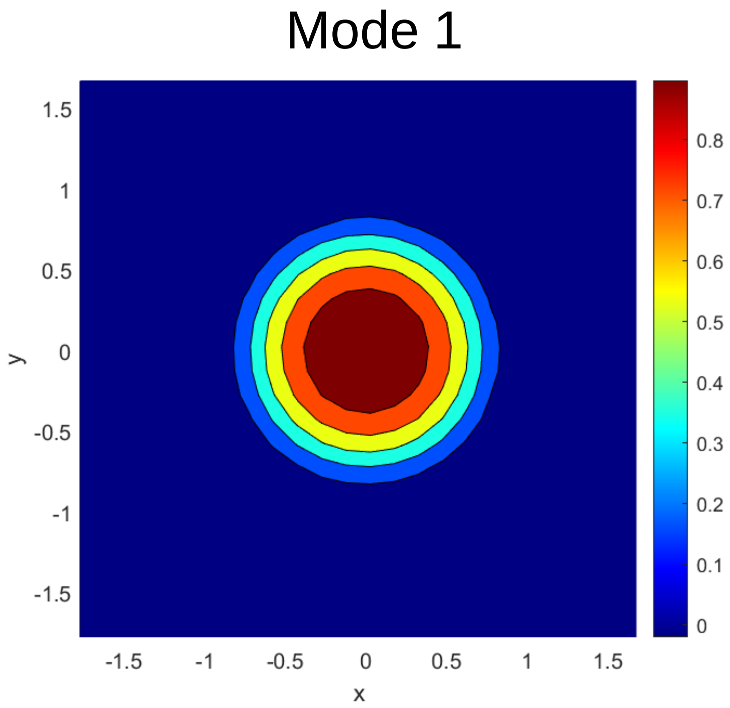
Mode 4



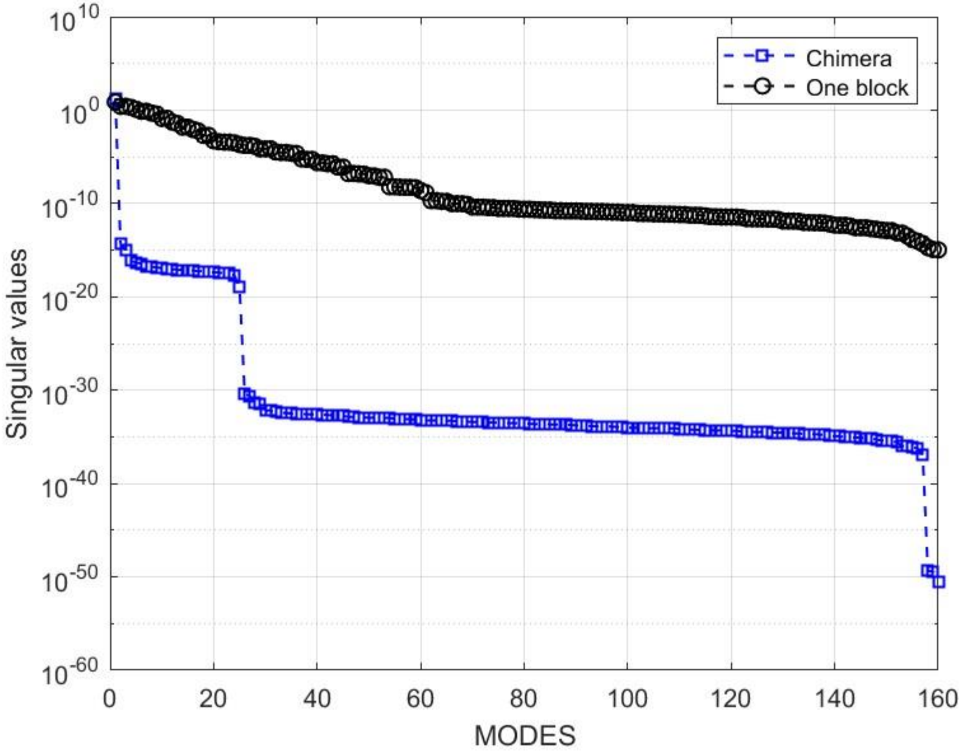
Mode 5



Pure advective problem (Chimera approach)



Comparison of decays of SVs



In-sample and out-of sample simulations

Error Chimera: $\mathcal{O}\left(10^{-6}\right)$ with 1 mode.
Error one-block: $\mathcal{O}(1)$ with 25 modes.

$N = 7137; N_{\mathrm{bg}} = 6561; N_{\mathrm{fg}} = 576;$
$N_{fg}^{HF} = 1$!!!!!!!!!!

Conclusions

- Hyper-reduced ADER-FV
- Collocated method
- Hyper-reduction as mesh-reduction AND extension tool
- Tool able to overcome the Kolmogorov n-width.

Future prespectives

- Extension to (in)compressible NS
- Extension to more classical FV approaches
- Extension to (pure) hyperbolic PDEs (need of good transmission of shocks and discontinuities along interface)

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Liberté
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Merci

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